Tanaka-Webster biharmonic hypersurfaces in Sasakian space forms

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Abstract. The article describe some recent results about concept of biharmonicity of hypersurfaces in the Sasakian space form, which is equipped with the Tanaka-Webster connection. There is a rich theory whose main message is the necessary and sufficient existence condition of the Tanaka-Webster biharmonic hypersurfaces. Also, it is included a brief nonexistence result of the Tanaka-Webster biharmonic Hopf hypersurfaces, where the gradient of the mean curvature is a principal direction.

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1 Introduction

A harmonic map $\psi : M \longrightarrow N$ between two Riemannian manifolds, where M is compact, is known as the critical point of the energy functional

$$E : C^{\infty}(M,N) \longrightarrow R, \quad E(\psi) = \frac{1}{2}\int_M \|d\psi\|^2 d\vartheta,$$

where $C^{\infty}(M,N)$ denotes space of the smooth maps. With respect to the similar idea authors [8, 9], introduced $k-$harmonic maps and proposed they are the critical points of E_k. Therefore, when $k=2$ the biharmonic maps represent as critical point of the bienergy

$$E_2 : C^{\infty}(M,N) \longrightarrow R, \quad E_2(\psi) = \frac{1}{2}\int_M |\tau(\psi)|^2 d\vartheta,$$

where the tension field associated to map ψ is given by $\tau(\psi) = \mathsf{trace}\nabla d\psi$. It is known, vanishing the tension field is a characterization of the harmonic maps. Later on, the first variation formula of E_2 was derived by Jiang [15] and given a new definition of 2-harmonic maps in the variational point of view, written as

$$\tau_2(\psi) = -J(\tau(\psi)) = -\Delta\tau(\psi) - \mathsf{trace}R^N(d\psi(.),\tau(\psi))d\psi(.) = 0, \quad (1.1)$$

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that is, $\tau(\psi) \in kerJ$, where J is an elliptic differential operator, called the Jacobi operator. Here $\Delta = -\mathsf{trace}\nabla^2$ stands for the Laplace-Beltrami operator, where ∇ is induced connection in the pull back bundle $\psi^{-1}(TN)$. Also, R^N is the curvature operator on N which is defined by $R^N(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ for all X and Y tangent to N. Since, each harmonic map is biharmonic because J is a linear operator, interesting is in non harmonic biharmonic maps which are named proper biharmonic.

Independently, by taking into account the harmonic mean curvature vector field, biharmonic notion of submanifold in the Euclidean space was defined by B. Y. Chen. Indeed, with respect to the characterization formula of the biharmonic Riemannian immersions into the Euclidean space, the biharmonic concept in the sense of Chen will be obtained, e.g, $\Delta H = 0$ where H denotes the mean curvature vector field (see [6]).

Into non-positive and positive curved spaces, nonexistence results for the proper biharmonic Riemannian immersions were obtained (see [2, 3, 7, 11, 12, 13]). Specialy, it was shown that does not exist a proper biharmonic hypersurfaces neither in the Euclidean space nor in the hyperbolic spaces H^{n+1} base on the number of distinct principal curvatures of the Weingarten operator.

Additionally, in spaces of the nonconstant sectional curvature there exist several classification results concerning the proper biharmonic hypersurfaces which has been investigated in [10, 14]. For example, all the proper biharmonic Hopf cylinders in 3-dimensional Sasakian space forms were classified. Morevere, all the proper-biharmonic Hopf cylinders over a homogeneous real hypersurfaces in the complex projective spaces were determined. In particular, authors in [16, 17] got some results about the biharmonic immersed hypersurfaces in the warped product space as well.

The idea is that in order to study the biharmonic notion of an immersed hypersurface inside of a Sasakian space form, it is equipped with the generalized Tanaka-Webster connection. This allows us, to obtain the necessary and sufficient condition of a generalized Tanaka-Webster biharmonic hypersurface in the Sasakian space forms. In this case, we examine the generalized Tanaka-Webster biharmonic pseudo Hopf hypersurfaces and determine the existance and nonexistence results of them, where the gradient of the mean curvature, $\mathsf{grad}|H|$, plays a significent role. Furthermore, the generalized Tanaka-Webster biharmonic pseudo Hopf hypersurfaces which are minimal are determine.

2 Preliminaries

In this section we introduce the notions and gather some known results that will be used throughout the paper. Indeed, an odd dimensional manifold M^{2m+1} equipped with tensor fields φ, ξ and η of type $(1,1)$, $(0,1)$ and $(1,0)$, respectively, is called an almost contact manifold where the following condition satisfies

$$\varphi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \eta(\varphi X) = 0,$$

for $X \in T(M)$, also the triple (φ, ξ, η) is named an almost contact structure. Now, M^{2m+1} is endowed a Riemannian metric g, in such a way that

$$\eta(X) = g(\xi, X), \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y),$$

where X and Y are vector fields on M^{2m+1}. If $g(X,\varphi Y)=d\eta(X,Y)$, then (φ,ξ,η,g) is called a contact metric structure. Now, $(\widetilde{M}^{2m+1},\varphi,\xi,\eta,g)$ is called a contact metric manifold. Also, a contact metric manifold $\widetilde{M}^{2m+1}$ is named a $K-$contact manifold, if ξ be a Killing vector field. Then we have

$$\nabla_X\xi=-\varphi X, \tag{2.1}$$

where ∇ is the Levi-Civita connection on $\widetilde{M}^{2m+1}$. A contact metric manifold $\widetilde{M}^{2m+1}$, is known as a Sasakian manifold, if and only if

$$(\nabla_X\varphi)Y=g(X,Y)\xi-\eta(Y)X. \tag{2.2}$$

A Sasakian manifold is a $k-$contact manifold as well [20].

