

## ON CHENEY-SHARMA CHLODOVSKY OPERATORS

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ABSTRACT. The main purpose of this paper is to show that Cheney-Sharma Chlodovsky operators preserve properties of the function of modulus of continuity and Lipschitz condition of a given Lipschitz continuous function  $f$ . Furthermore, we give a result for these operators when  $f$  is a convex function.

### 1. INTRODUCTION

The Bernstein polynomials  $B_n : C[0, 1] \rightarrow C[0, 1]$  are defined by

$$\sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0, 1], \quad n \in \mathbb{N}.$$

In 1982, Lindvall [21] proved that Bernstein polynomials preserve the Lipschitz constant and order of a Lipschitz continuous function using a probabilistic method. Later, Brown, Elliott and Paget [7] gave an elementary proof for the same result. Furthermore, using similar method as in [7], Li [20] showed that Bernstein polynomials preserve the properties of the function of modulus of continuity. These problems were investigated for some linear positive operators via elementary methods or probabilistic methods (see, for instance [3], [4], [5], [6], [12], [13] [18], [19], [28]).

In [10], Chlodovsky introduced the classical Bernstein-Chlodovsky polynomials as a generalization of Bernstein polynomials on unbounded sets. For every  $n \in \mathbb{N}$ ,  $f \in C[0, \infty)$  and  $x \in [0, \infty)$ , these polynomials  $C_n : C[0, \infty) \rightarrow C[0, \infty)$  defined by

$$C_n(f, x) := \begin{cases} \sum_{k=0}^n f\left(\frac{k}{n}b_n\right) \binom{n}{k} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k} & , \quad \text{if } 0 \leq x \leq b_n \\ f(x) & , \quad \text{if } x > b_n \end{cases} \quad (1.1)$$

where  $0 \leq x \leq b_n$  and  $\{b_n\}$  is a positive sequence with properties:  
 $\lim_{n \rightarrow \infty} b_n = \infty$ ,  $\lim_{n \rightarrow \infty} (b_n/n) = 0$ .

Recently, some authors have studied some Chlodovsky type polynomials which may be found in ([4], [8], [14], [15], [16], [17], [22]).

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It is known that the Abel-Jensen equalities are given by the following formulas (see [2], p.326)

$$(u + v + n\beta)^{n-1} = \sum_{k=0}^n \binom{n}{k} u (u + k\beta)^{k-1} v [v + (n - k)\beta]^{n-k-1} \quad (1.2)$$

and

$$(u + v) (u + v + m\beta)^{m-1} = \sum_{k=0}^m \binom{m}{k} u (u + k\beta)^{k-1} v [v + (m - k)\beta]^{m-k-1}, \quad (1.3)$$

where  $u, v$  and  $\beta \in \mathbb{R}$ . By means of these equalities Cheney-Sharma [9], introduced the following Bernstein type operators, for  $f \in C[0, 1]$ ,  $x \in [0, 1]$  and  $n \in \mathbb{N}$ ,

$$Q_n(f; x) = (1 + n\beta)^{1-n} \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x (x + k\beta)^{k-1} [t_n^{k,0}(x, 1)]^{n-k} \quad (1.4)$$

and

$$G_n(f; x) = (1 + n\beta)^{1-n} \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x (x + k\beta)^{k-1} (1 - x) [t_n^{k,0}(x, 1)]^{n-k-1}, \quad (1.5)$$

where  $\beta$  is a nonnegative real parameter and the notation

$$t_n^{k,j}(x, z) = 1 - \frac{x}{z} + (n - k - j)\beta$$

is used for convenience. It is obvious that for  $\beta = 0$  these operators reduce to the classical Bernstein operators. Cheney-Sharma proved that if  $n\beta_n \rightarrow \infty$  as  $n \rightarrow \infty$ , then for  $f \in C[0, 1]$ , these operators uniformly converge to  $f$  on  $[0, 1]$ . In [5], the authors showed that Cheney-Sharma operators preserve the Lipschitz constant and order of a Lipschitz continuous function as well as the properties of the function of modulus of continuity. They also gave a result for  $G_n(f; x)$  under the convexity of  $f$ . A Kantorovich type generalization of the Cheney-Sharma operators was studied in [23]. For these operators, we refer the readers to ([1], [6], [11], [24], [26], [27]).

