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SUMMATION FORMULA FOR GENERALIZED DISCRETE q-HERMITE II POLYNOMIALS

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Abstract. In this paper, we provide a family of generalized discrete q-Hermite II polynomials denoted by $\tilde{h}_{n,\alpha}(x,y|q)$. An explicit relations connecting them with the q-Laguerre and Stieltjes-Wigert polynomials are obtained. Summation formula is derived by using different analytical means on their generating functions.

1. Introduction

In their paper, Àlvarez-Nodarse et al [2], have introduced a q-extension of the discrete q-Hermite II polynomials as:

$$\begin{aligned}\mathcal{H}_{2n}^{(\mu)}(x;q): &= (-1)^n(q;q)_n\, L_n^{(\mu-1/2)}(x^2;q)\\ \mathcal{H}_{2n+1}^{(\mu)}(x;q): &= (-1)^n(q;q)_n\, x\, L_n^{(\mu+1/2)}(x^2;q)\end{aligned} \tag{1.1}$$

where $\mu > -1/2$, $L_n^{(\alpha)}(x;q)$ are the q-Laguerre polynomials given by

$$\begin{aligned}L_n^{(\alpha)}(x;q): &= \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\,{}_1\Phi_1\left(\begin{array}{c} q^{-n}\\ q^{\alpha+1}\end{array}\Big|q;-q^{n+\alpha+1}x\right)\\ &= \frac{1}{(q;q)_n}\,{}_2\Phi_1\left(\begin{array}{c} q^{-n},-x\\ 0\end{array}\Big|q;q^{n+\alpha+1}x\right)\end{aligned} \tag{1.2}$$

with $(a;q)_0 = 1$, $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$, $n = 1,2,\cdots$, the q-shifted factorial, and

$${}_r\Phi_s\left(\begin{array}{c} q^{-n},a_2,\cdots,a_r\\ b_1,b_2,\cdots,b_s\end{array}\Big|q;x\right) = \tag{1.3}$$

$$\sum_{k=0}^{\infty}\frac{(q^{-n};q)_k(a_2;q)_k\cdots(a_r;q)_k}{(b_1;q)_k(b_2;q)_k\cdots(b_s;q)_k}\frac{x^k}{(q;q)_k}\left[(-1)^k q^{k(k-1)/2}\right]^{1+s-r}$$

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the usual generalized basic or q-hypergeometric function of degree n in the variable x (see Slater [10, Chap. 3], Srivastava and Karlsson [11, p.347, Eq. (272)] for details). For $\mu = 0$ in (1.1), the polynomials $\mathcal{H}_n^{(0)}(x;q)$ correspond to the discrete q-Hermite II polynomials [1, 8], i.e., $\mathcal{H}_n^{(0)}(x;q^2) = q^{n(n-1)}\tilde{h}_n(x;q)$. They show that the polynomials $\mathcal{H}_n^{(\mu)}(x;q)$ satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} \mathcal{H}_n^{(\mu)}(x;q)\mathcal{H}_m^{(\mu)}(x;q)\omega(x)dx = \pi\, q^{-n/2}(q^{1/2};q^{1/2})_n(q^{1/2};q)_{1/2}\,\delta_{nm} \tag{1.4}$$

on the whole real line $\mathbb{R}$ with respect to the positive weight function $\omega(x) = 1/(-x^2;q)_\infty$. A detailed discussion of the properties of the polynomials $\mathcal{H}_n^{(\mu)}(x;q)$ can be found in [2].

Recently, Saley Jazmat et al [7], introduced a novel extension of discrete q-Hermite II polynomials by using new q-operators. This extension is defined as:

$$\begin{aligned} \tilde{h}_{2n,\alpha}(x;q) &= (-1)^n\, q^{-n(2n-1)}\frac{(q;q)_{2n}}{(q^{2\alpha+2};q^2)_n}\, L_n^{(\alpha)}\left(x^2q^{-2\alpha-1};q^2\right) \\ \tilde{h}_{2n+1,\alpha}(x;q) &= (-1)^n\, q^{-n(2n+1)}\frac{(q;q)_{2n+1}}{(q^{2\alpha+2};q^2)_{n+1}}\, x\, L_n^{(\alpha+1)}\left(x^2q^{-2\alpha-1};q^2\right). \end{aligned} \tag{1.5}$$