Let $(\widetilde{M}^{2m+1},\varphi,\xi,\eta,g)$ be a Sasakian manifold. The sectional curvature of 2-plane spanned by $\{X,\varphi X\}$ is called φ-sectional curvature, where X is orthogonal to ξ. Also, a Sasakian manifold which has constant φ-sectional curvature c is called a Sasakian space form and determined by $\overline{M}^{2m+1}(c)$. The curvature tensor field of a Sasakian space form is given by

$$\begin{aligned}\overline{R}(X,Y)Z \;=\; & -\frac{c-1}{4}\{\eta(Z)[\eta(Y)X-\eta(X)Y]\\ & +[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]\xi\\ & +g(\varphi X,Z)\varphi Y+2g(\varphi X,Y)\varphi Z-g(\varphi Y,Z)\varphi X\}\\ & +\frac{c+3}{4}\{g(Y,Z)X-g(X,Z)Y\}.\end{aligned} \tag{2.3}$$

A canonical affine connection defined on a non-degenerate pseudo-Hermitian CR-manifold at the end of $70's$ by [18] and [21] independently, which is well known the Tanaka-Webster connection. A generalization of this connection has been introduced by [4, 19] for the contact metric manifolds, written as

$$\hat{\nabla}_XY=\nabla_XY+(\nabla_X\eta)(Y)\xi-\eta(Y)\nabla_X\xi+\eta(X)\varphi Y, \tag{2.4}$$

for all $X,Y\in\Gamma(T(\widetilde{M}^{2m+1}))$, where ∇ denotes the Levi-Civita connection on $\widetilde{M}^{2m+1}$. Then $\hat{\nabla}$ is known the generalized Tanaka-Webster connection on a contact metric manifold $(\widetilde{M}^{2m+1},\varphi,\xi,\eta,g)$. Furthermore, it was shown that the generalized Tanaka-Webster connection $\hat{\nabla}$ is an unique linear connection, where the tensors ξ,η and g are all $\hat{\nabla}$-parallel, that is,

$$\hat{\nabla}\xi^*=0,\quad \hat{\nabla}\eta=0,\quad \hat{\nabla}g=0, \tag{2.5}$$

and whose torsion tensors satisfies

$$\hat{T}(X,Y)=2d\eta(X,Y)\xi,\quad \hat{T}(\xi,\varphi X)=-\varphi\hat{T}(\xi,X),$$

for all $X\in\Gamma(T(\widetilde{M}^{2m+1}))$.

At the end of this section, in order to illustrate the existence of the biharmonic hypersurfaces in the Sasakian space form, where ξ is tangent to them, we construct an example.

Example 2.1. Let R^3 be a hypersurface in the Euclidean space R^4. Let J be a standard almost complex structure in R^4 considered as C^2 and set $\xi = -JN$, where N is an unit normal vector field of R^3. Define φ by $\pi o J$, where π is the natural projection of the tangent space of R^4 in to the tangent space of R^3. Let (x, y, z) be the Euclidean coordinate in R^3, we consider

$$\eta = \frac{1}{2}(dz - ydx), \quad g = \eta \otimes \eta + \frac{1}{4}(dx^2 + dy^2),$$
$$\varphi(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}) = Y\frac{\partial}{\partial x} - X\frac{\partial}{\partial y} + Yy\frac{\partial}{\partial z}$$

where $\xi = 2\frac{\partial}{\partial z}$. Then $(R^3, \varphi, \eta, \xi, g)$ is called a Sasakian space form where its φ-sectional curvature is $c = -3$. Let $f \in C^\infty(R^3(-3))$ defines $f(x, y, z) = x + z$, then we consider the level set of f like $M^2 = f^{-1}(0) = \{(x, y, z) \in R^3; x + z = 0\}$ which is claimed as a minimal surface (as well as biharmonic) of $R^3(-3)$. In order to show this property, we choose an appropriate orthonormal frame field on $R^3(-3)$ such as

$$e_1 = 2(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}), \quad e_2 = -2\frac{\partial}{\partial y} \quad , e_3 = 2\frac{\partial}{\partial z}$$

then we calculate $\mathsf{grad} f = \sum_{i=1}^{3} e_i(f)e_i = 2((1 + y)e_1 + e_3)$. So, $N = \frac{\mathsf{grad} f}{|\mathsf{grad}\ f|} = \frac{1}{\sqrt{(1+y)^2+1}}((1 + y)e_1 + e_3)$ is an unit normal vector on M^2. Also, $-\varphi N = V = -\frac{1+y}{2\sqrt{(1+y)^2+1}}e_2$ is in $\Gamma(TM^2)$. Now, we take an orthonormal frame field $\{E_1 = \frac{V}{|V|} = -e_2, \quad E_2 = \sqrt{\frac{(1+y)^2}{1+(1+y)^2}}(-\frac{1}{(1+y)}e_1 + e_3)\}$ on M^2. Some easy computations show the following bracket relations, which we need to calculate the Weingarten operator A of M^2 in the Sasakian space $R^3(-3)$, as following

$$[e_1, e_2] = 2e_3, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0$$
$$\nabla_{e_1} e_2 = -\nabla_{e_2} e_1 = e_3, \quad \nabla_{e_1} e_3 = \nabla_{e_3} e_1 = -e_2, \quad \nabla_{e_2} e_3 = \nabla_{e_3} e_2 = e_1$$

after all, we can calculate

$$-AE_1 = \nabla_{E_1} N = \frac{1 - (y + 1)^2}{(1 + y)^2 + 1} E_2,$$
$$-AE_2 = \nabla_{E_2} N = \frac{1 - (y + 1)^2}{(1 + y)^2 + 1} E_1,$$

then we have

$$A = \begin{pmatrix} 0 & \frac{1-(y+1)^2}{(1+y)^2+1} \\ \frac{1-(y+1)^2}{(1+y)^2+1} & 0 \end{pmatrix}.$$