In [25], Chlodovsky-type Cheney-Sharma operators were defined as

$$G_n^*(f; x) = (1 + n\beta)^{1-n} \sum_{k=0}^n f\left(\frac{k}{n}b_n\right) \binom{n}{k} \frac{x}{b_n} \left(\frac{x}{b_n} + k\beta\right)^{k-1} \times \left(1 - \frac{x}{b_n}\right) [t_n^{k,0}(x, b_n)]^{n-k-1}, \quad (1.6)$$

for  $0 \leq x \leq b_n$  and  $f(x)$  for  $x > b_n$  and  $\{b_n\}$  satisfies the conditions given by (1.1). The authors also proved the weighted uniform convergence of  $G_n^*(f) \rightarrow f$ , obtained the rate of approximation in terms of the usual modulus of continuity and studied  $A$ -statistical convergence behaviours of these operators. For the case of  $b_n = 1$  the operators  $G_n^*(f; x)$  reduce to  $G_n(f; x)$ .

Also, from Lemma 2, in [25], the operators  $G_n^*(f; x)$  satisfy

$$G_n^*(1; x) = 1, \quad (1.7)$$

$$G_n^*(t; x) = x. \quad (1.8)$$

In the present paper, we show that the operators (1.6) preserve properties of the function of modulus of continuity when  $f$  is a function of modulus of continuity

and Lipschitz condition of a given Lipschitz continuous function  $f$ . Furthermore, we prove that

$$G_n^*(f) \geq f$$

for all convex functions  $f \in C[0, \infty)$ . In order to give some properties of the operators, defined by (1.6), let us recall some useful definitions.

**Definition 1.** *If a continuous, non-negative function,  $\omega$ , defined on  $[0, A]$ , satisfies the conditions:*

1.  $\omega(u+v) \leq \omega(u) + \omega(v)$  for  $u, v, u+v \in [0, A]$  ( $A > 0$ ), i.e.  $\omega$  is semi-additive,
2.  $\omega(u) \geq \omega(v)$  for  $u \geq v$ , i.e.  $\omega$  is non-decreasing,
3.  $\lim_{u \rightarrow 0^+} \omega(u) = \omega(0) = 0$ , then it is called a function of modulus of continuity.

**Definition 2.** *Let  $f$  be a real valued continuous function defined on  $[0, \infty)$ . Then  $f$  is said to be Lipschitz continuous of order  $\gamma$  on  $[0, \infty)$  if the following inequality holds*

$$|f(x) - f(y)| \leq M|x - y|^\gamma$$

for  $x, y \in [0, \infty)$  with  $M > 0$  and  $0 < \gamma \leq 1$ . The set of Lipschitz continuous functions is denoted by  $Lip_M(\gamma)$ .

**Definition 3.** *A real valued continuous function  $f$  is said to be convex on  $[0, \infty)$ , if the following inequality holds*

$$f\left(\sum_{k=1}^n \alpha_k x_k\right) \leq \sum_{k=1}^n \alpha_k f(x_k)$$

for all  $x_1, x_2, \dots, x_n \in [0, \infty)$  and for all non-negative numbers,  $\alpha_1, \alpha_2, \dots, \alpha_n$  such that  $\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$ .

## 2. MAIN RESULTS

In this section, we prove that Cheney-Sharma Chlodovsky operators preserve properties of the function of modulus of continuity when the attached function  $f$  is a function of modulus of continuity and the Lipschitz condition of a given Lipschitz continuous function  $f$ , as in [7] and [20]. Finally, we give an inequality for  $G_n^*(f; x)$ , using the convexity of  $f$ .

**Theorem 1.** *If  $\omega$  is a function of modulus of continuity then  $G_n^*(\omega; x)$  is also a function of modulus of continuity.*

*Proof.* Let  $x, y \in [0, b_n)$  and  $y \geq x$

$$\begin{aligned} G_n^*(\omega; y) &= (1 + n\beta)^{1-n} \sum_{j=0}^n \omega\left(\frac{j}{n}b_n\right) \binom{n}{j} \frac{y}{b_n} \left(\frac{y}{b_n} + j\beta\right)^{j-1} \\ &\quad \times \left(1 - \frac{y}{b_n}\right) [t_n^{j,0}(y, b_n)]^{n-j-1}. \end{aligned} \quad (2.1)$$

