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n-QUASI-(A,m)- ISOMETRIC OPERATORS ON A HILBERT SPACE

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Abstract. A bounded linear operator S on a Hilbert space $\mathcal{K}$ is said to be a n-quasi- (A,m)-isometric if

$$S^{*n}\Big(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}AS^k\Big)S^n=0,$$

for some positive operator A on $\mathcal{K}$ and for some positive integers m and n. This class of operators seems a natural generalization of n-quasi-m-isometric and (A,m)-isometric operators on a Hilbert space ([10, 12]). First, we extend some results obtained in several papers related to n-quasi-m-isometric operators on a Hilbert space. In particular, some structural properties of this class are established with the help of special kind of operator matrix representation associated with such operators. Then, we give a necessary and sufficient condition for an operator to be a n-quasi-(A,m)-isometry. Finally, we characterize the spectra of these operators.

1. Introduction and terminologies

The concept of m-isometries on a complex Hilbert space has attracted attention of many authors, specially J. Agler and M. Stankus [1] and other authors. A generalization of m-isometries to (A,m)-isometric operators on semi-Hilbertian spaces has been presented by Sid Ahmed *et al.* in [10]. Our goal in this paper is to study the class of n-quasi-m-isometric with respect to a semi-norm $\|.\|_A$ induced by a semi-inner product defined by a positive operator A. An operator in this class will be called n-quasi-(A,m)-isometric operator. We give several generalizations of many results on n-quasi-m-isometric operators from [12],[13],[14] and [15].

Throughout this paper $\mathcal{K}$ stands for a separable complex Hilbert space with inner product $\langle .|.\rangle$ and $\mathcal{L}[\mathcal{K}]$ is the Banach algebra of all bounded linear operators on $\mathcal{K}$. $I_{\mathcal{K}}$ denotes, as usual, the identity operator on $\mathcal{K}$. $\mathcal{L}(\mathcal{K})^+$ is the cone of positive (semi-definite) operators, i.e.,

$$\mathcal{L}(\mathcal{K})^+=\{A\in\mathcal{L}(\mathcal{K}):\langle A\xi,\ |\ \xi\rangle\geq 0,\ \forall\ \xi\in\mathcal{K}\ \}.$$

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For every $S \in \mathcal{L}[\mathcal{K}]$ its range is denoted by $\mathcal{R}(S)$, its null space by $\mathcal{N}(S)$ and its adjoint by S^*. If $\mathcal{M} \subset \mathcal{K}$ is a subspace, the subspace $\mathcal{M}$ is invariant for S if $S\mathcal{M} \subset \mathcal{M}$.

Also, let $\alpha(S) := \dim \mathcal{N}(S)$, $\beta(S) := \dim \mathcal{K}/\mathcal{R}(S)$ and let $\sigma(S)$ denote the spectrum of S, $\sigma_{ap}(S)$ the approximate point spectrum of S, $\pi_0(S)$ the eigenvalues of S, and $\pi_{0f}(S)$ the eigenvalues of finite multiplicity of S.

A positive operator $(A \neq 0)$ defines a positive semi-definite sesquilinear form:

$$\langle .|.\rangle_A : \mathcal{K}^2 \longrightarrow \mathbb{C},\ \langle \xi \mid \eta\rangle_A := \langle A\xi \mid \eta\rangle.$$

The map $\langle .|.\rangle_A$ induced a semi-norm on a certain subspace of $\mathcal{L}[\mathcal{K}]$, namely, on the subset

$$\mathcal{L}_A[\mathcal{K}] := \big\{ S \in \mathcal{L}[\mathcal{H}] :\ \ \exists\ \ k > 0 \ / \ \|S\xi\|_A \leq k\|\xi\|_A, \quad \forall\, \xi \in \mathcal{K}\ \big\}.$$

For $S \in \mathcal{L}[\mathcal{K}]$, it holds

$$\|S\|_A := \sup\Big\{ \Big(\frac{\|S\xi\|_A}{\|\xi\|_A}\Big),\ \ \xi \in \overline{\mathcal{R}(A)},\ \ \xi \neq 0\ \Big\} < \infty.$$

(See for more detail [5, 6, 7]).

Recall that an operator $S \in \mathcal{L}[\mathcal{K}]$ is said to be:

(i) n-Quasi-isometry for some integer $n \geq 1$ ([11],[18]) if

$$S^{*(n+1)}S^{n+1} - S^{*n}S^n = 0 \ \text{ or equivalently if } \ \ S^{*n}\big(S^*S - I_{\mathcal{K}}\big)S^n = 0.$$

If the relation is verified with $n = 1$, S is called a quasi-isometry (See [14] and [15]).

(ii) A-contraction if $S^*AS \leq A$ and n-quasi-contraction, for $n \geq 1$, if S is an $S^{*n}S^n$-contraction ([11]).

(iii) A-n-quasi-isometry for some integer $n \geq 1$ if

$$S^{*(n+1)}AS^{n+1} - S^{*n}AS^n = 0 \ \text{ or equivalently } \ S^{*n}\big(S^*AS - A\big)S^n = 0.$$

If the relation is verified with $n = 1$, S is called an A-quasi-isometry (see [18]).

(iv) m-isometry ([1]) if

$$\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}S = 0 \ \ \Big(\Leftrightarrow \sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\|S^k\xi\|^2 = 0,\ \forall\, \xi \in \mathcal{K}\Big).$$

(v) A-m-isometric operator (or (A,m)-isometric operator) (see [10]) if

$$\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}AS^k = 0 \ \ \Big(\Leftrightarrow \sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\|S^k\xi\|_A^2 = 0,\ \forall\, \xi \in \mathcal{K}\Big).$$

(vi) n-quasi-m-isometric operator (see [12],[17]) if

$$S^{*n}\Big(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}S^k\Big)S^n = 0.$$

The paper is organized as follow: In Section two, we introduce the concept of n-quasi-(A,m)-isometric operators. Several properties are proved by exploiting the special kind of operator matrix representation associated with such operators. In the course of our investigation, we find some properties of (A,m)-isometries and

n-quasi-m-isometries, which are retained by n-quasi-(A,m)-isometries. However, there are other ones which are shown to be no true for n-quasi-(A,m)-isometries in general. Several spectral properties of n-quasi-(A,m)-isometries are obtained in Section three, concerning the spectrum, the approximate spectrum and Weyl spectrum.