So, the shape operator presents that the mean curvature $|H| = 0$. In other words, the planes which are parallel to the xz-plane are biharmonic (harmonic) surfaces in the Sasakian space form $R^3(-3)$. Furthermore, the other non trivial example of the biharmonic surface in the Sasakian space form $R^3(-3)$ can be considered where $f \in C^\infty(R^3(-3))$ and $f(x, y, z) = x^2 + z^2$, then we take the level set of f like $M^2 = f^{-1}(1) = \{(x, y, z) \in R^3; x^2 + z^2 = 1\} \approx S^1 \times R$, similarly with respect to the above coordinate and orthonormal frame field, the cylinder $S^1 \times R$ is a minimal surface (as well as biharmonic) in $R^3(-3)$ as well.

3 Generalized Tanaka-Webster biharmonic hypersurfaces

Let $(\overline{M}^{2m+1}(c), \varphi, \xi, \eta, g)$ be a Sasakian space form with respect to constant $\varphi-$sectional curvature c, which is equipped with the generalized Tanaka-Webster connection and M^{2m} is an isometrically immersed hypersurface there. We suppose that ξ and $V = -\varphi N$ are tangent vector fields on M^{2m}, where N is a local unit normal vector on M^{2m}.

Now, in order to have the generalized Tanaka-Webster biharmonic notion of hypersurfaces in the Sasakian space form, we consider the following required lemma.

Lemma 3.1. *Let $\overline{M}^{2m+1}(c)$ be a Sasakian space form. Then, the generalized Tanaka-Webster connection holds in the following formula*

$$\nabla^\star_X Y = \overline{\nabla}_X Y + g(X, \varphi Y)\xi + \eta(Y)\varphi X + \eta(X)\varphi Y, \tag{3.1}$$

where X, Y and $\overline{\nabla}$ denote tangent vector fields and the Levi-Civita connection on $\overline{M}^{2m+1}(c)$, respectively.

Proof. By taking into the account the equations (2.4) and (2.1) we have

$$\begin{aligned} \nabla^\star_X Y &= \overline{\nabla}_X Y + [\overline{\nabla}_X \eta(Y) - \eta(\overline{\nabla}_X Y)]\xi + \eta(Y)\varphi X + \eta(X)\varphi Y \\ &= \overline{\nabla}_X Y + g(Y, \overline{\nabla}_X \xi)\xi + \eta(Y)\varphi X + \eta(X)\varphi(Y) \\ &= \overline{\nabla}_X Y + g(X, \varphi Y)\xi + \eta(Y)\varphi X + \eta(X)\varphi(Y). \end{aligned}$$

□

Now, under an isometric immersion we can express

$$\tau^\star_2(\psi) = -\Delta^\star H - \mathsf{trace} R^\star(d\psi(.), H)d\psi(.), \tag{3.2}$$

here $\Delta^\star$ stands for the Laplace-Beltrami operator on sections of the pull back bundle $\psi^{-1}(T(\overline{M}^{2m+1}))$ and $R^\star$ denotes the curvature tensor corresponding to $\nabla^\star$ on the Sasakian space form $\overline{M}^{2m+1}(c)$ and we utilize the following sign conventions

$$\Delta^\star X = -\mathsf{trace}\nabla^{\star^2} X, \quad \forall X \in \psi^{-1}(T(\overline{M}^{2m+1})), \tag{3.3}$$

$$R^\star(X, Y) = [\nabla^\star_X, \nabla^\star_Y] - \nabla^\star_{[X,Y]} \quad X, Y \in T(\overline{M}^{2m+1}). \tag{3.4}$$

After all, we can define the generalized Tanaka-Webster biharmonic hypersurfaces over a Sasakian space form as following

Definition 3.1. Let $\psi : M^{2m} \longrightarrow \overline{M}^{2m+1}(c)$ be an isometric immersion of a hypersurface M^{2m} in the Sasakian space form $\overline{M}^{2m+1}(c)$ associated to the generalized Tanaka-Webster connection $\nabla^\star$. Then, M^{2m} is called generalized Tanaka-webster biharmonic hypersurface if $\tau^\star_2(\psi) = 0$.

Now, we have all the necessary ingredients to prove the following results.

Lemma 3.2. *Let $\psi : M^{2m} \longrightarrow \overline{M}^{2m+1}(c)$ be an isometric immersion of a hypersurface M^{2m} in a Sasakian space form $\overline{M}^{2m+1}(c)$ associated to the generalized Tanaka-Webster connection $\nabla^\star$. Then*

$$\Delta^* H = \Delta H + 2g(\textit{grad}|H|, V)\xi - 2\eta(\textit{grad}|H|)V - 2H, \tag{3.5}$$

where ξ and V are tangent to M^{2m}.