Letting  $u = x/b_n$ ,  $v = (y-x)/b_n$  and  $n = j$  in (1.3), we obtain

$$\frac{y}{b_n} \left(\frac{y}{b_n} + j\beta\right)^{j-1} = \sum_{k=0}^j \binom{j}{k} \frac{x}{b_n} \left(\frac{x}{b_n} + k\beta\right)^{k-1} \frac{y-x}{b_n} \left[\frac{y-x}{b_n} + (j-k)\beta\right]^{j-k-1}$$

and, with use of (2.1), we have

$$\begin{aligned} G_n^*(\omega; y) &= (1+n\beta)^{1-n} \sum_{j=0}^n \sum_{k=0}^j \omega \left( \frac{j}{n} b_n \right) \binom{n}{j} \binom{j}{k} \frac{x}{b_n} \left( \frac{x}{b_n} + k\beta \right)^{k-1} \\ &\times \frac{y-x}{b_n} \left[ \frac{y-x}{b_n} + (j-k)\beta \right]^{j-k-1} \left( 1 - \frac{y}{b_n} \right) [t_n^{j,0}(y, b_n)]^{n-j-1}. \end{aligned}$$

In the last equality, let us change the order of the summations as follows:

$$\begin{aligned} G_n^*(\omega; y) &= (1+n\beta)^{1-n} \sum_{k=0}^n \sum_{j=k}^n \omega \left( \frac{j}{n} b_n \right) \frac{n!}{(n-j)!j!} \frac{j!}{k!(j-k)!} \frac{x}{b_n} \left( \frac{x}{b_n} + k\beta \right)^{k-1} \\ &\times \frac{y-x}{b_n} \left[ \frac{y-x}{b_n} + (j-k)\beta \right]^{j-k-1} \left( 1 - \frac{y}{b_n} \right) [t_n^{j,0}(y, b_n)]^{n-j-1}. \end{aligned} \quad (2.2)$$

Now, taking  $j-k=l$  in (2.2), we obtain

$$\begin{aligned} G_n^*(\omega; y) &= (1+n\beta)^{1-n} \sum_{k=0}^n \sum_{l=0}^{n-k} \omega \left( \frac{k+l}{n} b_n \right) \frac{n!}{k!l!(n-k-l)!} \frac{x}{b_n} \left( \frac{x}{b_n} + k\beta \right)^{k-1} \\ &\times \frac{y-x}{b_n} \left[ \frac{y-x}{b_n} + l\beta \right]^{l-1} \left( 1 - \frac{y}{b_n} \right) [t_n^{k,l}(y, b_n)]^{n-k-l-1}. \end{aligned} \quad (2.3)$$

In (1.3), replace  $u$ ,  $v$  and  $n$  by  $(y-x)/b_n$ ,  $1-y/b_n$  and  $n-k$ , respectively, to obtain

$$\begin{aligned} &\left( 1 - \frac{x}{b_n} \right) [t_n^{k,0}(x, b_n)]^{n-k-1} \\ &= \sum_{l=0}^{n-k} \binom{n-k}{l} \left( \frac{y-x}{b_n} \right) \left( \frac{y-x}{b_n} + l\beta \right)^{l-1} \left( 1 - \frac{y}{b_n} \right) [t_n^{k,l}(y, b_n)]^{n-k-l-1}, \end{aligned}$$

which implies

$$\begin{aligned} G_n^*(\omega; x) &= (1+n\beta)^{1-n} \sum_{k=0}^n \sum_{l=0}^{n-k} \omega \left( \frac{k}{n} b_n \right) \binom{n-k}{l} \binom{n-k}{k} \frac{x}{b_n} \left( \frac{x}{b_n} + k\beta \right)^{k-1} \\ &\times \left( \frac{y-x}{b_n} \right) \left( \frac{y-x}{b_n} + l\beta \right)^{l-1} \left( 1 - \frac{y}{b_n} \right) [t_n^{k,l}(y, b_n)]^{n-k-l-1}. \end{aligned} \quad (2.4)$$

According to the equations (2.3) and (2.4) one has

$$\begin{aligned}
& G_n^*(\omega; y) - G_n^*(\omega; x) \\
&= (1+n\beta)^{1-n} \sum_{k=0}^n \sum_{l=0}^{n-k} \left[ \omega\left(\frac{k+l}{n}b_n\right) - \omega\left(\frac{k}{n}b_n\right) \right] \frac{n!}{k!l!(n-k-l)!} \\
&\times \frac{x}{b_n} \left(\frac{x}{b_n} + k\beta\right)^{k-1} \left(\frac{y-x}{b_n}\right) \left(\frac{y-x}{b_n} + l\beta\right)^{l-1} \left(1 - \frac{y}{b_n}\right) [t_n^{k,l}(y, b_n)]^{n-k-l-1}.
\end{aligned} \tag{2.5}$$