2. n-Quasi-(A,m)-isometric operators

In the sequel, $A \in \mathcal{L}[\mathcal{K}]$ will denote a positive operator. Let m and n be two natural numbers, we define the n-quasi-(A,m)-isometric operator as follows:
Definition 2.1. *Let $S \in \mathcal{L}[\mathcal{K}]$, S is said to be an n-quasi-(A,m)-isometric operator if there exists a positive operator A on $\mathcal{K}$ such that*

$$S^{*n}\Big(\sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}S^{*k}AS^{k}\Big)S^{n}=0. \tag{2.1}$$

It is clear that each (A,m)-isometric operator is an n-quasi-(A,m)-isometric operator and each n-quasi-(A,m)-isometric operator is a $(n+1)$-quasi-(A,m)-isometric operator.

Remark. *We make the following remarks*

(i) If $A = I_{\mathcal{K}}$ every n-quasi-(A,m)-isometric operator is a n-quasi-m-isometric operator.

(ii) Every A-quasi-isometry (or quasi-A-isometry) is an n-quasi-(A,m)-isometric.

(iii) If S is an invertible n-quasi-(A,m)-isometric operator then S is an (A,m)-isometry.

(iv) 1-quasi-(A,m)-isometry is simply quasi-(A,m)-isometry.

(v) If $SA = AS$ and A is injective, then every n-quasi-(A,m)-isometric operator is a n-quasi-m-isometric operator.

The following remark is a consequence of definitions of n-quasi-m-isometric operator and n-quasi-(A,m)-isometric operator.

Remark. *Let $S \in \mathcal{L}[\mathcal{K}]$ and $A \in \mathcal{L}[\mathcal{K}]^+$. The following observations hold:*

(i) S is an n-quasi-m-isometric operator if and only if S is an $(S^{*n}S^n, m)$-isometric operator.

(ii) S is an n-quasi-(A,m)-isometric operator if and only if S is an $(S^{*n}AS^n, m)$ -isometric operator.

The following example shows that for fixed operator A, a n-quasi-(A,m)-isometry property is not necessary an (A,m)-isometry for some positive integers n and m.

Example 2.1. *Let us consider the operators on $\mathbb{C}^3$: $S = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. By Computing the products $S^{*(n+j)}AS^{n+j}$ and S^jAS^j for*

$j=0,1,\cdots,m$, *we show that*

$$S^{*n}\Big(\sum_{0\leq j\leq m}(-1)^{m-j}\binom{m}{j}S^{*j}AS^{j}\Big)S^{n}=0$$

and

$$\sum_{0\leq j\leq m}(-1)^{m-j}\binom{m}{j}S^{*j}AS^{j}\neq 0.$$

Therefore, S is a n-quasi-(A,m)-isometric operator but not an (A,m)-isometry.

The following example shows that an n-quasi-(A,m)-isometric operator need not be an n-quasi-m-isometric operator for some positive integers n and m and vice versa.

Example 2.2. *Let us consider the operators on $\mathbb{C}^3$: $S=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$ and $A=\begin{pmatrix}0&0&0\\0&1&1\\0&1&1\end{pmatrix}$.*

A direct calculation shows that S is a quasi-A-isometric but not a quasi-isometry.

Theorem 2.1. *Let $S\in\mathcal{L}[\mathcal{K}]$ and $A\in\mathcal{L}[\mathcal{K}]^+$, then S is an n-quasi-(A,m)-isometric operator if and only if*

$$\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\|S^{n+k}\xi\|_A^2=0,\ \ \forall\ \ \xi\in\mathcal{K}.$$

Proof. follows immediately from Definition 2.8. □

Proposition 2.2. *Let $S\in\mathcal{L}[\mathcal{K}]$. If S is a n-quasi-(A,m)-isometric operator and $\mathcal{R}(S^n)$ is dense, then S is an (A,m)-isometric operator.*

Proof. Immediate consequence of Theorem 2.1. □

Theorem 2.3. *Let $A,S\in\mathcal{L}[\mathcal{K}]$ with $A\geq 0$ and let $\mathcal{M}$ be an invariant closed subspace for S, P the orthogonal projection on $\mathcal{M}$ and $A_{\mathcal{M}}=PA_{/\mathcal{M}}$. If S is a n-quasi-(A,m)-isometric operator, then $S_{/\mathcal{M}}$ is also a n-quasi-$(A_{\mathcal{M}},m)$-isometric operator.*

Proof. Let $\xi\in\mathcal{M}$, we have

$$\|\xi\|_{A_{\mathcal{M}}}^2=\|\xi\|_A^2.$$

Thus

$$\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\|S_{/\mathcal{M}}^{n+k}\xi\|_{A_{\mathcal{M}}}^2=\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\|S^{n+k}\xi\|_A^2.=0.$$

This means that $S_{/\mathcal{M}}$ is an n-quasi-$(A_{\mathcal{M}},m)$-isometric operator. □

As a consequence of Theorem 2.3, we have the following result:

Corollary 2.4. *Let $S\in\mathcal{L}[\mathcal{K}]$ and let $\mathcal{M}$ be a reducing subspace for S and $A\mathcal{M}\subseteq\mathcal{M}$. If S is a n-quasi-(A,m)-isometric operator, then $S_{/\mathcal{M}}$ is also a n-quasi-$(A_{/\mathcal{M}},m)$-isometric operator.*

Proof. We have $A_{/\mathcal{M}} = A_{\mathcal{M}}$. This yields the desired result, by applying Theorem 2.1. □

The next proposition gives a necessary condition for an operator to be a n-quasi-(A,m)-isometry.

Proposition 2.5. *Let $A \in \mathcal{L}[\mathcal{K}]^+$ and $S \in \mathcal{L}_A[\mathcal{K}]$. If S is a n-Quasi-(A,m)-isometric operator such that $\|S^n\|_A \neq 0$, then*

$$2^{\frac{1}{m}} \leq 1 + \|S\|_A^2. \tag{2.2}$$

In particular, if S is a n-quasi-A-isometric operator, then

$$1 \leq \|S\|_A.$$

Proof. Since S is a n-quasi-(A,m)-isometric operator, then we have, by Theorem 2.1,

$$\sum_{0\leq k\leq m} (-1)^{m-k}\binom{m}{k}\|S^{n+k}\xi\|_A^2 = 0, \;\; \forall \;\; \xi \in \mathcal{H}.$$

Which can be written

$$\|S^n\xi\|_A^2 = \sum_{1\leq k\leq m} (-1)^{m-k}\binom{m}{k}\|S^{n+k}\xi\|_A^2, \;\; \forall \;\; \xi \in \mathcal{H}.$$

