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SOME PROPERTIES OF A QUARTER-SYMMETRIC METRIC CONNECTION ON A SASAKIAN MANIFOLD

(DEDICATED IN OCCASION OF THE 65-YEARS OF PROFESSOR R.K. RAINA)

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Abstract. The object of the present paper is to study a quarter-symmetric metric connection on a Sasakian manifold. The existence of the connection is given on a Riemannian manifold. We deduce the relation between the Riemannian connection and the quarter-symmetric metric connection on a Sasakian manifold. We study the projective curvature tensor with respect to the quarter-symmetric metric connection and also characterized $\xi-$projectively flat and $\phi-$projectively flat Sasakian manifold with respect to the quarter-symmetric metric connection. Finally we study locally $\phi-$symmetric Sasakian manifold with respect to the quarter-symmetric metric connection.

1. Introduction

In this paper we undertake a study of quarter-symmetric metric connection on a Sasakian manifold. In 1975, S. Golab[6] defined and studied quarter-symmetric connection in a differentiable manifold with affine connection.

A linear connection $\tilde{\nabla}$ on an n-dimensional Riemannian manifold(M, g) is called a quarter-symmetric connection[6] if its torsion tensor T of the connection $\tilde{\nabla}$

$$T(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y]$$

satisfies

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.1}$$

where η is a 1-form and ϕ is a $(1,1)$ tensor field.

In particular, if $\phi(X) = X$, then the quarter-symmetric connection reduces to the semi-symmetric connection[5]. Thus the notion of quarter-symmetric connection generalizes the idea of the semi-symmetric connection.

If moreover, a quarter-symmetric connection $\tilde{\nabla}$ satisfies the condition

$$(\tilde{\nabla}_X g)(Y,Z) = 0 \tag{1.2}$$

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for all $X, Y, Z \in T(M)$, where $T(M)$ is the Lie algebra of vector fields of the manifold M, then $\tilde{\nabla}$ is said to be a quarter-symmetric metric connection, otherwise it is said to be a quarter-symmetric non-metric connection.

After S. Golab[6], S. C. Rastogi ([12],[13]) continued the systematic study of quarter-symmetric metric connection.

In 1980, R. S. Mishra and S. N. Pandey[9] studied quarter-symmetric metric connection in Riemannian, Kaehlerian and Sasakian manifolds.

In 1982, K. Yano and T. Imai[18] studied quarter-symmetric metric connection in Hermitian and Kaehlerian manifols.

In 1991, S. Mukhopadhyay, A. K. Roy and B. Barua[10] studied a quarter-symmetric metric connection on a Riemannian manifold (M, g) with an almost complex structure ϕ.

In 1997, U. C. De and S. C. Biswas[2] studied a quarter-symmetric metric connection on a $SP-$Sasakian manifold. Also in 2008, Sular, Ozgur and De[14] studied a quarter-symmetric metric connection in a Kenmotsu manifold.

Apart from conformal curvature tensor, the projective curvature tensor is another important tensor from the differential geometric point of view. Let M be an $n-$dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighborhood of M and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 3$, M is locally projectively flat if and only if the well known projective curvature tensor P vanishes. Here P is defined by [8]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\}, \tag{1.3}$$

for $X, Y, Z \varepsilon T(M)$, where R is the curvature tensor and S is the Ricci tensor. In fact, M is projectively flat (that is $P = 0$) if and only if the manifold is of constant curvature (pp. 84-85 of [17]). Thus, the projective curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

A Sasakian manifold is said to be an Einstein manifold if its Ricci tensor S satisfies the condition

$$S(X,Y) = \lambda g(X,Y)$$

where λ is a constant.