By Definition 1, we have

$$\omega\left(\frac{k+l}{n}b_n\right) - \omega\left(\frac{k}{n}b_n\right) \leq \omega\left(\frac{l}{n}b_n\right),$$

therefore,

$$\begin{aligned}
& G_n^*(\omega; y) - G_n^*(\omega; x) \\
&\leq (1+n\beta)^{1-n} \sum_{k=0}^n \sum_{l=0}^{n-k} \omega\left(\frac{l}{n}b_n\right) \frac{n!}{k!l!(n-k-l)!} \frac{x}{b_n} \left(\frac{x}{b_n} + k\beta\right)^{k-1} \\
&\times \left(\frac{y-x}{b_n}\right) \left(\frac{y-x}{b_n} + l\beta\right)^{l-1} \left(1 - \frac{y}{b_n}\right) [t_n^{k,l}(y, b_n)]^{n-k-l-1}.
\end{aligned}$$

By changing the order of the summations, we have

$$\begin{aligned}
& G_n^*(\omega; y) - G_n^*(\omega; x) \\
&\leq (1+n\beta)^{1-n} \sum_{l=0}^n \sum_{k=0}^{n-l} \omega\left(\frac{l}{n}b_n\right) \frac{n!(n-l)!}{l!(n-l)!k!(n-k-l)!} \frac{x}{b_n} \left(\frac{x}{b_n} + k\beta\right)^{k-1} \\
&\times \left(\frac{y-x}{b_n}\right) \left(\frac{y-x}{b_n} + l\beta\right)^{l-1} \left(1 - \frac{y}{b_n}\right) [t_n^{k,l}(y, b_n)]^{n-k-l-1} \\
&= (1+n\beta)^{1-n} \sum_{l=0}^n \omega\left(\frac{l}{n}b_n\right) \binom{n}{l} \left(\frac{y-x}{b_n}\right) \left(\frac{y-x}{b_n} + l\beta\right)^{l-1} \\
&\times \sum_{k=0}^{n-l} \binom{n-l}{k} \frac{x}{b_n} \left(\frac{x}{b_n} + k\beta\right)^{k-1} \left(1 - \frac{y}{b_n}\right) [t_n^{k,l}(y, b_n)]^{n-k-l-1}.
\end{aligned} \tag{2.6}$$

In (1.3), if we take  $u, v$  and  $n$ , by  $x/b_n, 1 - y/b_n$  and  $n - l$ , respectively, then we have

$$\begin{aligned}
& \left(1 - \frac{y-x}{b_n}\right) [t_n^{l,0}(y-x, b_n)]^{n-l-1} \\
&= \sum_{k=0}^{n-l} \binom{n-l}{k} \left(\frac{x}{b_n}\right) \left(\frac{x}{b_n} + k\beta\right)^{k-1} \left(1 - \frac{y}{b_n}\right) [t_n^{k,l}(y, b_n)]^{n-k-l-1}.
\end{aligned}$$

Rearranging the equality (2.6), we obtain

$$\begin{aligned} & G_n^*(\omega; y) - G_n^*(\omega; x) \\ & \leq (1 + n\beta)^{1-n} \sum_{l=0}^n \omega\left(\frac{l}{n}b_n\right) \binom{n}{l} \left(\frac{y-x}{b_n}\right) \left(\frac{y-x}{b_n} + l\beta\right)^{l-1} \\ & \quad \times \left(1 - \frac{y-x}{b_n}\right) [t_n^{l,0}(y-x, b_n)]^{n-l-1} \\ & = G_n^*(\omega; y-x). \end{aligned}$$

Hence  $\{G_n^*\}$  is semi-additive. We also infer from (2.5) that  $\{G_n^*\}$  is non decreasing. From the definition of  $\{G_n^*\}$  it is clear that

$$\lim_{x \rightarrow 0} G_n^*(\omega)(x) = G_n^*(\omega)(0) = \omega(0) = 0.$$

So the proof is completed.  $\square$

**Theorem 2.** *If  $f \in Lip_M(\gamma)$ , then  $G_n^*(f; x) \in Lip_M(\gamma)$ .*

*Proof.* Let  $x, y \in [0, b_n)$  and  $y \geq x$ . According to (2.5) we can write

$$\begin{aligned} & |G_n^*(f; y) - G_n^*(f; x)| \\ & \leq (1 + n\beta)^{1-n} \sum_{k=0}^n \sum_{l=0}^{n-k} \left| f\left(\frac{k+l}{n}b_n\right) - f\left(\frac{k}{n}b_n\right) \right| \frac{n!}{k!l!(n-k-l)!} \frac{x}{b_n} \left(\frac{x}{b_n} + k\beta\right)^{k-1} \\ & \quad \times \left(\frac{y-x}{b_n}\right) \left(\frac{y-x}{b_n} + l\beta\right)^{l-1} \left(1 - \frac{y}{b_n}\right) [t_n^{k,l}(y, b_n)]^{n-k-l-1}. \end{aligned}$$