This implies

$$\|S^n\xi\|_A^2 \leq \left(\sum_{1\leq k\leq m} \binom{m}{k}\|S^{n+k}\|_A^2\right)\|\xi\|_A^2, \;\; \forall \;\; \xi \in \mathcal{H},$$

which gives

$$\|S^n\xi\|_A^2 \leq \left(\sum_{1\leq k\leq m} \binom{m}{k}\|S^{n}\|_A^2\|S\|_A^{2k}\right)\|\xi\|_A^2, \;\; \forall \;\; \xi \in \mathcal{H}.$$

Therefore

$$\|S^n\xi\|_A \leq \|S^n\|_A\left(\sum_{1\leq k\leq m} \binom{m}{k}\|S\|_A^{2k}\right)^{\frac{1}{2}}\|\xi\|_A, \;\; \forall \;\; \xi \in \mathcal{H}.$$

Which can be written

$$\|S^n\xi\|_A \leq \|S^n\|_A\left((1+\|S\|_A^2)^m - 1\right)^{\frac{1}{2}}\|\xi\|_A, \;\; \forall \;\; \xi \in \mathcal{H}.$$

Thus

$$\|S^n\|_A \leq \|S^n\|_A\big((1+\|S\|_A^2)^m - 1\big)^{\frac{1}{2}}.$$

Hence

$$1 \leq \big((1+\|S\|_A^2)^m - 1\big)^{\frac{1}{2}}.$$

Which yields

$$2^{\frac{1}{m}} \leq 1 + \|S\|_A^2.$$

Applying (2.2) for $m = 1$, we get

$$1 \leq \|S\|_A.$$

Thus, if S is an n-quasi-A-isometric operator, then

$$1 \leq \|S\|_A.$$

This ends the proof. $\square$

Set

$$\mathbf{Q}_l^A(S) := \sum_{0\le k\le l}(-1)^{l-k}\binom{l}{k}S^{*k}AS^k, \quad l\in\mathbb{N}. \tag{2.3}$$

Proposition 2.6. *Let $S\in\mathcal{L}[\mathcal{K}]$, then the following identity holds*

$$\mathbf{Q}_{m+1}^A(S) = S^*\mathbf{Q}_m^A(S)S - \mathbf{Q}_m^A(S). \tag{2.4}$$

In particular, every n-quasi-(A,m)-isometric operator is a n-quasi-(A,k)-isometric operator for each $k\ge m$.

Proof. In view of (2.9), we can write

$$\begin{aligned}\mathbf{Q}_{m+1}^A(S) &= (-1)^{m+1}A + \sum_{1\le k\le m}(-1)^{m+1-k}\binom{m+1}{k}S^{*k}AS^k + S^{*m+1}AS^{m+1}\\ &= (-1)^{m+1}A + \sum_{1\le k\le m}(-1)^{m+1-k}\left(\binom{m}{k}+\binom{m}{k-1}\right)S^{*k}AS^k + S^{*m+1}AS^{m+1}\\ &= -\mathbf{Q}_m^A(S) + S^*\mathbf{Q}_m^A(S)S.\end{aligned}$$

Thus, we obtain (2.4).

On the other hand, we have from (2.4) that $S^{*n}\mathbf{Q}_{m+1}^A(S)S^n = 0$ whenever $\mathbf{Q}_m^A(S) = 0$. $\square$

The outline of the following example is inspired of the paper [16].

Example 2.3. *Let $\mathcal{K} = l^2(\mathbb{C}) := \left\{(\xi_p)_p \subset \mathbb{C} \ / \ \sum_{p\ge 0}|\xi_p|^2 < \infty\ \right\}$ and consider $(e_p)_p$ be an orthonormal basis of $\mathcal{K}$. Define $S\in\mathcal{L}[\mathcal{K}]$ by $Se_n = \left(\dfrac{n+3}{n+1}\right)^{\frac{1}{2}}e_{n+1}$ and $Ae_n = \dfrac{n+1}{n+2}e_n$. By computing integers powers of S we may easily check that*

$$\|S^3e_n\|_A^2 - 2\|S^2e_n\|_A^2 + \|Se_n\|_A^2 \neq 0$$

and similarly, we find that

$$\|S^4e_n\|_A^2 - 3\|S^3e_n\|_A^2 + 3\|S^2e_n\|_A^2 - \|Se_n\|_A^2 = 0,\ \ \forall\, n\in\mathbb{N}.$$

In view of Theorem 2.1, we see that S is a quasi-$(A,3)$-isometry but cannot be a quasi-$(A,2)$-isometry.

Definition 2.2. *Let $S\in\mathcal{L}[\mathcal{K}]$ be a n-quasi-(A,m)-isometric operator. We define the (A,m)-exponent of S, denoted $e_{A,m}(S)$, to be the smallest $k\in\mathbb{N}$ such that S is a k-quasi-(A,m)-isometric.*

We give the following example to show that there exists an $(n+1)$- quasi-(A,m)-isometric operator, but not an n-quasi-(A,m)-isometric operator.

Example 2.4. *Let* $\mathcal{K}=\mathbb{C}^3$, $A=\begin{pmatrix}0&0&0\\0&1&1\\0&1&1\end{pmatrix}\in\mathcal{L}(\mathcal{K})^+$ *and* $S=\begin{pmatrix}0&0&-1\\1&0&0\\0&1&0\end{pmatrix}\in\mathcal{L}(\mathcal{K})$. *By computing* $S^{*j}\big(S^*AS-A\big)S^j$ *for* $j=1,2$ *we show that*

$$S^*\big(S^*AS-A\big)S\neq 0 \ \text{ and } \ S^{*2}\big(S^*AS-A\big)S^2=0.$$

Hence, the following conclusions hold:

(i) S is not an A-isometry neither an A-quasi-isometry.

(ii) S is a 2-quasi-$(A,1)$-isometry (or 2-quasi-A-isometry) but not a A-quasi-isometry.

Now we are ready to give a sufficient condition for a n-quasi-(A,m)-isometric operator to be a n_0-quasi-(A,m)-isometric operator for $n\geq n_0$.