The paper is organized as follows:

After preliminaries, in section 3 we prove the existence of the quarter-symmetric metric connection. In the next section we establish the relation between the Riemannian connection and the quarter-symmetric metric connection on a Sasakian manifold. Section 5 deals with the projective curvature tensor with respect to the quarter-symmetric metric connection and in the next section we prove that for a Sasakian manifold the Riemannian connection ∇ is $\xi-$projectively flat if and only if the quarter-symmetric metric connection $\tilde{\nabla}$ is so. We also study $\phi-$projectively flat Sasakian manifold and prove that if a Sasakian manifold is $\phi-$projectively flat then the manifold is an $\eta-$Einstein manifold with respect to the quarter-symmetric metric connection. Finally we characterized locally $\phi-$symmetric Sasakian manifold with respect to the quarter-symmetric metric connection.

2. PRELIMINARIES

An $n(=2m+1)-$dimensional smooth manifold M is said to be a contact manifold if it carries a global $1-$form η such that $\eta \wedge (d\eta)^m \neq 0$ everywhere on M. For a given contact $1-$form η there exist a unique vectore field ξ (the Reeb vector field) such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$. Polarizing $d\eta$ on the contact subbundle $\eta = 0$, one obtains a Riemannian metric g and a $(1,1)-$tensor field ϕ such that

$$a) \quad d\eta(X,Y) = g(\phi X, Y) \quad b) \quad \eta(X) = g(X,\xi) \quad c) \quad \phi^2 = -X + \eta(X)\xi \tag{2.1}$$

g is called an associated metric of η and (ϕ, η, ξ, g) a contact metric structure. The tensor $h = \frac{1}{2}\pounds_\xi \phi$ is known to be self-adjoint, anti-commutes with ϕ, and satisfies: $Tr.h = Tr.h\phi = 0$. A contact metric structure is said to be $K-$*contact* if ξ is a Killing with respect to g, equivalently, $h = 0$. If in such a manifold the relation

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X \tag{2.2}$$

holds, where ∇ denotes the Levi-Civita connection of g, then M is called a Sasakian manifold. The contact structure on M is said to be normal if the almost complex structure on $M \times R$ defined by $J(X, fd/dt) = (\phi X - f\xi, \eta(X)d/dt)$, where f is a real function on $M \times R$, is integrable. Also, a normal contact metric manifold is a Sasakian manifold. It is well known that every Sasakian manifold is $K-$ contact but converse is not true in general. However, a 3-dimensional $K-$contact manifold is Sasakian[7].

Let R and r denote respectively the curvature tensor of type $(1,3)$ and scalar curvature of M. It is known that in a contact metric manifold M the Riemannian metric may be so chosen that the following relations hold [1],[19].

$$a) \quad \phi\xi = 0 \quad b) \quad \eta(\xi) = 1 \quad c) \quad \eta.\phi = 0 \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.4}$$

for any vector field X,Y. If M is a Sasakian manifold, then besides (2.2),(2.3) (2.4) and (2.5) the following relations hold:

$$\nabla_X \xi = -\phi X \tag{2.5}$$

$$(\nabla_X \eta)Y = g(X, \phi Y) \tag{2.6}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.7}$$

$$R(\xi, X)Y = (\nabla_X \phi)Y \tag{2.8}$$

$$S(X,\xi) = (n-1)\eta(X). \tag{2.9}$$

$$S(\phi X, \phi Y) = S(X,Y) - (n-1)\eta(X)\eta(Y). \tag{2.10}$$

for any vector fields X, Y.

3. **Existence of a quarter-symmetric metric connection**

Let X and Y be any two vector fields on (M, g). Let us define a connection $\tilde{\nabla}_X Y$ by the following equation:

$$\begin{aligned} 2g(\tilde{\nabla}_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z) \\ &- g([Y,Z],X) + g([Z,X],Y) + g(\eta(Y)\phi X \\ &- \eta(X)\phi Y, Z) + g(\eta(Y)\phi Z - \eta(Z)\phi Y, X) \\ &+ g(\eta(X)\phi Z - \eta(Z)\phi X, Y), \end{aligned} \tag{3.1}$$

which holds for all vector fields $X, Y, Z \in T(M)$.