Proof. Let $\nabla^\star$ and $\overline{\nabla}$ denote the generalized Tanaka-Webster and the Levi-Civita connections on $\overline{M}^{2m+1}(c)$, respectively. Also, let us denote by ∇ the Levi-Civita connection on M^{2m}. We consider a parallel local orthonormal frame $\{e_\alpha\}_{\alpha=1}^{2m}$ at $p \in M^{2m}$. Then, from the equations (3.1), (3.3) and with respect to this fact the tensors φ, η and g are all $\nabla^\star$-parallel also by applying the Weingarten equation $\overline{\nabla}_{e_\alpha} N = -Ae_\alpha + \nabla^\perp_{e_\alpha} N$ we have

$$\begin{aligned}
\Delta^\star H &= -\sum_{\alpha=1}^{2m} \nabla^\star_{e_\alpha} \nabla^\star_{e_\alpha} H \\
&= -\sum_{\alpha=1}^{2m} \nabla^\star_{e_\alpha}\big(\overline{\nabla}_{e_\alpha} H + g(e_\alpha, \varphi H)\xi + \eta(H)\varphi e_\alpha + \eta(e_\alpha)\varphi H\big) \\
&= -\sum_{\alpha=1}^{2m} \big\{\overline{\nabla}_{e_\alpha}\overline{\nabla}_{e_\alpha} H + g(e_\alpha, \varphi(\overline{\nabla}_{e_\alpha} H))\xi + \eta(\overline{\nabla}_{e_\alpha} H)\varphi e_\alpha \\
&\quad + \eta(e_\alpha)\varphi(\overline{\nabla}_{e_\alpha} H) + g(e_\alpha, \varphi(\nabla^\star_{e_\alpha} H)\xi) + \eta(e_\alpha)\varphi(\nabla^\star_{e_\alpha} H)\big\} \\
&= \Delta H - 2\big(\sum_{\alpha=1}^{2m} g(e_\alpha, e_\alpha|H|\varphi N) - \sum_{\alpha=1}^{2m} g(e_\alpha, \varphi A_H e_\alpha)\big)\xi \\
&\quad -2\big(\sum_{\alpha=1}^{2m} \eta(e_\alpha)e_\alpha|H|\varphi N - \sum_{\alpha=1}^{2m} \eta(e_\alpha)\varphi A_H e_\alpha\big) \\
&\quad + \sum_{\alpha=1}^{2m} g(A_H e_\alpha, \xi)\varphi e_\alpha + H.
\end{aligned} \tag{3.6}$$

The next step is to compute all terms of the equation (3.6) as following

$$\sum_{\alpha=1}^{2m} g(e_\alpha, e_\alpha|H|\varphi N)\xi = -g(\mathsf{grad}|H|, V)\xi, \tag{3.7}$$

and because the tensors φ is skew symmetric then

$$\sum_{\alpha=1}^{2m} g(e_\alpha, \varphi A_H e_\alpha)\xi = \mathsf{trace}(\varphi A_H)\xi = 0. \tag{3.8}$$

Also, for the other terms we have

$$\begin{aligned}
\sum_{\alpha=1}^{2m} \eta(e_\alpha)e_\alpha|H|\varphi N &= \sum_{\alpha=1}^{2m} \eta(e_\alpha)g(\mathsf{grad}|H|, e_\alpha)\varphi N \\
&= \eta(\mathsf{grad}|H|)\varphi N \\
&= -\eta(\mathsf{grad}|H|)V.
\end{aligned} \tag{3.9}$$

Now, in order to calculate what is remainder we need to consider

$$\sum_{\alpha=1}^{2m} \eta(\overline{\nabla}_{e_\alpha} H)\varphi e_\alpha = \sum_{\alpha=1}^{2m} g(\overline{\nabla}_{e_\alpha} H, \xi)\varphi e_\alpha = \sum_{\alpha=1}^{2m} g(H, \varphi e_\alpha)\varphi e_\alpha,$$

also

$$\sum_{\alpha=1}^{2m} \eta(\overline{\nabla}_{e_\alpha} H)\varphi e_\alpha = \sum_{\alpha=1}^{2m} g(\overline{\nabla}_{e_\alpha} H, \xi)\varphi e_\alpha = -\sum_{\alpha=1}^{2m} g(A_H e_\alpha, \xi)\varphi e_\alpha,$$

so

$$A_H \xi = -|H|V. \tag{3.10}$$

Then the last two terms of (3.6) follow

$$\sum_{\alpha=1}^{2m} \eta(e_\alpha)\varphi A_H e_\alpha = \varphi A_H \xi = -\varphi|H|V = -H, \tag{3.11}$$

and

$$\begin{aligned}
\sum_{\alpha=1}^{2m} g(A_H e_\alpha, \xi)\varphi e_\alpha &= \sum_{\alpha,\beta=1}^{2m} g(A_H e_\alpha, \xi)(g(\varphi e_\alpha, e_\beta)e_\beta + g(\varphi e_\alpha, N)N) \\
&= -|H| \sum_{\alpha,\beta=1}^{2m} g(e_\alpha, V)(-g(e_\alpha, \varphi e_\beta)e_\beta - g(e_\alpha, \varphi N)N) \\
&= -|H| \sum_{\beta=1}^{2m} (g(V, \varphi e_\beta)e_\beta + g(V,V)N) \\
&= -H. \qquad (3.12)
\end{aligned}$$

After all, from the equations (3.7), (3.8), (3.9), (3.11) and (3.12) we have the result as it was claimed. □

Lemma 3.3. *Let $\psi : M^{2m} \longrightarrow \overline{M}^{2m+1}(c)$ be an isometric immersion of $2m$-dimensional hypersurface M^{2m} in a Sasakian space form $\overline{M}^{2m+1}(c)$. Let the ambient manifold equipped with the ∇^* associated to the generalized Tanaka-Webster connection, then*

$$\textsf{trace}R^*(d\psi(.), H)d\psi(.) = kH, \tag{3.13}$$

where $k = \frac{1}{4}(15 - 6m - c(3 + 2m))$ and H denotes the mean curvature vector field of M^{2m}.