For $\alpha = -1/2$ in (1.5), the polynomials $\tilde{h}_{n,-\frac{1}{2}}(x;q)$ correspond to the discrete q-Hermite II polynomials, i.e., $\tilde{h}_{n,-\frac{1}{2}}(x;q) = \tilde{h}_n(x;q)$. The generalized discrete q-Hermite II polynomials (1.5) satisfy the orthogonality relation

$$\int_{-\infty}^{+\infty} \tilde{h}_{n,\alpha}(x;q)\tilde{h}_{m,\alpha}(x;q)\omega_\alpha(x;q)|x|^{2\alpha+1}d_qx \tag{1.6}$$

$$= \frac{2q^{-n^2}(1-q)(-q,-q,q^2;q^2)_\infty}{(-q^{-2\alpha-1},-q^{2\alpha+3},q^{2\alpha+2};q^2)_\infty}\frac{(q;q)_n^2}{(q;q)_{n,\alpha}}\,\delta_{n,m}$$

on the whole real line $\mathbb{R}$ with respect to the positive weight function $\omega_\alpha(x) = 1/(-q^{-2\alpha-1}\,x^2;q^2)_\infty$. A detailed discussion of the properties of the polynomials $\tilde{h}_{n,\alpha}(x;q)$ can be found in [7].

Srivastava and Jain [12, 6], investigated multilinear generating functions for q-Hermite, q-Laguerre polynomials and other special functions. Relevant connections of these multilinear generating functions with various known results for the classical or q-Hermite polynomials are also indicated. They also proved many combinatorial q-series identities by applying the theory of q-hypergeometric functions (see [6], for more details).

Motivated by Saley Jazmat's [7] and Srivastava et *al* [12, 6] works, our interest in this paper is to introduce new family of "*generalized discrete q-Hermite II polynomials (in short gdq-H2P)* $\tilde{h}_{n,\alpha}(x,y|q)$" which is an extension of the generalized discrete q-Hermite II polynomials $\tilde{h}_{n,\alpha}(x;q)$ and investigate summation formulae.

The paper is organized as follows. In Section 2, we recall notations to be used in the sequel. In Section 3, we define a gdq-H2P $\tilde{h}_{n,\alpha}(x,y|q)$ and investigate several properties. In Section 4, we derive summation and inversion formulae for gdq-H2P $\tilde{h}_{n,\alpha}(x,y|q)$. In Section 5, concluding remarks are given.

2. Notations and Preliminaries

For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper. We refer to the general references [4, 8] and [7] for the definitions and notations. Throughout this paper, we assume that $0 < q < 1$, $\alpha > -1$.

For a complex number a, the q-shifted factorials are defined by:

$$(a;q)_0 = 1;\ (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), n = 1, 2, \cdots;\ (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \tag{2.1}$$

and the q-number is defined by:

$$[n]_q = \frac{1-q^n}{1-q}, \quad n!_q := \prod_{k=1}^{n}[k]_q, \quad 0!_q := 1,\ n \in \mathbb{N}. \tag{2.2}$$

Let x and y be two real or complex numbers, the Hahn [5] q-addition $\oplus_q$ of x and y is given by:

$$\begin{aligned}\big(x \oplus_q y\big)^n : &= (x+y)(x+qy)\dots(x+q^{n-1}y)\\ &= (q;q)_n \sum_{k=0}^{n} \frac{q^{\binom{k}{2}} x^{n-k} y^k}{(q;q)_k (q;q)_{n-k}}, \quad n \geq 1, \quad \big(x \oplus_q y\big)^0 := 1, \end{aligned} \tag{2.3}$$

while the q-subtraction $\ominus_q$ is given by

$$\big(x \ominus_q y\big)^n := \big(x \oplus_q (-y)\big)^n. \tag{2.4}$$

The generalized q-shifted factorials [7] are defined by the recursion relations

$$[n+1]_{q,\alpha}! = [n+1+\theta_n(2\alpha+1)]_q\,[n]_{q,\alpha}! \tag{2.5}$$

and

$$(q;q)_{n+1,\alpha} = (1-q)[n+1+\theta_n(2\alpha+1)]_q (q;q)_{n,\alpha}, \tag{2.6}$$

where

$$\theta_n = \begin{cases} 1 & \text{if n even} \\ 0 & \text{if n odd.} \end{cases} \tag{2.7}$$

Remark that, for $\alpha = -1/2$, we have

$$(q;q)_{n,-1/2} = (q;q)_n, \quad [n]_{q,-1/2}! = (1-q)^n (q;q)_n. \tag{2.8}$$

We denote

$$(q;q)_{2n,\alpha} = (q^2;q^2)_n (q^{2\,\alpha+2};q^2)_n, \tag{2.9}$$

and

$$(q;q)_{2n+1,\alpha} = (q^2;q^2)_n (q^{2\,\alpha+2};q^2)_{n+1}. \tag{2.10}$$

The two Euler's q-analogues of the exponential functions are given by [4]

$$E_q(x) = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}\, x^k}{(q;q)_k} = (-x;q)_\infty \tag{2.11}$$

and

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k} = \frac{1}{(x;q)_\infty}, \quad |x| < 1. \tag{2.12}$$