It can easily be verified that the mapping

$$(X, Y) \longrightarrow \tilde{\nabla}_X Y$$

satisfies the following equalities:

$$\tilde{\nabla}_X (Y + Z) = \tilde{\nabla}_X Y + \tilde{\nabla}_X Z, \tag{3.2}$$

$$\tilde{\nabla}_{X+Y} Z = \tilde{\nabla}_X Z + \tilde{\nabla}_Y Z, \tag{3.3}$$

$$\tilde{\nabla}_{fX} Y = f \tilde{\nabla}_X Y \tag{3.4}$$

and

$$\tilde{\nabla}_X (fY) = f \tilde{\nabla}_X Y + (Xf) Y \tag{3.5}$$

for all $X, Y, Z \in T(M)$ and $f \in F(M)$, the set of all differentiable mappings over M. From (3.2),(3.3),(3.4) and (3.5) we can conclude that $\tilde{\nabla}$ determine a linear connection on (M, g).

Now we have

$$2g(\tilde{\nabla}_X Y, Z) - 2g(\tilde{\nabla}_Y X, Z) = 2g([X, Y], Z) + 2g(\eta(Y)\phi X - \eta(X)\phi Y, Z). \tag{3.6}$$

Hence,

$$\tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] = \eta(Y)\phi X - \eta(X)\phi Y$$

or,

$$T(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y. \tag{3.7}$$

Also we have

$$2g(\tilde{\nabla}_X Y, Z) + 2g(\tilde{\nabla}_X Z, Y) = 2Xg(Y, Z),$$

or,

$$(\tilde{\nabla}_X g)(Y, Z) = 0,$$

that is,

$$\tilde{\nabla} g = 0. \tag{3.8}$$

From (3.7) and (3.8) it follows that $\tilde{\nabla}$ determines a quarter-symmetric metric connection on (M, g). It can be easily verified that $\tilde{\nabla}$ determines a unique quarter-symmetric metric connection on (M, g). Thus we have

Theorem 3.1. *Let M be a Riemannian manifold and η be a 1-form on it. Then there exist a unique linear connection $\tilde{\nabla}$ satisfying (3.7) and (3.8).*

Remark: The above theorem prove the existence of a quarter-symmetric metric connection on (M, g).

4. **Relation between the Riemannian connection and the quarter-symmetric metric connection**

Let $\tilde{\nabla}$ be a linear connection and ∇ be a Riemannian connection of an almost contact metric manifold M such that

$$\tilde{\nabla}_X Y = \nabla_X Y + U(X, Y), \tag{4.1}$$

where U is a tensor of type $(1, 1)$. For $\tilde{\nabla}$ to be a quarter-symmetric metric connection in M, we have [6]

$$U(X, Y) = \frac{1}{2}[T(X, Y) + T'(X, Y) + T'(Y, X)], \tag{4.2}$$

where

$$g(T'(X, Y), Z) = g(T(Z, X), Y). \tag{4.3}$$

From (1.1) and (4.3) we get

$$T'(X, Y) = g(\phi Y, X)\xi - \eta(X)\phi Y \tag{4.4}$$

and using (1.1) and (4.4) in (4.2) we obtain

$$U(X, Y) = -\eta(X)\phi Y.$$

Hence a quarter-symmetric metric connection $\tilde{\nabla}$ in a Sasakian manifold is given by

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{4.5}$$

Conversely, we show that a linear connection $\tilde{\nabla}$ on a Sasakian manifold defined by

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y, \tag{4.6}$$

denotes a quarter-symmetric metric connection.