Proof. Let $\overline{R}$ and R^* be the curvature tensor of $\overline{M}^{2m+1}(c)$ associated to the Levi-Civita connection $\overline{\nabla}$ and ∇^*, respectively. Then, by applying the equation (3.4) we have

$$R^\star(X,H)X = \overline{R}(X,H)X - 3g(X, \varphi H)\varphi X - \eta^2(X)\varphi^2 H, \tag{3.14}$$

wherer in an appropriate local orthonormal frame field $\{e_\alpha\}_{\alpha=1}^{2m-1}\cup\{\xi,V\}$ on $\overline{M}^{2m+1}(c)$, that follows

$$\begin{aligned}\mathrm{trace}R^\star(d\psi(.),H)d\psi(.) &= \mathrm{trace}\overline{R}(d\psi(.),H)d\psi(.)\\ &\quad -3g(V,\varphi H)\varphi V-\eta^2(\xi)\varphi^2 H\\ &= \sum_{\alpha=1}^{2m-1}\overline{R}(e_\alpha,H)e_\alpha\\ &\quad +\overline{R}(\xi,H)\xi+\overline{R}(V,H)V+4H,\end{aligned}$$

then the equation (2.3) with the straightforward computation shows that

$$\overline{R}(e_\alpha,H)e_\alpha=-\frac{c+3}{4}g(e_\alpha,e_\alpha)H,$$

and

$$\overline{R}(\xi,H)\xi=-H,\quad \overline{R}(V,H)V=-cH.$$

Hence

$$\mathrm{trace}R^\star(d\psi(.),H)d\psi(.)=-(2m-1)\frac{c+3}{4}H+(1-c)H.$$

□

After all we obtain the main result of this section

Theorem 3.4. *Let $\psi: M^{2m}\longrightarrow \overline{M}^{2m+1}(c)$ be an isometric immersion of $2m-$ dimensional hypersurface M^{2m} in the Sasakian space form $\overline{M}^{2m+1}(c)$ equipped with the ∇^* associated to the generalized Tanaka-Webster connection. Then M^{2m} is a generalized Tanaka-Webster biharmonic hypersurface if and only if*

$$(3.15)\quad \begin{cases}\Delta^\perp H=\mathrm{trace}B((.),A_H(.))+lH;\\ \mathrm{trace}A_{\nabla^\perp_{(.)}H}(.)+m|H|\mathrm{grad}|H|+g(\mathrm{grad}|H|,V)\xi-\eta(\mathrm{grad}|H|)V=0,\end{cases}$$

where $l=\frac{-(2m+3)c-6m+7}{4}$ is constant, B,A and H denote the second fundamental form, shape operator and mean curvature vector field of M^{2m} in $\overline{M}^{2m+1}(c)$, respectively.

Proof. By applying, the Definition 3.1, Lemmas 3.2 and 3.3 directly, we have

$$(3.16)\,\tau_2^*(\psi)=-(\Delta H+2g(\mathrm{grad}|H|,V)\xi-2\eta(\mathrm{grad}|H|)V-2H)-kH=0,$$

where

$$\begin{aligned}(3.17)\quad \Delta H &= -\sum_{\alpha=1}^{2m}\overline{\nabla}_{e_\alpha}\overline{\nabla}_{e_\alpha}H\\ &= -\Delta^\perp H+\mathrm{trace}B(.,A_H.)+\mathrm{trace}A_{\nabla^\perp_{(.)}H}(.)+\mathrm{trace}\nabla A_H(.,.),\end{aligned}$$

in more details

$$\begin{aligned}\overline{\nabla}_{e_\alpha}\overline{\nabla}_{e_\alpha}H &= \nabla^\perp_{e_\alpha}\nabla^\perp_{e_\alpha}H \\ &\quad -A_{\nabla^\perp_{e_\alpha}H}e_\alpha - \nabla_{e_\alpha}A_H(e_\alpha) - B(e_\alpha, A_H(e_\alpha)),\end{aligned}$$

and

$$\begin{aligned}\mathsf{trace}\nabla A_H(.,.) &= m\mathsf{grad}|H|^2 \\ &\quad +(\mathsf{trace}\overline{R}(d\psi(.), H)d\psi(.))^\top + \mathsf{trace}A_{\nabla^\perp_{(.)}H}(.),\end{aligned}$$

where, as respect to the Lemma 3.3 the tangent part of $\mathsf{trace}\overline{R}(d\psi(.), H)d\psi(.)$ vanishes. Now, putting the above equations together and replacing them in the equation (3.16) at follow splitting the normal and tangent parts of it, the result obtains. □

Regarding the mean curvature, from Theorem 3.4. we can have the following result.