By the hypothesis, we can write

$$\begin{aligned} & |G_n^*(f; y) - G_n^*(f; x)| \\ & \leq M(1 + n\beta)^{1-n} \sum_{k=0}^n \sum_{l=0}^{n-k} \left(\frac{l}{n}b_n\right)^\gamma \frac{n!}{k!l!(n-k-l)!} \frac{x}{b_n} \left(\frac{x}{b_n} + k\beta\right)^{k-1} \\ & \quad \left(\frac{y-x}{b_n}\right) \left(\frac{y-x}{b_n} + l\beta\right)^{l-1} \left(1 - \frac{y}{b_n}\right) [t_n^{k,l}(y, b_n)]^{n-k-l-1} \end{aligned}$$

and by changing the order of the summations we have

$$\begin{aligned} & |G_n^*(f; y) - G_n^*(f; x)| \\ & \leq M(1 + n\beta)^{1-n} \sum_{l=0}^n \left(\frac{l}{n}b_n\right)^\gamma \binom{n}{l} \left(\frac{y-x}{b_n}\right) \left(\frac{y-x}{b_n} + l\beta\right)^{l-1} \\ & \quad \times \sum_{k=0}^{n-l} \binom{n-l}{k} \frac{x}{b_n} \left(\frac{x}{b_n} + k\beta\right)^{k-1} \left(1 - \frac{y}{b_n}\right) [t_n^{k,l}(y, b_n)]^{n-k-l-1}. \end{aligned}$$

From (1.3), we obtain

$$\begin{aligned} & |G_n^*(f; y) - G_n^*(f; x)| \\ & \leq M(1 + n\beta)^{1-n} \sum_{l=0}^n \left(\frac{l}{n}b_n\right)^\gamma \binom{n}{l} \left(\frac{y-x}{b_n}\right) \left(\frac{y-x}{b_n} + l\beta\right)^{l-1} \\ & \quad \times \left(1 - \frac{y-x}{b_n}\right) [t_n^{l,0}(y-x, b_n)]^{n-l-1}. \end{aligned}$$

Considering the Hölder inequality with  $p = 1/\gamma$  and  $q = 1/(1 - \gamma)$ , we have

$$\begin{aligned}
& |G_n^*(f; y) - G_n^*(f; x)| \\
& \leq M \left\{ (1 + n\beta)^{1-n} \sum_{l=0}^n \frac{l}{n} b_n \binom{n}{l} \left( \frac{y-x}{b_n} \right) \left( \frac{y-x}{b_n} + l\beta \right)^{l-1} \right. \\
& \quad \times \left. \left( 1 - \frac{y-x}{b_n} \right) [t_n^{l,0}(y-x, b_n)]^{n-l-1} \right\}^\gamma \\
& \quad \times \left\{ (1 + n\beta)^{1-n} \sum_{l=0}^n \binom{n}{l} \left( \frac{y-x}{b_n} \right) \left( \frac{y-x}{b_n} + l\beta \right)^{l-1} \right. \\
& \quad \times \left. \left( 1 - \frac{y-x}{b_n} \right) [t_n^{l,0}(y-x, b_n)]^{n-l-1} \right\}^{1-\gamma} \\
& = M \{G_n^*(t; y-x)\}^\gamma \cdot \{G_n^*(1; y-x)\}^\gamma \\
& = M (y-x)^\gamma.
\end{aligned}$$

From (1.7), (1.8) and by the hypothesis, we get

$$|G_n^*(f; y) - G_n^*(f; x)| \leq |y-x|^\gamma.$$

This proves the theorem.  $\square$

**Theorem 3.** *If  $f$  is convex, then  $G_n^*(f; x) \geq f(x)$ .*

*Proof.* Suppose that

$$\alpha_k = (1 + n\beta)^{1-n} \binom{n}{k} \frac{x}{b_n} \left( \frac{x}{b_n} + k\beta \right)^{k-1} \left( 1 - \frac{x}{b_n} \right) (t_n^{k,0}(x, b_n))^{n-k-1}$$

and

$$x_k = \frac{k}{n} b_n, \quad k = 0, 1, 2, \dots, n.$$

It is easy to see that

$$\sum_{k=0}^n \alpha_k = G_n^*(1; x) = 1.$$

Therefore we obtain

$$G_n^*(f; x) = \sum_{k=0}^n \alpha_k f(x_k) \geq f\left(\sum_{k=0}^n \alpha_k x_k\right) = f(G_n^*(t; x)) = f(x),$$

which completes the proof.  $\square$

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