Using (4.6) the torsion tensor of the connection $\tilde{\nabla}$ is given by

$$\begin{aligned} aT(X, Y) &= \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] \\ &= \eta(Y)\phi X - \eta(X)\phi Y. \end{aligned} \tag{4.7}$$

The above equation shows that the connection $\tilde{\nabla}$ is a quarter-symmetric connection [6]. Also we have

$$\begin{aligned} a(\tilde{\nabla}_X g)(Y, Z) &= Xg(Y, Z) - g(\tilde{\nabla}_X Y, Z) - g(Y, \tilde{\nabla}_X Z) \\ &= \eta(X)[g(\phi Y, Z) + g(\phi Z, Y)] \\ &= 0. \end{aligned} \tag{4.8}$$

In virtue of (4.7) and (4.8) we conclude that $\tilde{\nabla}$ is a quarter-symmetric metric connection. Therefore equation (4.5) is the relation between the Riemannian connection and the quarter-symmetric metric connection on a Sasakian manifold.

5. Curvature tensor of a Sasakian manifold with respect to the quarter-symmetric metric connection

A relation between the curvature tensor of M with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ and the Riemannian connection ∇ is given by [4].

$$\begin{aligned} a\tilde{R}(X,Y)Z &= R(X,Y)Z - 2d\eta(X,Y)\phi Z + \eta(X)g(Y,Z)\xi \\ &- \eta(Y)g(X,Z)\xi + \{\eta(Y)X - \eta(X)Y\}\eta(Z), \end{aligned} \tag{5.1}$$

where $R(X,Y)Z$ is the Riemannian curvature of the manifold. Also from (5.1) we obtain

$$\tilde{S}(Y,Z) = S(Y,Z) - 2d\eta(\phi Z, Y) + g(Y,Z) + (n-2)\eta(Y)\eta(Z), \tag{5.2}$$

where $\tilde{S}$ and S are the Ricci tensors of the connections $\tilde{\nabla}$ and ∇ respectively. From (5.2) it is clear that in a Sasakian manifold the Ricci tensor with respect to the quarter-symmetric metric connection is symmetric.

Again contracting (5.2) we have

$$\tilde{r} = r + 2(n-1),$$

where $\tilde{r}$ and r are the scalar curvature of the connections $\tilde{\nabla}$ and ∇ respectively.

6. Projective curvature tensor on a Sasakian manifold

The generalized projective curvature tensor of a Sasakian manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ is defined by [8]

$$\begin{aligned} \tilde{P}(X,Y)Z &= \tilde{R}(X,Y)Z + \frac{1}{n+1}[\tilde{S}(X,Y)Z - \tilde{S}(Y,X)Z] \\ &+ \frac{1}{n^2-1}[\{n\tilde{S}(X,Z) + \tilde{S}(Z,X)\}Y \\ &- \{n\tilde{S}(Y,Z) + \tilde{S}(Z,Y)\}X]. \end{aligned} \tag{6.1}$$

Since the Ricci tensor $\tilde{S}$ of the manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ is symmetric, the projective curvature tensor $\tilde{P}$ reduces to

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{n-1}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y]. \tag{6.2}$$

Using (5.1) and (5.2), (6.2) reduces to

$$\begin{aligned} a\tilde{P}(X,Y)Z &= P(X,Y)Z - 2d\eta(X,Y)\phi Z - \{\eta(Y)g(X,Z) \\ &- \eta(X)g(Y,Z)\}\xi + \frac{2}{n-1}[d\eta(\phi Z,Y)X - d\eta(\phi Z,X)Y] \\ &+ \frac{1}{n-1}[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\ &- g(Y,Z)X + g(X,Z)Y], \end{aligned} \tag{6.3}$$

where P is the projective curvature tensor defined by (1.3).

$\xi-$conformally flat $K-$contact manifolds have been studied by Zhen, Cabrerizo and Fernandez [20]. Analogous to the definition of $\xi-$conformally flat $K-$contact manifold we define the $\xi-$projectively flat Sasakian manifold.

Definition 6.1 A Sasakian manifold M is called $\xi-$projectively flat if the condition $P(X,Y)\xi = 0$ holds on M.