Corollary 3.5. *Let M^{2m} be a generalized Tanaka-Webster biharmonic hypersurface with the constant mean curvature. Tthen the φ-sectional curvature holds*

$$c > \frac{-6m+7}{2m+3}.$$

Proof. According to the assumption M^{2m} has constant mean curvature $|H| = constant \neq 0$. In this case, the Theorem 3.4 yields

$$|B|^2 = -l = -\frac{7 - c(2m+3) - 6m}{4},$$

which implies the result. □

4 Generalized Tanaka-Webster biharmonic pseudo Hopf hypersurfaces

Let $x : (M^{2m}, g) \to (\overline{M}^{2m+1}(c), \overline{g})$ be an isometric immersion from a real hypersurface M^{2m} in the Sasakian space form $\overline{M}^{2m+1}(c)$. We underline the ambient manifold is equipped with the generalized Tanaka-Webster connection $\nabla^\star$ as well. Let $\overline{\nabla}$ and ∇ denote the Levi-Civita connections on $\overline{M}^{2m+1}(c)$ and M^{2m}, respectively. We recall that ξ and V are tangent on M^{2m}. Also, we suppose that $T(M^{2m}) = D \oplus D^\perp$, where D is a maximal $\varphi-$invariant distribution and $D^\perp = \mathsf{Span}\{\xi, V\}$ in such away that the Weingarten operator A satisfies $AD^\perp \subseteq D^\perp$ and $AD \subseteq D$. A hypersurface M^{2m} is called a pseudo-Hopf hypersurface, provided that the Weingarten operator A be invariant on $\mathsf{Span}\{V, \xi\}$ (see [1]). Supposed that $W_1, W_2 \in \mathsf{Span}\{\xi, V\}$ are eigenvectors of the Weingarten operator A in which $AW_1 = \gamma_1 W_1$ and $AW_2 = \gamma_2 W_2$ where

$$W_1 = \xi\cos\theta + V\sin\theta, \qquad W_2 = -\xi\sin\theta + V\cos\theta, \tag{4.1}$$

for some $0 < \theta < \frac{\pi}{2}$, where $\gamma_1 = -\tan\theta$ and $\gamma_2 = \cot\theta$. Let $AV = \alpha\xi + \beta V$, then we have $\alpha = -1$ and $\beta = \frac{\cos 2\theta}{\cos\theta\sin\theta}$.

Lemma 4.1. *Let* $x : (M^{2m}, g) \longrightarrow (\overline{M}^{2m+1}(c), \overline{g})$ *be an isometric immersion of a hypersurface* M^{2m} *in the Sasakian space form* $\overline{M}^{2m+1}(c)$*. Then for* X, Y *and* Z *tangent on* $T(M^{2m})$ *the Coddazi equation holds*

$$\begin{aligned} &g((\nabla_X A)Y - (\nabla_Y A)X, Z) = \\ &-\frac{c-1}{4}\overline{g}\big(\overline{g}(\varphi X, Z)Y + 2\overline{g}(\varphi X, Y)Z - \overline{g}(\varphi Y, Z)X, V\big), \end{aligned} \tag{4.2}$$

where ∇ *and* A *are the Levi-Civita connection and shape operator of* M^{2m}.

Lemma 4.2. *Let* $x : M^{2m} \longrightarrow \overline{M}^{2m+1}(c)$ *be an isometric immersion of a pseudo Hopf hypersurface* M^{2m} *in the Sasakian space form* $\overline{M}^{2m+1}(c)$*. If the Weingarten operator* A *for some* $X \in D$ *satisfies* $AX = \lambda X$*, then*

$$A\varphi X = \frac{2\beta\lambda + c + 3}{4\lambda - 2\beta}\varphi X. \tag{4.3}$$

Proof. Let $X, Y \in D$ be the eigenvectors of the Weingarten operator A. We consider $AV = -\xi + \beta V$ and take the covariant derivative of both sides, then

$$(\nabla_X A)V + A\nabla_X V = X(\beta)V + \beta\nabla_X V + \varphi X,$$

where ∇ denotes the Levi-Civita connection on M^{2m} and $\nabla_X V = \varphi AX$ [1]. Hence, we get

$$g((\nabla_X A)V, Y) + g(A\varphi AX, Y) = \beta g(\varphi AX, Y) + g(\varphi X, Y),$$

similarly

$$g((\nabla_Y A)V, X) + g(A\varphi AY, X) = \beta g(\varphi AY, X) + g(\varphi Y, X),$$

so

$$\begin{aligned} g((\nabla_X A)Y - (\nabla_Y A)X, V) + 2g(A\varphi AX, Y) = \\ \beta g(\varphi AX, Y) + \beta g(A\varphi X, Y) + 2g(\varphi X, Y), \end{aligned}$$

then, by applying the Lemma 4.1, we obtain

$$\begin{aligned} -\frac{c-1}{2}g(\varphi X, Y) + 2g(A\varphi AX, Y) = \\ \beta g(\varphi AX, Y) + \beta g(A\varphi X, Y) + 2g(\varphi X, Y), \end{aligned}$$

consequently

$$(2\lambda - \beta)g(A\varphi X, Y) = (\beta\lambda + 2 + \frac{c-1}{2})g(\varphi X, Y),$$

which shows φX is an eigenvector corresponding to the eigenvalue $\frac{2\beta\lambda+c+3}{4\lambda-2\beta}$. □

Now we discuss about the generalized Tanaka-Webster biharmonic pseudo Hopf hypersurfaces in more details.

Theorem 4.3. *There exists no generalized Tanaka-Webster biharmonic pseudo Hopf hypersurface in such away that* $\mathrm{grad}|H|$ *is in the direction of the vector field in* D.