For $m \geq 1$ and by means of the generalized q-shifted factorials, we define two generalized q-exponential functions as follows

$$\tilde{E}_{q^m,\alpha}(x) := \sum_{k=0}^{\infty} \frac{q^{mk(k-1)/2}\, x^k}{(q^m;q^m)_{k,\alpha}}, \tag{2.13}$$

and

$$\tilde{e}_{q^m,\alpha}(x) := \sum_{k=0}^{\infty} \frac{x^k}{(q^m;q^m)_{k,\alpha}}, \quad |x| < 1. \tag{2.14}$$

Remark that, for $m = 1$ and $\alpha = -\frac{1}{2}$, we have:

$$\tilde{E}_{q,\alpha}(x) = E_q(x), \quad \tilde{e}_{q,\alpha}(x) = e_q(x). \tag{2.15}$$

For $m = 2$, the following elementary result is useful in the sequel to establish the summation formulae for gdq-H2P:

$$\tilde{e}_{q^2,-\frac{1}{2}}(x)\tilde{E}_{q^2,-\frac{1}{2}}(y) = \tilde{e}_{q^2,-\frac{1}{2}}(x \oplus_{q^2} y), \tag{2.16}$$

$$\tilde{e}_{q,-\frac{1}{2}}(x)\tilde{E}_{q^2,-\frac{1}{2}}(-y) = \tilde{e}_q(x \ominus_{q,q^2} y), \;\; \tilde{e}_{q^2,-\frac{1}{2}}(x)\tilde{E}_{q^2,-\frac{1}{2}}(-x) = 1, \tag{2.17}$$

where

$$(a \ominus_{q,q^2} b)^n := n!_q \sum_{k=0}^{n} \frac{(-1)^k q^{k(k-1)}}{(n-k)!_q \; k!_{q^2}} a^{n-k} b^k, \;\; (a \ominus_{q,q^2} b)^0 := 1. \tag{2.18}$$

3. Generalized discrete q-Hermite II polynomials

In this section, we introduce a sequence of gdq-H2P $\{\tilde{h}_{n,\alpha}(x,y|q)\}_{n=0}^{\infty}$. Several properties related to these polynomials are derived.

Definition 3.1. *For x, $y \in \mathbb{R}$, the gdq-H2P $\{\tilde{h}_{n,\alpha}(x,y|q)\}_{n=0}^{\infty}$ are defined by:*

$$\tilde{h}_{n,\alpha}(x,y|q) := (q;q)_n \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k q^{-2nk+k(2k+1)}\; x^{n-2k}\; y^k}{(q;q)_{n-2k,\alpha}\, (q^2;q^2)_k} \tag{3.1}$$

and

$$\tilde{h}_{n,\alpha}(x,0|q) := \frac{(q;q)_n}{(q;q)_{n,\alpha}} x^n. \tag{3.2}$$

Remark that,

(1) for $y = 1$, we get

$$\tilde{h}_{n,\alpha}(x,1|q) = \tilde{h}_{n,\alpha}(x;q) \tag{3.3}$$

where $\tilde{h}_{n,\alpha}(x;q)$ is the generalized discrete q-Hermite II polynomial [7];

(2) for $\alpha = -1/2$ and $y = 1$, we have

$$\tilde{h}_{n,-1/2}(x,1|q) = \tilde{h}_n(x;q). \tag{3.4}$$

where $\tilde{h}_n(x;q)$ is the discrete q-Hermite II polynomial [1, 8].

(3) Indeed since

$$\lim_{q\to 1} \frac{(q^a;q)_n}{(1-q)^n} = (a)_n \tag{3.5}$$

one readily verifies that

$$\lim_{q\to 1} \frac{\tilde{h}_{n,-\frac{1}{2}}(\sqrt{1-q^2}x, 1|q)}{(1-q^2)^{n/2}} = \frac{h_n^{\alpha+\frac{1}{2}}(x)}{2^n} \tag{3.6}$$

where $h_n^{\alpha+\frac{1}{2}}(x)$ is the Rosenblums generalized Hermite polynomial [9].