Theorem 2.7. *Let* $S\in\mathcal{L}[\mathcal{K}]$ *be an* n*-quasi-*(A,m)*-isometric operator. The following statements hold.*

(1) *If* $\mathcal{N}(S^*)=\mathcal{N}(S^{*2})$, *then* S *is a quasi-*(A,m)*-isometry.*

(2) *If there exists a positive integer* n_0 *for which* $n\geq n_0$ *and* $\mathcal{N}(S^{*n_0})=\mathcal{N}(S^{*n})$, *then* S *is a* n_0*-quasi-*(A,m)*-isometric operator.*

Proof. (1) Under the assumption that $\mathcal{N}(S^*)=\mathcal{N}(S^{*2})$ it is enough to show that $\mathcal{N}(S^*)=\mathcal{N}(S^{*n})$ for all $n\in\mathbb{N}$, $n\geq 1$. By hypothesis, S is a n-quasi-(A,m)-isometry, then we have

$$S^{*n}\bigg(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}AS^k\bigg)S^n=0.$$

Since $\mathcal{N}(S^*)=\mathcal{N}(S^{*n})$, then a direct computation shows that

$$S^{*}\bigg(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}AS^k\bigg)S=0,$$

which implies that S is a quasi-(A,m)-isometry.

(2) Assume there exists $n_0\in\mathbb{N}$ for which $n\geq n_0$ and $\mathcal{N}(S^{*n_0})=\mathcal{N}(S^{*n})$. By using the fact that $\mathbf{Q}_m^A(S)$ is self-adjoint we obtain

$$S^{*n_0}\mathbf{Q}_m^A(S)S^{n_0}=0\Longleftrightarrow S^{*n}\mathbf{Q}_m^A(S)S^n=0.$$

The statement (ii) follows, and this completes the proof. □

Lemma 2.8. *Let* $S\in\mathcal{L}[\mathcal{K}]$ *ba a quasi-*$(A,2)$*-isometry, then* S *is a* S^*AS*-contractive operator i.e.;* $S^*\big(S^*AS\big)S\geq S^*AS$ $\big($*i.e.;* $S^{*2}AS^2\geq S^*AS\big)$.

Proof. Since S is a quasi-$(A,2)$-isometry it follows that

$$S^{*3}AS^3=2S^{*2}AS^2-S^*AS. \tag{2.5}$$

We use the induction to show that for all $k\geq 1$ we have

$$S^{*k+2}AS^{k+2}=(k+1)S^{*2}AS^2-kS^*AS. \tag{2.6}$$

For $k = 1$, the equality is obvious. Suppose (2.6) holds for k and prove it for $k+1$. Indeed, we have by multiplying the equation (2.6) on the left by S^* and on the right by S we get

$$\begin{aligned} S^{*k+3}AS^{k+3} &= (k+1)S^{*3}AS^3 - kS^{*2}AS^2 \\ &= (k+1)\Big(2S^{*2}AS^2 - S^*AS\Big) - kS^{*2}AS^2 \\ &= (k+2)S^{*2}AS^2 - (k+1)S^*AS \end{aligned}$$

This proves (2.6) for $k+1$. In particular,

$$(k+1)S^{*2}AS^2 \geq kS^*AS \text{ or equivalently } (1+\frac{1}{k})S^{*2}AS^2 \geq S^*AS.$$

Taking $k \longrightarrow \infty$, we get $S^{*2}AS^2 \geq S^*AS$ or equivalently $S^*\big(S^*AS\big)S \geq S^*AS$. Since S^*AS is a positive operator, the desired result follows immediately. $\square$

Theorem 2.9. *Let $S \in \mathcal{L}[\mathcal{K}]$ and $n, m \in \mathbb{N}$. Then S is a n-quasi-(A, m)-isometric operator if and only if $S = B^{-1}SB$ is a n-quasi-(B^*AB, m)-isometric operator for every invertible $B \in \mathcal{L}[\mathcal{K}]$.*

Proof. A little calculation yields

$$\begin{aligned} & S^{*n}\bigg(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}B^*ABS^k\bigg)S^n \\ = \;& B^*S^{*n}\big(B^*\big)^{-1}\bigg(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}B^*S^{*k}\big(B^*\big)^{-1}B^*ABB^{-1}S^kB\bigg)B^{-1}S^nB \\ = \;& B^*S^{*n}\bigg(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}AS^k\bigg)S^nB. \end{aligned}$$

It follows immediately that S is a n-quasi-(A, m)-isometric operator if and only if S is a n-quasi-(B^*AB, m)-isometric operator. $\square$

In the following theorem, we give a necessary and sufficient condition for S to be a n-quasi-(A, m)-isometric operator.

Theorem 2.10. *Let $S \in \mathcal{L}[\mathcal{K}]$. If $S^n \neq 0$ does not a dense range such that $\overline{\mathcal{R}(S^n)}$ is an invariant subspace for A, then the following statements are equivalent.*

(1) *S is a n-quasi-(A, m)-isometric operator.*

(2) $S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}$ *on $\mathcal{K} = \overline{\mathcal{R}(S^n)} \oplus \mathcal{N}(S^{*n})$, where S_1 is an (A_1, m)-isometric operator and $S_3^n = 0$, with $A_1 = A_{|\overline{\mathcal{R}(S^n)}}$.*

Proof. This idea comes from proof of [17, Theorem 2.1]. Since $\overline{\mathcal{R}(S^n)}$ is invariant for A and S, we have $A = A_1 \oplus A_2$ with $A_1 = A_{|\overline{\mathcal{R}(S^n)}}$ and $A_2 = A_{|\mathcal{N}(S^{*n})}$ and S has a matrix decomposition $S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}$, relative to the decomposition $\mathcal{H} = \overline{\mathcal{R}(S^n)} \oplus \mathcal{N}(S^{*n})$.

Assume that S is a n-quasi-(A,m)-isometric operator and let $P = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}$ be the projection onto $\overline{\mathcal{R}(S^n)}$, where $I_1 = I|\overline{\mathcal{R}(S^n)}$, it follows that

$$P\Big(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}AS^k\Big)P = 0$$

and so that

$$\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S_1^{*k}A_1S_1^k = 0.$$

Hence S_1 is an (A_1,m)-isometric operator.
On the other hand, let $x = x_1 + x_2 \in \mathcal{H} = \overline{\mathcal{R}(S^n)} \oplus \mathcal{N}(S^{*n})$. A simple computation shows that

$$\begin{aligned}\langle S_3^n x_2, x_2\rangle &= \langle S^n(I-P)x, (I-P)x\rangle \\ &= \langle (I-P)x, S^{*n}(I-P)x\rangle = 0.\end{aligned}$$

So, $S_3^n = 0$.
(2) $\Rightarrow$ (1) Suppose that $S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}$ onto $\mathcal{H} = \overline{\mathcal{R}(S^n)} \oplus \mathcal{N}(S^{*n})$, with

$$\Lambda_m(A_1,S) := \sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S_1^{*k}A_1S_1^k = 0 \;\text{ and }\; S_3^n = 0.$$