From (6.3) it is clear that $\tilde{P}(X,Y)\xi = P(X,Y)\xi$.

So we have the following:

Theorem 6.1. *For a Sasakian manifold the Riemannian connection ∇ is $\xi-$projectively flat if and only if the quarter-symmetric metric connection $\tilde{\nabla}$ is so.*

Analogous to the definition of $\phi-$conformally flat contact manifold [3], we define $\phi-$projectively flat Sasakian manifold.

Definition 6.2 A Sasakian manifold satisfying the condition

$$\phi^2 P(\phi X, \phi Y)\phi Z = 0 \tag{6.4}$$

is called $\phi-$projectively flat[11].

Let us assume that M is a $\phi-$projectively flat Sasakian manifold with respect to the quarter-symmetric metric connection. It can be easily seen that $\phi^2\tilde{P}(\phi X, \phi Y)\phi Z = 0$ holds if and only if

$$g(\tilde{P}(\phi X, \phi Y)\phi Z, \phi W) = 0, \tag{6.5}$$

for $X, Y, Z, W \varepsilon T(M)$.

Using (6.2) and (6.5), $\phi-$projectively flat means

$$g(\tilde{R}(\phi X, \phi Y)\phi Z, \phi W) = \frac{1}{n-1}\{\tilde{S}(\phi Y, \phi Z)g(\phi X, \phi W) - \tilde{S}(\phi X, \phi Z)g(\phi Y, \phi W)\}. \tag{6.6}$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of the vector fields in M and using the fact that $\{\phi e_1, \phi e_2......, \phi e_{n-1}, \xi\}$ is also a local orthonormal basis, putting $X = W = e_i$ in (6.6) and summing up with respect to $i = 1, 2, ..., n-1$, we have

$$\sum_{i=1}^{n-1} g(\tilde{R}(\phi e_i, \phi Y)\phi Z, \phi e_i)$$

$$= \frac{1}{n-1}\sum_{i=1}^{n-1}\{\tilde{S}(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - \tilde{S}(\phi e_i, \phi Z)g(\phi Y, \phi e_i)\}. \tag{6.7}$$

Using (2.1), (2.3), (2.6) and (5.2), it can be easily verified that

$$\begin{aligned} a\sum_{i=1}^{n-1} g(\tilde{R}(\phi e_i, \phi Y)\phi Z, \phi e_i) &= \sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) - 2g(\phi Y, \phi Z) \\ &= S(Y,Z) - R(\xi, Y, Z, \xi) \\ &- (n-1)\eta(Y)\eta(Z) - 2g(\phi Y, \phi Z) \\ &= \tilde{S}(Y,Z) - 6g(Y,Z) - 2(n-4)\eta(Y)\eta(Z), \end{aligned} \tag{6.8}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n-1, \tag{6.9}$$

$$\sum_{i=1}^{n-1} \tilde{S}(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = \tilde{S}(\phi Y, \phi Z). \tag{6.10}$$

So using (6.8), (6.9) and (6.10) the equation (6.7) becomes

$$\tilde{S}(Y,Z) - 6g(Y,Z) - 2(n-4)\eta(Y)\eta(Z) = \frac{n-2}{n-1}\tilde{S}(\phi Y, \phi Z). \tag{6.11}$$

Using (2.9),and (5.2), (6.11) reduces to

$$\tilde{S}(Y,Z) = 6(n-1)g(Y,Z) - 4(n-1)\eta(Y)\eta(Z). \tag{6.12}$$

Hence we can state the following:

Theorem 6.2. *If a Sasakian manifold is $\phi-$projectively flat with respect to the quarter-symmetric metric connection then the manifold is an $\eta-$Einstein manifold with respect to the quarter-symmetric metric connection.*

7. Locally $\phi-$symmetric Sasakian manifold with respect to the quarter-symmetric metric connection

Definition 7.1 A Sasakian manifold is said to be locally $\phi-$symmetric if

$$\phi^2(\nabla_W R)(X,Y)Z = 0, \tag{7.1}$$

for all vector fields W, X, Y, Z orthogonal to ξ. This notion was introduced by Takahashi[16].