Lemma 3.2. *The following recursion relation for gdq-H2P* $\{\tilde{h}_{n,\alpha}(x,y|q)\}_{n=0}^{\infty}$ *holds true.*

$$\begin{aligned}\frac{1-q^{n+1+\theta_n(2\alpha+1)}}{1-q^{n+1}}\tilde{h}_{n+1,\alpha}(x,y|q) \quad (3.7)\\ = x\tilde{h}_{n,\alpha}(x,y|q) - y\,q^{-2n+1}(1-q^n)\tilde{h}_{n-1,\alpha}(x,y|q).\end{aligned}$$

Proof. To prove the assertion (3.7), we consider separately even and odd cases of the expression

$$x\tilde{h}_{n,\alpha}(x,y|q) - y\,q^{-2n+1}(1-q^n)\tilde{h}_{n-1,\alpha}(x,y|q). \quad (3.8)$$

For n even, we have:

$$x\tilde{h}_{2n,\alpha}(x,y|q) = \frac{(q;q)_{2n}}{(q;q)_{2n,\alpha}}x^{2n+1} + (q;q)_{2n}\sum_{k=1}^{n}\frac{(-1)^k q^{-2nk+k(2k+1)}x^{2n-2k+1}\,y^k}{(q;q)_{2n-2k,\alpha}\,(q^2;q^2)_k}.$$

The right-hand side of the last relation can be written as

$$\begin{aligned}\frac{(q;q)_{2n}}{(q;q)_{2n,\alpha}}x^{2n+1} + (q;q)_{2n} \quad (3.9)\\ \times\sum_{k=1}^{n}\frac{(-1)^k q^{-2k(2n+1)+k(2k+1)}x^{2n+1-2k}\,y^k}{(q;q)_{2n+1-2k,\alpha}\,(q^2;q^2)_k}\left[q^{2k}(1-q^{2n+2+2\alpha-2k})\right].\end{aligned}$$

In the same way,

$$\begin{aligned}-\,y\,q^{-4n+1}\,(1-q^{2n})\,\tilde{h}_{2n-1,\alpha}(x,y|q) = -y\,q^{-4n+1}\,(q;q)_{2n}\\ \times\sum_{k=0}^{n-1}\frac{(-1)^k q^{-2k(2n+1)+k(2k+1)}x^{2n+1-2(k+1)}\,y^k}{(q;q)_{2n+1-2(k+1),\alpha}\,(q^2;q^2)_k}. \quad (3.10)\end{aligned}$$

Change k to $k-1$ in (3.10), one obtains

$$(q;q)_{2n}\sum_{k=1}^{n}\frac{(-1)^k q^{-2k(2n+1)+k(2k+1)}x^{2n+1-2k}\,y^k}{(q;q)_{2n+1-2k,\alpha}\,(q^2;q^2)_k}(1-q^{2k}). \quad (3.11)$$

Then combining (3.9) and (3.11), we have

$$\begin{aligned}x\tilde{h}_{2n,\alpha}(x,y|q) - y\,q^{-4n+1}\,(1-q^{2n})\,\tilde{h}_{2n-1,\alpha}(x,y|q) = \quad (3.12)\\ \frac{(q;q)_{2n}}{(q;q)_{2n,\alpha}}x^{2n+1} + (q;q)_{2n}\sum_{k=1}^{n}\frac{(-1)^k q^{-2k(2n+1)+k(2k+1)}x^{2n+1-2k}\,y^k}{(q;q)_{2n+1-2k,\alpha}\,(q^2;q^2)_k}\\ \times\left[q^{2k}(1-q^{2n+2+2\alpha-2k}) + (1-q^{2k})\right].\end{aligned}$$

After simplification, it is equal to

$$\begin{aligned}\frac{(q;q)_{2n}}{(q;q)_{2n,\alpha}}x^{2n+1}+\\ (1-q^{2n+2+2\alpha})(q;q)_{2n}\sum_{k=1}^{n}\frac{(-1)^k q^{-2k(2n+1)+k(2k+1)}x^{2n+1-2k}\,y^k}{(q;q)_{2n+1-2k,\alpha}\,(q^2;q^2)_k}.\end{aligned}$$

The last expression can be written as

$$\frac{1-q^{2n+2+2\alpha}}{1-q^{2n+1}}\tilde{h}_{2n+1,\alpha}(x,y|q). \quad (3.13)$$

Summarizing the above calculations in (3.12)-(3.13), we get the assertion (3.7) for n even. In the odd case, the proof follows the same steps as the even case. □

Theorem 3.3. *We have:*

$$\lim_{\alpha\to+\infty} \tilde{h}_{2n,\alpha}(x,y|q) = q^{-n(2n-1)}(q;q)_{2n}\,(-y)^n\, S_n\left(x^2y^{-1}q^{-1};q^2\right) \tag{3.14}$$

and

$$\lim_{\alpha\to+\infty} \tilde{h}_{2n+1,\alpha}(x,y|q) = q^{-n(2n+1)}(q;q)_{2n+1}\,x\,(-y)^n\, S_n\left(x^2y^{-1}q^{-1};q^2\right) \tag{3.15}$$

where $S_n(x;q)$ are the Stieltjes-Wigert polynomials [8].

In order to prove Theorem 3.3, we need the following Lemma.