Since $S^k = \begin{pmatrix} S_1^k & \sum\limits_{j=0}^{k-1} S_1^jS_2S_3^{k-1-j} \\ 0 & S_3^k \end{pmatrix}$ we have

$$\begin{aligned}
&S^{*n}\Big(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}AS^kA\Big)S^n \\
=\;& \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}^{*n}\Big\{I + \sum_{1\leq k\leq m}(-1)^{m-k}\binom{m}{k}\begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}^{*k}\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}^k\Big\} \\
&\times\begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}^n \\
=\;& \begin{pmatrix} S_1^{*n} & 0 \\ \sum\limits_{j=0}^{n-1} S_3^{*n-1-j}S_2^*S_1^{*j} & S_3^{*n} \end{pmatrix} \\
&\times\Big\{I + \sum_{1\leq k\leq m}(-1)^k\binom{m}{k}\begin{pmatrix} S_1^{*k}A_1 & 0 \\ \sum\limits_{j=0}^{k-1} S_3^{*k-1-j}S_2^*S_1^{*j}C_1 & S_3^{*k}A_2 \end{pmatrix}\begin{pmatrix} S_1^k & \sum\limits_{j=0}^{k-1} S_1^jS_2S_3^{k-1-j} \\ 0 & S_3^k \end{pmatrix}\Big\} \\
&\times\begin{pmatrix} S_1^n & \sum\limits_{j=1}^{n-1} S_1^jS_2S_3^{n-1-j} \\ 0 & S_3^n \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} S_1^{*n} & 0 \\ \sum_{j=1}^{n-1} S_3^{*n-1-j} S_2^* S_1^{*j} & 0 \end{pmatrix}$$

$$\times \left\{ \begin{pmatrix} \Lambda_m(A_1,S_1) & C \\ D & B \end{pmatrix} \right\} \times \begin{pmatrix} S_1^n & \sum_{j=1}^{n-1} S_1^j S_2 S_3^{n-1-j} \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} S_1^{*n}\Lambda_m(A_1,S_1)S_1^n & S_1^{*n}\Lambda_m(A_1,S_1)\sum_{j=1}^{n-1} S_1^j S_2 S_3^{n-1-j} \\ \left(\sum_{j=1}^{n-1} S_1^j S_2 S_3^{n-1-j}\right)^*\Lambda_m(A_1,S_1)S_1^n & \left(\sum_{j=1}^{n-1} S_1^j S_2 S_3^{n-1-j}\right)^*\Lambda_m(A_1,S_1)\left(\sum_{j=1}^{n-1} S_1^j S_2 S_3^{n-1-j}\right) \end{pmatrix}.$$

Since $\Lambda_m(A_1,S_1)=0$, it follows that $S^{*n}\left(\sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}S^{*k}AS^k\right)S^n = 0.$ Thus S is a n-quasi-(m,A)-isometric operator. □

In [2], it was proved that if S is an (A,m)-isometric operator on a Hilbert space, then so is S^k for each positive integer k. In the following corollary we extend this result to n-quasi-(A,m)-isometric operators.

Corollary 2.11. *Under the same hypothesis as in Theorem 2.10, if S is a n-quasi-(A,m)-isometric operator, so is S^k for each positive integer k.*

Proof. If $\mathcal{R}(S^n)$ is dense then S is an (A,m)-isometric operator and so is S^k by [2, thoerem 1]. Now, assume that $\mathcal{R}(S^n)$ is not dense, by Theorem 2.10 we write the matrix representation of S on $\mathcal{H}=\overline{\mathcal{R}(S^n)}\oplus\mathcal{N}(S^{*n})$ as follows $S=\begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}$ where $S_1=S_{|\overline{\mathcal{R}}(S^n)}$ is an (A_1,m)-isometric operator and $S_3^n=0$. We notice that

$$S^k = \begin{pmatrix} S_1^k & \sum_{j=0}^{k-1} S_1^j S_2 S_3^{k-1-j} \\ 0 & S_3^k \end{pmatrix}$$

where S_1^k is an (A_1,m)-isometric operator ([2, Theroem 1]) and $(S_3^k)^n=0$. Hence S^k is an n-quasi-(A,m)-isometric operator by Theorem 2.10.

□

Recall that from [10], an operator $S\in\mathcal{L}[\mathcal{K}]$ is said to be A-power bounded, if $\sup_k \|S^k\|_A<\infty$ or equivalently, there exists $M>0$ so that for every positive integer k and every $\xi\in\mathcal{R}(A)$, one has

$$\|S^k\xi\|_A \le M\|\xi\|_A.$$

In [2, Theorem 1], it was proved that every A-power bounded (A,m)-isometric operator is A-isometric.

Theorem 2.12. *Under the same hypotheses as in Theorem 2.10, if $S \in \mathcal{L}[\mathcal{K}]$ is an n-quasi-(A,m)-isometric operator which is A-power bounded, Then S is a n-quasi-A-isometry.*

Proof. We consider the following two cases:

Case 1: If $\overline{\mathcal{R}(S^n)}$ is dense, then S is an (A,m)-isometric operator which is A-power bounded, thus S is an A-isometry by [2, Theorem 2] and so that S is a n-quasi-A-isometry.

Case 2: If $\overline{\mathcal{R}(S^n)}$ is not dense. By Theorem 2.5, we write the matrix representation of S on $\mathcal{K} = \overline{\mathcal{R}(S^n)} \oplus \mathcal{N}(S^{*n})$ as follows $S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}$ where $S_1 = S|\overline{\mathcal{R}(S^n)}$ is an (A_1,m)-isometric operator and $S_3^n = 0$. By taking into account that S is A-power bounded, it is easily seen that S_1 is A_1-power bounded from which we deduce that S_1-is an A_1-isometry. The result now follows by applying the statement (2) of Theorem 2.10. $\square$

Corollary 2.13. *Under the same hypothesis as in Theorem 2.10, if S is a n-quasi-(A,m)-isometric operator such that $S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}$ on $\mathcal{H} = \overline{\mathcal{R}(S^n)} \oplus \mathcal{N}(S^{*n})$ and $\lambda \in \mathbb{C}$, $\lambda \neq 0$ then $\mathcal{R}(S-\lambda I)$ is closed if and only if $\mathcal{R}(S_1-\lambda I)$ is closed.*

Proof. Assume that $\mathcal{R}(S-\lambda I)$ is closed and let $(\xi_k)_k$ be a sequence in $\overline{\mathcal{R}(S^n)}$ such that $(S_1-\lambda I)\xi_k \to \xi$ as $k \to \infty$. Then $(S-\lambda I)(\xi_k \oplus 0) \to \xi \oplus 0$. By the assumption, there exists $a \oplus b \in \mathcal{K} = \overline{\mathcal{R}(S^n)} \oplus \mathcal{N}(S^{*n})$ such that $\xi \oplus 0 = (S-\lambda I)(a \oplus b)$. This means that $\xi = (S_1-\lambda I)a + S_2 b$ and $(S_3-\lambda I)b = 0$. Since $S_3^n = 0$, it follows that $\lambda^n b = 0$ and hence $b = 0$. Therefore $\xi = (S_1-\lambda I)a$ and so that $\mathcal{R}(S_1-\lambda I)$ is closed.