Proof. We denote by $\psi : M^{2m} \to \overline{M}^{2m+1}(c)$ an isometric immersion where M^{2m} is a generalized Tanaka-Webster biharmonic pseudo Hopf hypersurface in a Sasakian space form $\overline{M}^{2m+1}(c)$. According to the assumption $\mathrm{grad}|H|$ is in the direction of vectors in D, that is, $AD \subset D$. Then, by applying Theorem 3.4 directly, we get

$$\begin{cases} \Delta^{\perp} H = \mathrm{trace} B((.), A_H(.)) + lH; \\ \mathrm{trace} A_{\nabla^{\perp}_{(.)} H}(.) + m|H|\mathrm{grad}|H| = 0, \end{cases} \tag{4.4}$$

where $A\mathrm{grad}|H| = -m|H|\mathrm{grad}|H|$ is deduced by the second term. Also, the Lemma 4.2 allows the Weingarten operator A of M^{2m} takes the following form in a suitable orthogonal frame field $\{e_1, ..., e_{m-1}, e_m = \varphi e_1, ..., e_{2m-2} = \varphi e_{m-1}, e_{2m-1} = W_1, e_{2m} = W_2\}$ in which

$$\begin{aligned} Ae_i &= \lambda_i e_i, \quad i = 1, ..., m-1 \\ A\varphi e_i &= \overline{\lambda_i} \varphi e_i, \quad i = 1, ..., m-1 \\ AW_1 &= -\gamma_1 W_1, \quad AW_2 = \gamma_2 W_2 \end{aligned} \tag{4.5}$$

where, λ_i and $\overline{\lambda_i} = \frac{2\beta\lambda_i + c + 3}{4\lambda_i - 2\beta}$ are the eigenvalues corresponding to the eigenvectors e_i and φe_i, respectively. We recall that $\gamma_1 = -\tan\theta$ and $\gamma_2 = \cot\theta$, consequently we get $\gamma_1\gamma_2 = -1$. Let $e_1 = \frac{\mathrm{grad}|H|}{|\mathrm{grad}|H||}$. Assume that $\mathrm{grad}|H|$ is given by $\mathrm{grad}|H| = \sum_{i=1}^{2m} e_i(|H|)e_i$. Then

$$e_1(|H|) \neq 0, \qquad e_i(|H|) = 0, \qquad i = 2, ..., 2m. \tag{4.6}$$

Also, it is written

$$\nabla_{e_i} e_j = \textstyle\sum_{k=1}^{2m} \omega_{ij}^k e_k, \tag{4.7}$$

where 1-forms ω_{ij}^k are called the connection forms. From $\nabla_{e_k}\langle e_i, e_j\rangle = 0$ it follows

$$\omega_{ki}^i = 0, \quad i = j \tag{4.8}$$

$$\omega_{ki}^j + \omega_{kj}^i = 0 \quad , \quad i \neq j, \quad i, j, k = 1, ..., 2m. \tag{4.9}$$

Morevere, from the Codazzi equation, and taking (4.5) and (4.7) we get

$$e_k(\lambda_i)e_i + (\lambda_i - \lambda_j)\omega_{ki}^j e_j = e_i(\lambda_k)e_k + (\lambda_k - \lambda_j)\omega_{ik}^j e_j,$$

where, multiply both sides of it to e_j, we arrive at

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j \tag{4.10}$$

$$(\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j, \tag{4.11}$$

for distinct $i, j, k = 1, ..., 2m$. From $\lambda_1 = -m|H|$ and (4.6) we obtain

$$e_1(\lambda_1) \neq 0, \qquad e_i(\lambda_1) = 0, \qquad i = 2, ..., 2m \tag{4.12}$$

and

$$0 = [e_i, e_j]\lambda_1 = (\nabla_{e_i} e_j - \nabla_{e_j} e_i)\lambda_1, \quad 2 \le i, j \le 2m, i \ne j \tag{4.13}$$

which implies

$$\omega_{ij}^1 = \omega_{ji}^1, \tag{4.14}$$

for distinct $i, j = 2, ..., 2m$. It is claimed that, $\lambda_j \ne \lambda_1$ for $j = 2, ..., 2m$. Since, if $\lambda_j = \lambda_1$ for $j \ne 1$, utilize (4.10) and put $i = 1$

$$0 = (\lambda_1 - \lambda_j)\omega_{j1}^j = e_1(\lambda_j) = e_1(\lambda_1),$$

which contradicts to (4.12). For $j = 1$ and $k, i \ne 1$ the equation (4.11) yields

$$(\lambda_i - \lambda_1)\omega_{ki}^1 = (\lambda_k - \lambda_1)\omega_{ik}^1 \tag{4.15}$$

which together with (4.14) follow

$$\omega_{ij}^1 = 0, \qquad i \ne j, \qquad i, j = 2, ..., 2m \tag{4.16}$$

that combining with the equation (4.9), implies $\omega_{i1}^j = 0$, $i \ne j$, $i, j = 2, ..., 2m$.