Lemma 3.4. *For $\alpha > -1$, the sequence of gdq-H2P $\{\tilde{h}_{n,\alpha}(x,y|q)\}_{n=0}^{\infty}$ can be written in terms of q-Laguerre polynomials $L_n^{(\alpha)}(x;q)$ as*

$$\tilde{h}_{2n,\alpha}(x,y|q) = q^{-n(2n-1)}\frac{(q;q)_{2n}}{(q^{2\alpha+2};q^2)_n}\,(-y)^n\, L_n^{(\alpha)}\left(x^2y^{-1}q^{-2\alpha-1};q^2\right) \tag{3.16}$$

and

$$\tilde{h}_{2n+1,\alpha}(x,y|q) = q^{-n(2n+1)}\frac{(q;q)_{2n+1}}{(q^{2\alpha+2};q^2)_{n+1}}\,x\,(-y)^n\, L_n^{(\alpha+1)}\left(x^2y^{-1}q^{-2\alpha-1};q^2\right). \tag{3.17}$$

In order to prove Lemma 3.4, we need the following Proposition.

Proposition 3.5. *For $\alpha > -1$, the sequence of gdq-H2P $\{\tilde{h}_{n,\alpha}(x,y|q)\}_{n=0}^{\infty}$ can be written in terms of basic hypergeometric functions as*

$$\tilde{h}_{n,\alpha}(x,y|q) = \frac{(q;q)_n}{(q;q)_{n,\alpha}}\,x^n\,{}_2\Phi_1\left(\begin{matrix} q^{-n},q^{-n-2\alpha} \\ 0\end{matrix}\,\middle|\, q^2;\ -\frac{y\,q^{2\alpha+3}}{x^2}\right). \tag{3.18}$$

Proof. In fact, for n even, and by using

$$(q;q)_{2n-2k,\alpha} = (q^2;q^2)_{n-k}(q^{2\alpha+2};q^2)_{n-k}, \tag{3.19}$$

the gdq-H2P $\tilde{h}_{n,\alpha}(x,y|q)$ defined in (3.1) can be rewritten as

$$\tilde{h}_{2n,\alpha}(x,y|q) = (q;q)_{2n}\sum_{k=0}^{n}\frac{(-1)^kq^{-4nk+k(2k+1)}x^{2n-2k}\,y^k}{(q^2;q^2)_{n-k}(q^{2\alpha+2};q^2)_{n-k}\,(q^2;q^2)_k}. \tag{3.20}$$

From the formula [8, p.9, Eq. (0.2.12)]

$$(a;q)_{n-k} = \frac{(a;q)_n}{(a^{-1}q^{1-n};q)_k}\left(-\frac{q}{a}\right)^k q^{\binom{k}{2}-nk}, \tag{3.21}$$

we have for $a = q^2$ and $q^{2\alpha+2}$,

$$\tilde{h}_{2n,\alpha}(x,y|q) = \frac{(q;q)_{2n}\,x^{2n}}{(q;q)_{2n,\alpha}}\sum_{k=0}^{n}\frac{(-1)^kq^{-4nk+k(2k+1)}(q^{-2n},q^{-2n-2\alpha};q^2)_k}{(q^2;q^2)_kq^{4\binom{k}{2}-4nk-2\alpha k}}\left(\frac{y}{x^2}\right)^k.$$

After simplification, the last equation reads

$$\tilde{h}_{2n,\alpha}(x,y|q) = \frac{(q;q)_{2n}}{(q;q)_{2n,\alpha}}x^{2n}\sum_{k=0}^{n}\frac{(q^{-2n},q^{-2n-2\alpha};q^2)_k}{(q^2;q^2)_k}\left(-\frac{y\,q^{2\alpha+3}}{x^2}\right)^k. \tag{3.22}$$

In the odd case, the proof follows the same steps as the even case. □

Now, we are in position to prove Lemma 3.3.

Proof. (of Lemma 3.3) For n even, the relation (3.18) becomes:

$$\tilde{h}_{2n,\alpha}(x,y|q) = \frac{(q;q)_{2n}}{(q;q)_{2n,\alpha}}\, x^{2n}\, {}_2\Phi_1\left(\begin{array}{c} q^{-2n}, q^{-2n-2\alpha} \\ 0 \end{array} \Big|\, q^2;\ -\frac{y\, q^{2\alpha+3}}{x^2} \right). \quad (3.23)$$