Conversely, assume that $\mathcal{R}(S_1-\lambda I)$ is closed and let $(\xi_k \oplus \tau_k)_k$ be a sequence in $\mathcal{K} = \overline{\mathcal{R}(S^n)} \oplus \mathcal{N}(S^{*n})$ such that $(S-\lambda I)(\xi_k \oplus \tau_k) \to a \oplus b$, i.e.,

$$\begin{cases} (S_1-\lambda I)\xi_k + S_2\tau_k \to a \\ (S_3-\lambda)\tau_k \to b \end{cases}$$

Since $\lambda \notin \sigma(S_3)$, it follows that $\tau_k \to (S_3-\lambda)^{-1}b$ and so that

$$(S_1-\lambda I)\xi_k \to a - S_2(S_3-\lambda)^{-1}b.$$

From the assumptions, there exist $u \oplus v \in \mathcal{K} = \overline{\mathcal{R}(S^n)} \oplus \mathcal{N}(S^{*n})$ such that

$$a = (S_1-\lambda I)u + S_2(S_3-\lambda)^{-1}b \quad \text{and } b = (S_3-\lambda I)v,$$

which means

$$a = (S_1-\lambda I)u + S_2 v \ \text{ and } \ b = (S_3-\lambda I)v.$$

Consequently, $(S-\lambda I)(u \oplus v) = a \oplus b$ and hence $\mathcal{R}(S-\lambda I)$ is closed. $\square$

Proposition 2.14. *Under the same hypothesis as in Theorem 2.10, if S is a n-quasi-(A,m)-isometric operator such that $S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}$ on $\mathcal{K} = \overline{\mathcal{R}(S^n)} \oplus \mathcal{N}(S^{*n})$ and $\lambda \in \mathbb{C}$, $\lambda \neq 0$, then the following statements hold:*

(1) $\alpha(S-\lambda I) = \alpha(S_1-\lambda I)$.

(2) $\beta(S^*-\overline{\lambda}I)=\beta(S_1^*-\overline{\lambda}I)$.

Proof. (1) Since $S=\begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}$ it is clear that $\mathcal{N}(S-\lambda I)=\mathcal{N}(S_1-\lambda I)\cup\{0\}$ and it follows that

$$\alpha(S-\lambda I)=\alpha(S_1-\lambda I).$$

(2) Note that $\xi\oplus\eta\in\mathcal{N}(S^*-\overline{\lambda}I)$ if and only if

$$\xi\in\mathcal{N}(S_1^*-\overline{\lambda}I)\quad\text{and}\quad\eta=(S_3^*-\overline{\lambda})^{-1}S_2^*\xi.$$

Consider $(\xi_j\oplus\eta_j)_{1\leq j\leq k}$ be a family of linearly independent vectors in $\mathcal{N}(S^*-\overline{\lambda}I)$.

Then by the above observation we have

$$\xi_j\in\mathcal{N}(S_1^*-\overline{\lambda}I)\ \text{ and }\ \eta_j=\big(S_3^*-\overline{\lambda}\big)^{-1}S_2^*\xi_j\quad\text{for all}\quad j=1,2,...,k.$$

Now, assume that $\sum\limits_{1\leq j\leq k}\alpha_j\xi_j=0$, then $\sum\limits_{1\leq j\leq k}\alpha_j\eta_j=0$ and so $\sum\limits_{1\leq j\leq k}\alpha_j(\xi_j\oplus\eta_j)=0$.

Since $(\xi_j\oplus\eta_j)_{1\leq j\leq k}$ are linearly independent vectors of $\mathcal{K}$, it follows that $\alpha_j=0$ for $j=1,2,...,k$ which means that the vectors $(\xi_j)_{1\leq j\leq k}$ are linearly independent. Hence

$$\dim\mathcal{N}(S^*-\overline{\lambda}I)\leq\dim\mathcal{N}(S_1^*-\overline{\lambda}I).$$

Conversely, let $(\xi_j)_{1\leq j\leq k}$ be linearly independent vectors in $\mathcal{N}(S_1^*-\overline{\lambda}I)$.

Taking $\eta_j=\big(S_3^*-\overline{\lambda}\big)^{-1}S_2^*\xi_j$ for $j=1,\cdots,k.$, the vectors $(\xi_j\oplus\eta_j)_{1\leq j\leq k}$ belong to $\mathcal{N}(S^*-\overline{\lambda}I)$. Therefore the linear independence of these vectors follows from that of $(\xi_j)_{1\leq j\leq k}$. Consequently,

$$\dim\mathcal{N}(S^*-\overline{\lambda}I)\geq\dim\mathcal{N}(S_1^*-\overline{\lambda}I).$$

Hence

$$\dim\mathcal{N}(S^*-\overline{\lambda}I)=\dim\mathcal{N}(S_1^*-\overline{\lambda}I).$$

Consequently, $\beta(S^*-\overline{\lambda}I)=\beta(S_1^*-\overline{\lambda}I)$. □

For the concepts of SVEP and Bishop's property (β), we refer the interested readers to [3, 4].

Theorem 2.15. *Let $S\in\mathcal{L}[\mathcal{K}]$ be an (A,m)-isometric operator and let $0\notin\sigma_p(A)$, then S has the single-valued extension property.*

Proof. Let $\mu_0\in\mathbb{C}$ and let $\mathbb{U}$ be any open neighborhood of μ_0 in $\mathbb{C}$. Assume that $g:\mathbb{U}\to\mathcal{K}$ is any analytic function on $\mathbb{U}$ such that

$$(S-\mu)g(\mu)\equiv 0\ \text{ on }\ \mathbb{U}. \tag{2.7}$$

From (2.7), it follows that $(S^k-\mu^k)g(\mu)=0$ on $\mathbb{U}$ for all positive integers k.