After all, by summarizing and considering the above equations and applying appropriate connections we will reach to a contradiction. Indeed, by using the equation (4.7) we have

$$\begin{aligned}
\nabla_{e_m} W_2 &= \nabla_{e_m}(-\xi \sin\theta + V \cos\theta) \\
&= -e_m(\sin\theta)\xi - \sin(\theta)\nabla_{e_m}\xi + e_m(\cos\theta)V + \cos(\theta)\nabla_{e_m} V \\
&= -e_m(\sin\theta)(W_1 \sin\theta + W_2 \cos\theta) - \sin\theta(-\varphi e_m) \\
&+ e_m(\cos\theta)(W_1 \cos\theta - W_2 \sin\theta) + \cos\theta(\varphi A e_m) \\
&= (-e_m(\sin\theta)\sin\theta + e_m(\cos\theta)\cos\theta)W_1 - (\sin\theta + \overline{\lambda_1}\cos\theta)e_1 \\
&- (e_m(\sin\theta)\cos\theta + e_m(\cos\theta)\sin\theta)W_2.
\end{aligned} \tag{4.17}$$

On the one hand, from (4.16) the connection form $\omega_{m2m}^1 = 0$, which is associated to the vector field e_1. Then, (4.17) follows

$$0 = \omega_{m2m}^1 = \sin\theta + \overline{\lambda_1}\cos\theta,$$

which implies

$$\overline{\lambda_1} = -\tan\theta, \tag{4.18}$$

where $\overline{\lambda}_1$ is an eigenvalue of the Weingarten operator corresponding to the eigenvector $e_m = \varphi e_1$.

Similarly, by computing the $\nabla_{e_m} W_1$ and applying the equation (4.16) we have

$$0 = \omega_{m2m-1}^1 = \cos\theta - \overline{\lambda_1}\sin\theta,$$

in which

$$\overline{\lambda_1} = \cot\theta. \tag{4.19}$$

Finally, a contradiction is made by (4.18) and (4.19). Hence, the generalized Tanaka-Webster biharmonic pseudo Hopf hypersurfaces do not exist in a Sasakian space form whenever $\mathsf{grad}|H|$ be an eigenvector of the Weingarten operator in the distribution D. □

Also, we have the following

Proposition 4.4. *Let M^{2m} be a generalized Tanaka-Webster biharmonic pseudo Hopf hypersurface, where $\mathsf{grad}|H| \in D^{\perp}$. Then M^{2m} is either a minimal hypersurface ($|H| = 0$) or $|H| = -\frac{\gamma_1+\gamma_2}{m}$.*

Proof. According to the assumption $\mathrm{grad}|H| \in D^{\perp}$, so we can suppose that $\mathsf{grad}|H| = \alpha\xi + \beta V$. By using the Theorem 3.4 we have

$$\begin{cases} \Delta^{\perp} H = \mathsf{trace} B((.), A_H(.)) + lH; \\ A\mathsf{grad}|H| + m|H|\mathsf{grad}|H| + \beta\xi - \alpha V = 0, \end{cases} \tag{4.20}$$

where the second term yields

$$0 = A(\alpha\xi + \beta V) + m|H|(\alpha\xi + \beta V) + \beta\xi - \alpha V. \tag{4.21}$$

Now, by taking into account the equation (4.1) we have

$$\begin{aligned} AW_1 &= \gamma_1 W_1 \\ &= \cos\theta A\xi + \sin\theta AV, \end{aligned}$$

so with respect to the $\gamma_1 = -\mathsf{tan}\theta$ we have

$$AV = -\xi + (\gamma_1 + \gamma_2)V, \tag{4.22}$$

then from the equations (4.22) and (4.21) we get

$$0 = (m|H|\alpha)\xi + (-2\alpha + m|H|\beta + \beta(\gamma_1 + \gamma_2))V,$$

consequently it is obtained that

$$\begin{aligned} 0 &= m|H|\alpha \\ 0 &= -2\alpha + m|H|\beta + \beta(\gamma_1 + \gamma_2), \end{aligned}$$

those show, either $|H| = 0$ or $\alpha = 0$. When $|H| = 0$ the generalized Tanaka-Webster pseudo Hopf hypersurface M^{2m} is minimal, clearly. Also, we get $|H| = -\frac{(\gamma_1+\gamma_2)}{m}$, provided that $\alpha = 0$. More precisely, where $\alpha = 0$ then the Theorem 3.4 follows

$$\begin{aligned} &\Delta^{\perp} H = \mathsf{trace} B((.), A_H(.)) + kH \\ &A\mathsf{grad}|H| + m|H|\mathsf{grad}|H| + \beta\xi = 0, \end{aligned}$$

in which the second term, where $\mathsf{grad}|H| = \beta V$, implies

$$0 = \beta AV + m|H|\beta V + \beta\xi,$$

and now, by applying (4.22), we have

$$|H| = -\frac{\gamma_1 + \gamma_2}{m}.$$

□

Proposition 4.5. *The generalized Tanaka-Webster biharmonic pseudo Hopf hypersurfaces have the constant mean curvature provided that* $\mathsf{grad}|H|$ *is in the direction* ξ.

Proof. Let M^{2m} be a generalized Tanaka-Webster biharmonic hypersurface in $\overline{M}^{2m+1}(c)$. According to the assumption $\mathsf{grad}|H| = \alpha\xi$, where H is the mean curvature vector field of M^{2m}. Then in this case the Theorem 3.4 implies

$$\begin{cases} \Delta^{\perp} H = \mathsf{trace} B((.), A_H(.)) + lH; \\ A\mathsf{grad}|H| + m|H|\mathsf{grad}|H| - \alpha V = 0, \end{cases} \tag{4.23}$$

where the second term yields

$$\alpha A\xi + m|H|\alpha\xi - \alpha V = 0,$$

by taking into account the equation (3.10) then $\alpha = 0$ and $\mathsf{grad}|H| = 0$, that states $|H| =$ constant. □

Immediately, it follows

Corollary 4.6. *There exists not a generalized Tanaka-Webster biharmonic pseudo Hopf hypersurface with the constant mean curvature where the* φ*-sectional curvature holds* $c < -3$.

Proof. Obviously, it is followed by the Corollary 3.5 and Proposition 4.5. □

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