By taking $a^{-1} = q^{-2\alpha-2}$ and $z = -q^{2n+1}\, x^2 y^{-1}$ and the formula [8, p.17, Eq. (0.6.17)]

$${}_2\Phi_1\left(\begin{array}{c} q^{-n}, a^{-1}q^{1-n} \\ 0 \end{array} \Big|\, q;\ \frac{aq^{n+1}}{z} \right) = (a;q)_n (qz^{-1})^n\, {}_1\Phi_1\left(\begin{array}{c} q^{-n} \\ a \end{array} \Big|\, q; z \right) \quad (3.24)$$

we have

$${}_2\Phi_1\left(\begin{array}{c} q^{-2n}, q^{-2n-2\alpha} \\ 0 \end{array} \Big|\, q^2;\ -\frac{y\, q^{2\alpha+3}}{x^2} \right) = \quad (3.25)$$

$$(q^{2\alpha+2};q^2)_n \left(-\frac{y}{x^2}\right)^n q^{-2n^2+n}\, {}_1\Phi_1\left(\begin{array}{c} q^{-2n} \\ q^{2+2\alpha} \end{array} \Big|\, q^2;\ -\frac{q^{2n+1}\, x^2}{y} \right).$$

By using (1.2), the relation (3.25) can be written as

$$q^{-2n^2+n}\, (q^2;q^2)_n \left(-\frac{y}{x^2}\right)^n L_n^{(\alpha)}\left(x^2 y^{-1} q^{-2\alpha-1}; q^2\right). \quad (3.26)$$

The assertion (3.16) of Lemma 3.3 follows by summarizing the above calculations in (3.23)-(3.26).
In the odd case, the proof follows the same steps as the even case. □

Proof. (of Theorem 3.4) By taking the limit $\alpha \to +\infty$ in the assertions (3.16) and (3.17) of Lemma 3.3, respectively, we get the assertions (3.14) and (3.15) of Theorem 3.4. □

4. Connection formulae for the generalized discrete q-Hermite II polynomials $\{\tilde{h}_{n,\alpha}(x,y|q)\}_{n=0}^{\infty}$

We begin this section with the following theorem:

Theorem 4.1. *The sequence of gdq-H2P $\{\tilde{h}_{n,\alpha}(x,y|q)\}_{n=0}^{\infty}$, which is defined by the relation (3.1), satisfies the connection formula*

$$\tilde{h}_{n,\alpha}(x,\omega|q) = (q;q)_n \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{q^{-2nk+k(2k+1)}\, (-\omega \oplus_{q^2} y)^k}{(q^2;q^2)_k\, (q;q)_{n-2k}} \tilde{h}_{n-2k,\alpha}(x,y|q). \quad (4.1)$$

To prove Theorem 4.1, we need the following Lemma.

Lemma 4.2. *The following generating function for gdq-H2P $\{\tilde{h}_{n,\alpha}(x,y|q)\}_{n=0}^{\infty}$ holds true.*

$$\tilde{e}_{q^2,-\frac{1}{2}}(-yt^2)\tilde{E}_{q,\alpha}(xt) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}\, t^n}{(q;q)_n} \tilde{h}_{n,\alpha}(x,y|q), \quad |yt| < 1. \quad (4.2)$$

Proof. Let us consider the function

$$f_q(t;x,y) := \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}\, t^n}{(q;q)_n} \tilde{h}_{n,\alpha}(x,y|q). \quad (4.3)$$

By replacing in (4.3) gdq-H2P $\tilde{h}_{n,\alpha}(x,y|q)$ by its explicit expression (3.1) we obtain

$$f_q(t;x,y) = \sum_{n=0}^{\infty} t^n \left(\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k q^{\binom{n}{2}-2nk+k(2k+1)} x^{n-2k}\, y^k}{(q;q)_{n-2k,\alpha}\, (q^2;q^2)_k} \right). \quad (4.4)$$

The right-hand side of (4.4) also reads

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k q^{\binom{n-2k}{2}}(yt^2)^k(xt)^{n-2k}}{(q;q)_{n-2k,\alpha}\,(q^2;q^2)_k}. \tag{4.5}$$

Next, changing $n-2k$ by $r,\ r=0,1,\cdots$, the last relation becomes

$$\sum_{n=0}^{\infty}\frac{\left(-yt^2\right)^n}{(q^2;q^2)_n}\sum_{r=0}^{\infty}\frac{q^{\binom{r}{2}}\,(xt)^r}{(q;q)_{r,\alpha}}. \tag{4.6}$$

Hence,

$$f_q(t;x,y)=\tilde{e}_{q^2,-\frac{1}{2}}(-yt^2)\tilde{E}_{q,\alpha}(xt). \tag{4.7}$$

□

Now, we are in position to prove Theorem 4.1.