Since S is an (A,m)-isometric operator, we obtain

$$0=\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}AS^kg(\mu)=\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}A\big(S^k-\mu^k+\mu^k\big)g(\mu).$$

So that

$$\begin{aligned} 0 &= \sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\langle S^{*k}A\mu^k g(\mu)\mid g(\mu)\rangle \\ &\Rightarrow \sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}|\mu|^{2k}\langle Ag(\mu)\mid g(\mu)\rangle = 0 \\ &\Rightarrow \big(1-|\mu|^2\big)^m\langle Ag(\mu)\mid g(\mu)\rangle = 0 \quad \forall\ \mu\in\mathbb{U}. \end{aligned}$$

Hence, $\langle Ag(\mu)\mid g(\mu)\rangle = 0 = \|A^{\frac{1}{2}}g(\mu)\| \quad \forall\ \mu\in\mathbb{U}$.

Since $0\notin\sigma_p(A)$ we have $g(\mu)=0$ on $\mathbb{U}$. Thus S has the SVEP at every $\mu_0\in\mathbb{C}$, i.e., S has the SVEP. $\square$

Theorem 2.16. *Under the same hypothesis as in Theorem 2.10, if S is a n-quasi-(A,m)-isometric operator such that $0\notin\sigma_p(A)$, then S has the single valued extension property.*

Proof. We consider the following two cases:

Case 1: If $\overline{\mathcal{R}(S^n)}$ is dense, then S is an (A,m)-isometric operator, thus S has SVEP by Theorem 2.15.

Case 2: If $\overline{\mathcal{R}(S^n)}$ is not dense. By Theorem 2.10 we write the matrix representation of S on $\mathcal{H}=\overline{\mathcal{R}(S^n)}\oplus\mathcal{N}(S^{*n})$ as follows $S=\begin{pmatrix} S_1 & S_2\\ 0 & S_3\end{pmatrix}$ where $S_1=S|\overline{\mathcal{R}(S^n)}$ is an (A_1,m)-isometric operator and $S_3^n=0$.

Assume that $(S-\mu)g(\mu)=0$ where $g(\mu)=g_1(\mu)\oplus g_2(\mu)$ on $\mathcal{H}=\overline{\mathcal{R}(S^n)}\oplus\mathcal{N}(S^{*n})$. Obviously we can write

$$\begin{pmatrix} S_1-\mu & S_2\\ 0 & S_3-\mu\end{pmatrix}\begin{pmatrix} g_1(\mu)\\ g_2(\mu)\end{pmatrix}=\begin{pmatrix} (S_1-\mu)g_1(\mu)+S_2g_2(\mu)\\ (S_3-\mu)g_2(\mu)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

Since S_3 is nilpotent, it follows that S_3 has SVEP and hence $g_2(\mu)=0$. We deduce that $(S_1-\mu)g_1(\mu)=0$. Under the condition that S_1 is an (A_1,m)-isometric operator, S_1 has the single valued extension property by Theorem 2.15, then $g_1(\mu)=0$. Consequently, $g\equiv 0$, so that S has SVEP as required. $\square$

Definition 2.3. *An operator $S\in\mathcal{L}[\mathcal{K}]$ is said to be a n-quasi-(X,m)-isometric operator if there exists some operator $X\in\mathcal{L}[\mathcal{K}]$ such that*

$$S^{*n}\Big(\sum_{0\leq j\leq m}(-1)^{m-j}\binom{m}{j}S^{*j}XS^j\Big)S^n=0.$$

for some positive integer m.

Remark. *If $X=I$, then S is just a n-quasi-m-isometric operator.*

(ii) *If X is a positive operator A, then S is just a n-quasi-(A,m)-isometric operator.*

Example 2.5. *Let* $S = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^3)$. *A simple computation shows that*

$$S^{*n}\Big(\sum_{0\leq j\leq m}(-1)^{m-j}\binom{m}{j}S^{*j}XS^j\Big)S^n = 0.$$

Therefore S is a n-quasi-(X, m)-isometric operator.

Let $S, X \in \mathcal{L}[\mathcal{K}]$. We define the X-covariance operator of S by

$$\Delta_S^X := \sum_{0\leq k\leq m-1}(-1)^{m-1-k}\binom{m-1}{k}S^{*k}XS^k.$$

Proposition 2.17. *Let S be a n-quasi-(X, m)-isometry, then S is a n-quasi-Δ_S^X-isometry.*

Proof. Since S is an n-quasi-(X, m)-isometry, it follows that

$$S^{*n}\Big(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}XS^k\Big)S^n = 0$$

or equivalently

$$S^{*n}\Big(-\Delta_S^X + S^*\Delta_S^X S\Big)S^n = 0.$$

Consequently,

$$S^{*n+1}\Delta_S^X S^{n+1} = S^{*n}\Delta_S^X S^n$$

and hence S is an n-quasi-Δ_S^X-isometry. □

3. Spectral properties of n-quasi-(A, m)-isometries

In this section, we study some spectral properties of some n-quasi-(A, m)-isometries. In [10, Proposition 4.1], the authors proved that if S is an (A, m)-isometry such that $0 \notin \sigma_{ap}(A)$ then the approximate point spectrum of S lies in the unit circle of the complex plane $\mathbb{C}$. i.e

$$\sigma_{ap}(S) \subset \partial\mathbb{D} := \{z \in \mathbb{C} \ / \ |z| = 1 \ \}.$$

The following theorem generalized [10, Proposition 4.1].

Theorem 3.1. *Let $S \in \mathcal{L}[\mathcal{K}]$, be a n-quasi-(A, m)-isometric operator where A is a positive operator on $\mathcal{K}$. If $0 \notin \sigma_{ap}(A)$, then $\sigma_{ap}(S) \subset \partial\mathbb{D} \cup \{0\}$.*