Analogous to the definition of $\phi-$symmetric Sasakian manifold with respect to the Riemannian connection, we define locally $\phi-$symmetric Sasakian manifold with respect to the quarter-symmetric metric connection by

$$\phi^2(\tilde{\nabla}_W \tilde{R})(X,Y)Z = 0, \tag{7.2}$$

for all vector fields W, X, Y, Z orthogonal to ξ.

Using (4.5) we can write

$$(\tilde{\nabla}_W \tilde{R})(X,Y)Z = (\nabla_W \tilde{R})(X,Y)Z - \eta(W)\phi\tilde{R}(X,Y)Z. \tag{7.3}$$

Now differentiating (5.1) with respect to W, we obtain

$$\begin{aligned} a(\nabla_W \tilde{R})(X,Y)Z &= (\nabla_W R)(X,Y)Z - 2d\eta(X,Y)(\nabla_W \phi)Z - \{(\nabla_W \eta)(Y)g(X,Z) \\ &- (\nabla_W \eta)(X)g(Y,Z)\}\xi - \{\eta(Y)g(X,Z) \\ &- \eta(X)g(Y,Z)\}(\nabla_W \xi) + (\nabla_W \eta)(Y)\eta(Z)X \\ &+ (\nabla_W \eta)(Z)\eta(Y)X - (\nabla_W \eta)(X)\eta(Z)Y \\ &- (\nabla_W \eta)(Z)\eta(X)Y. \end{aligned} \tag{7.4}$$

Using (2.2),(2.4) and (2.5) we have

$$\begin{aligned} a(\nabla_W \tilde{R})(X,Y)Z &= (\nabla_W R)(X,Y)Z - 2d\eta(X,Y)\{g(Z,W)\xi \\ &- 2g(\phi X,Y)\eta(Z)W\} + g(\phi W,Y)g(X,Z)\xi \\ &- g(\phi W,X)g(Y,Z)\xi + \eta(Y)g(X,Z)\phi W \\ &- \eta(X)g(Y,Z)\phi W - g(\phi W,Y)\eta(Z)X \\ &- g(\phi W,Z)\eta(Y)X + g(\phi W,X)\eta(Z)Y \\ &+ g(\phi W,Z)\eta(X)Y. \end{aligned} \tag{7.5}$$

Using (7.5) and (2.3) in (7.3) we get

$$\begin{aligned} a\phi^2(\tilde{\nabla}_W\tilde{R})(X,Y)Z &= \phi^2(\nabla_W R)(X,Y)Z + -2d\eta(X,Y)\{\eta(Z)W \\ &- \eta(Z)\eta(W)\xi\} - \eta(Y)g(X,Z)\phi W \\ &+ \eta(X)g(Y,Z)\phi W + g(\phi W,Y)\eta(Z)X \\ &- g(\phi W,Y)\eta(Z)\eta(X)\xi + g(\phi W,Z)\eta(Y)X \\ &- g(\phi W,X)\eta(Z)Y + g(\phi W,X)\eta(Z)\eta(Y)\xi \\ &- g(\phi W,Z)\eta(X)Y - \eta(W)\phi^2(\phi\tilde{R})(X,Y)Z. \end{aligned} \tag{7.6}$$

If we take W, X, Y, Z orthogonal to ξ, (7.6) reduces to

$$\phi^2(\tilde{\nabla}_W\tilde{R})(X,Y)Z = \phi^2(\nabla_W R)(X,Y)Z.$$

Hence we can state the following:

Theorem 7.1. *For a Sasakian manifold the Riemannian connection ∇ is locally $\phi-$ symmetric if and only if the quarter-symmetric metric connection $\tilde{\nabla}$ is so.*

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