Proof. (of Theorem 4.1) Replacing t by $u\oplus_q t$ in (4.2), we find the following generating function

$$\tilde{E}_{q,\alpha}\Big[(u\oplus_q t)x\Big]\tilde{e}_{q^2,-\frac{1}{2}}\Big[-y(u\oplus_q t)^2\Big]=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(u\oplus_q t)^n}{(q;q)_n}\tilde{h}_{n,\alpha}(x,y|q) \tag{4.8}$$

which by using (2.17), becomes

$$\tilde{E}_{q,\alpha}\Big[(u\oplus_q t)x\Big]=\tilde{E}_{q^2,-\frac{1}{2}}\Big[y(u\oplus_q t)^2\Big]\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(u\oplus_q t)^n}{(q;q)_n}\tilde{h}_{n,\alpha}(x,y|q). \tag{4.9}$$

Replacing y by ω and (4.9), respectively, in (4.8), we get

$$\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(u\oplus_q t)^n}{(q;q)_n}\tilde{h}_{n,\alpha}(x,\omega|q)= \tag{4.10}$$

$$=\tilde{e}_{q^2,-\frac{1}{2}}\Big[-\omega(u\oplus_q t)^2\Big]\tilde{E}_{q^2,-\frac{1}{2}}\Big[y(u\oplus_q t)^2\Big]\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(u\oplus_q t)^n}{(q;q)_n}\tilde{h}_{n,\alpha}(x,y|q).$$

By using (2.17), the last relation reads

$$\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(u\oplus_q t)^n}{(q;q)_n}\tilde{h}_{n,\alpha}(x,\omega|q) \tag{4.11}$$

$$=\tilde{e}_{q^2,-\frac{1}{2}}\Big[(-\omega\oplus_{q^2}y)(u\oplus_q t)^2\Big]\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(u\oplus_q t)^n}{(q;q)_n}\tilde{h}_{n,\alpha}(x,y|q).$$

According to (2.12), the right-hand side of (4.11) can be written as

$$\sum_{r=0}^{\infty}\frac{(-\omega\oplus_{q^2}y)^r(u\oplus_q t)^{2r}}{(q^2;q^2)_r}\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(u\oplus_q t)^n}{(q;q)_n}\tilde{h}_{n,\alpha}(x,y|q). \tag{4.12}$$

Let us substitute $n+2r=k\implies r\le\lfloor k/2\rfloor$ in (4.12), then we have:

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(q^{\binom{n-2k}{2}}(-\omega\oplus_{q^2}y)^k}{(q^2;q^2)_k\,(q;q)_{n-2k}}\tilde{h}_{n-2k,\alpha}(x,y|q)\right)(u\oplus_q t)^n. \tag{4.13}$$

Next, replacing (4.13) in (4.11), we obtain

$$\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(u\oplus_q t)^n}{(q;q)_n}\tilde{h}_{n,\alpha}(x,\omega|q)= \tag{4.14}$$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(q^{\binom{n-2k}{2}}(-\omega\oplus_{q^2} y)^k}{(q^2;q^2)_k\,(q;q)_{n-2k}}\tilde{h}_{n-2k,\alpha}(x,y|q)\right)(u\oplus_q t)^n.$$

Finally, on equating the coefficients of like powers of $(u\oplus_q t)^n/(q;q)_n$ in (4.14), we get the desired identity. □

We have the following special cases of Theorem 4.1 of particular interest.

Corollary 4.3. *Letting:*

(i) $y=0$ *in the assertion (4.1) of Theorem 4.1, we get the definition of gdq-H2P (3.1), i.e.,*

$$\tilde{h}_{n,\alpha}(x,\omega|q)=(q;q)_n\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k q^{-2nk+k(2k+1)}\,x^{n-2k}\,\omega^k}{(q^2;q^2)_k\,(q;q)_{n-2k,\alpha}}; \tag{4.15}$$

(ii) $\omega=0$ *in the assertion (4.1) of Theorem 4.1, and using (3.2), we get the inversion formula for gdq-H2P*

$$x^n=(q;q)_{n,\alpha}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{q^{-2nk+3k^2}\,y^k}{(q^2;q^2)_k\,(q;q)_{n-2k}}\tilde{h}_{n-2k,\alpha}(x,y|q). \tag{4.16}$$

iii) *For $y=1$, the summation formulae (4.1) can be expressed in terms of generalized discrete q-Hermite II polynomials $\tilde{h}_{n,\alpha}(x;q)$. Also, the summation formulae (4.1) can be written in terms of discrete q-Hermite II polynomials $\tilde{h}_n(x;q)$ by choosing $y=1$ and $\alpha=-1/2$.*

5. Concluding remarks

In the previous sections, we have introduced gdq-H2P $\tilde{h}_{n,\alpha}(x,y|q)$ and derived several properties. Also, we have derived implicit summation formula for gdq-H2P $\tilde{h}_{n,\alpha}(x,y|q)$ by using different analytical means on their generating function. This process can be extended to summation formulae for more generalized forms of q-Hermite polynomials. This study is still in progress.

We note that the generating function of even and odd gdq-H2P $\tilde{h}_{n,\alpha}(x,y|q)$ are given by

$$\sum_{n=0}^{\infty}\frac{(-t^2)^n q^{n(2n-1)}}{(q;q)_{2n}}\tilde{h}_{2n,\alpha}(x,y|q)=\frac{q^{\alpha(\alpha+\frac{1}{2})}(q^2;q^2)_\infty}{(q^{2\alpha+2};q^2)_\infty}\,\frac{x^{-\alpha}J_\alpha^{(2)}(2xq^{-\alpha-\frac{1}{2}};q^2)}{(y\,t^2;q^2)_\infty}$$

and

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(2n+1)}\,t^{2n+1}}{(q;q)_{2n+1}}h_{2n+1,\alpha}(x,y|q)=\frac{q^{\alpha(\alpha+1)}(q^2;q^2)_\infty}{(q^{2\alpha+2};q^2)_\infty}\,\frac{x^{-\alpha}J_\alpha^{(2)}(2xq^{-\alpha};q^2)}{(y\,t^2;q^2)_\infty}$$

where $J_\nu^{(2)}(z;q)$ is the q-analogue of the Bessel function [8].
Indeed, it is well known that from (4.2), the generating function of gdq-H2P

$\tilde{h}_{n,\alpha}(x,y|q)$ is given by

$$\tilde{E}_{q,\alpha}(xt)\tilde{e}_{q^2,-\frac{1}{2}}(-yt^2) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}t^n}{(q;q)_n}\tilde{h}_{n,\alpha}(x,y|q) \tag{5.1}$$

which on separating the power in the right-hand side into their even and odd terms by using the elementary identity

$$\sum_{n=0}^{\infty} f(n) = \sum_{n=0}^{\infty} f(2n) + \sum_{n=0}^{\infty} f(2n+1) \tag{5.2}$$

becomes

$$\tilde{E}_{q,\alpha}(xt)\tilde{e}_{q^2,-\frac{1}{2}}(-yt^2) = \tag{5.3}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(2n-1)}\,t^{2n}}{(q;q)_{2n}}\tilde{h}_{2n,\alpha}(x,y|q) + \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}\,t^{2n+1}}{(q;q)_{2n+1}}\tilde{h}_{2n+1,\alpha}(x,y|q).$$

Now replacing t by $i\,t$ in (5.3) and equating the real and imaginary parts of the resultant equation, we get the generating function of even and odd gdq-H2P $h_{n,\alpha}(x,y|q)$ as

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(2n-1)}\,t^{2n}}{(q;q)_{2n}}\tilde{h}_{2n,\alpha}(x,y|q) = Cos_{q,\alpha}(xt)\tilde{e}_{q^2,-\frac{1}{2}}(yt^2) \tag{5.4}$$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(2n+1)}\;t^{2n+1}}{(q;q)_{2n+1}}\tilde{h}_{2n+1,\alpha}(x,y|q) = Sin_{q,\alpha}(xt)\tilde{e}_{q^2,-\frac{1}{2}}(yt^2) \tag{5.5}$$

where the generalized q-Cosine and q-Sine are defined as:

$$Cos_{q,\alpha}(x) : \;=\; \sum_{k=0}^{\infty} \frac{(-1)^n q^{n(2n-1)}\,x^{2n}}{(q;q)_{2n,\alpha}}, \tag{5.6}$$

$$Sin_{q,\alpha}(x) : \;=\; \sum_{k=0}^{\infty} \frac{(-1)^n q^{n(2n+1)}\,x^{2n+1}}{(q;q)_{2n+1,\alpha}}. \tag{5.7}$$

By using (2.9) and (2.10), respectively, the relations (5.6) and (5.7) can be expressed in terms of basic hypergeometric functions as

$$Cos_{q,\alpha}(x) \;=\; {}_0\Phi_1\left(\begin{array}{c} - \\ q^{2\alpha+2} \end{array} \middle|\; q^2;\; -qx^2\right) \tag{5.8}$$

$$Sin_{q,\alpha}(x) \;=\; \frac{x}{1-q^{2\alpha+2}}\;{}_0\Phi_1\left(\begin{array}{c} - \\ q^{2\alpha+4} \end{array} \middle|\; q^2;\; -q^2x^2\right). \tag{5.9}$$

The q-analogue of the Bessel function is defined [8, p.20, Eq.(0.7.14)] by

$$J_{\nu}^{(2)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\left(\frac{z}{2}\right)^{\nu}\;{}_0\Phi_1\left(\begin{array}{c} - \\ q^{\nu+1} \end{array} \middle|\; q;\; -\frac{q^{\nu+1}z^2}{4}\right) \tag{5.10}$$

from which the generating functions of (5.8) and (5.9) follow.

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