Proof. Let $\lambda \in \sigma_{ap}(S)$ and $0 \notin \sigma_{ap}(A)$. Then there exists a sequence $(\xi_p)_{p\geq 1} \subset \mathcal{K}$, with $||\xi_p|| = 1$ such that $(S - \lambda I_{\mathcal{K}})\xi_p \to 0$ as $p \to \infty$. By induction for each integer $k \geq 0$, we have $(S^k - \lambda^k I_{\mathcal{K}})\xi_p \to 0$ as $p \to \infty$. Since, S is an n-quasi-(A, m)-isometric operator, one has

$$\begin{aligned}
0 &= \langle S^{*n} \sum_{0\le k\le m} (-1)^{m-k}\binom{m}{k} S^{*k}AS^kS^n\xi_p \mid \xi_p\rangle \\
&= \langle \sum_{0\le k\le m} (-1)^{m-k}\binom{m}{k} S^{*n+k}AS^kS^n\xi_p \mid \xi_p\rangle \\
&= \langle \sum_{0\le k\le m} (-1)^{m-k}\binom{m}{k} A\Big((S^{n+k}-\lambda^{n+k})\xi_p + \lambda^{n+k}\xi_p\Big) \mid (S^{n+k}-\lambda^{n+k})\xi_p + \lambda^{n+k}\xi_p\rangle \\
&= \langle \sum_{0\le k\le m} (-1)^{m-k}\binom{m}{k} A(S^{n+k}-\lambda^{n+k})\xi_p \mid (S^{n+k}-\lambda^{n+k})\xi_p\rangle \\
&\quad +\langle \sum_{0\le k\le m} (-1)^{m-k}\binom{m}{k} A(S^{n+k}-\lambda^{n+k})\xi_p \mid \lambda^{n+k})\xi_p\rangle \\
&\quad +\langle \sum_{0\le k\le m} (-1)^{m-k}\binom{m}{k} A\lambda^{n+k}\xi_p \mid (S^{n+k}-\lambda^{n+k})\xi_p\rangle \\
&\quad +\langle \sum_{0\le k\le m} (-1)^{m-k}\binom{m}{k} A\lambda^{n+k}\xi_p \mid \lambda^{n+k}\xi_p\rangle.
\end{aligned}$$

As $\lim\limits_{p\to\infty}(S-\lambda I_{\mathcal{K}})\xi_p \to 0$, $\lim\limits_{p\to\infty}(S^{n+k}-\lambda^{n+k}I_{\mathcal{K}})\xi_p \to 0$, for $k=0,1,\cdots,m$. Then we have by taking $p\to\infty$

$$0 = \sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}\big(|\lambda|^2\big)^{n+k}\lim_{p\to\infty}\langle A\xi_p \mid \xi_p\rangle$$

or equivalently,

$$|\lambda|^{2n}\big(1-|\lambda|^2\big)^m \lim_{p\to\infty}\langle A\xi_p \mid \xi_p\rangle = 0.$$

Since $0\notin\sigma_{ap}(A)$, it must be the case that $\lim\limits_{p\to\infty}\langle A\xi_p \mid \xi_p\rangle \neq 0$, and so $|\lambda|^{2n}\big(1-|\lambda|^2\big)^m = 0$. Consequently, $\lambda=0$ or $|\lambda|=1$. This completes the proof. $\square$

Remark. *If the condition $0\notin\sigma_{ap}(A)$ is not satisfied , the conclusion of Theorem 3.1 cannot be true as show by the following example.*

Example 3.1. *For example, on $\mathcal{K}=\mathbb{C}^2$ the matrix operator $S=\begin{pmatrix}0 & 0\\ 1 & \beta\end{pmatrix}$ where $|\beta|^2=\dfrac{1+\sqrt{5}}{2}$ is a n-quasi-(A,m)-isometry with $A=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$. It is easily to check that $\sigma(S)=\{0,\beta\}$.*

Recall that two vectors ξ and $\eta\in\mathcal{K}$ are said to be A-orthogonal if $\langle A\xi \mid \eta\rangle = 0$.

The following proposition extend [14, Theorem 2.5].

Proposition 3.2. *Let $S\in\mathcal{L}[\mathcal{H}]$ be a n-quasi-(A,m)-isometric operator. If $0\notin\sigma_{ap}(A)$, then the following statements hold:*

(i) $\sigma_p(S)^* = \{\ \overline{\lambda},\ \lambda\in\sigma_p(S)\ \}\subset\sigma_p(S^*)$.

(ii) $\sigma_{ap}(S)^* = \{\ \overline{\lambda},\ \lambda \in \sigma_{ap}(S)\ \} \subset \sigma_{ap}(S^*)$,

(iii) Eigenvectors of S corresponding to distinct eigenvalues are A-orthogonal.

(iv) Let λ and $\mu \in \sigma_{ap}(S)$ such that $\lambda \neq \mu$. If $(\xi_p)_p$ and $(\eta_p)_p$ are two sequences of unit vectors in $\mathcal{K}$ such that $\|(S-\lambda)\xi_p\| \to 0$ and $\|(S-\mu)\eta_p\| \to 0$ (as $p \to \infty$,) then we have

$$\langle A\xi_p|\ \eta_p\rangle \to 0 \ \text{ (as } \ p \to \infty).$$

Proof. (i) Let $\lambda \in \sigma_p(S)$. Suppose $\lambda = 0$. If $0 \in \mathbb{C} - \sigma_p(S^*)$. Since S is a n-quasi-(A,m)-isometric operator, we have

$$S^{*n}\mathbf{Q}_m^A(S)S^n = 0$$

and it follows that

$$\begin{aligned} 0 = S^{*n}\Big(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}AS^k\Big)S^n \quad &\Rightarrow \quad 0 = \Big(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}AS^k\Big)S^n \\ &\Rightarrow \quad 0 = S^{*n}\Big(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}AS^k\Big) \\ &\Rightarrow \quad 0 = \sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}AS^k. \end{aligned}$$

Thus, S is an (A,m)-isometric. But this will contradict the fact that $0 \in \sigma_p(S)$.

Consider now $\lambda \neq 0$. Choose a non-zero vector $\xi \in \mathcal{K}$ such that $S\xi = \lambda\xi$. Since S is a n-quasi-(A,m)-isometric operator, we have

$$\begin{aligned} S\xi = \lambda\xi \quad &\Rightarrow \quad S^{*n}\Big(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}AS^k\Big)\lambda^n\xi = 0 \\ &\Rightarrow \quad S^{*n}\big(I_{\mathcal{K}} - \lambda S^*\big)^m A\xi = 0 \\ &\Rightarrow \quad \big(I_{\mathcal{K}} - \lambda S^*\big)^m S^{*n}A\xi = 0. \end{aligned}$$

If $(I_{\mathcal{K}} - \lambda S^*)$ is bounded from below, then so is $(I_{\mathcal{K}} - \lambda S^*)^m$ and hence there exists a positive constant $C > 0$ such that

$$\|(I_{\mathcal{K}} - \lambda S^*)^m\xi\| \geq C\|\xi\|, \ \ \forall\ \xi \in \mathcal{K}.$$

In particular

$$\|(I_{\mathcal{K}} - \lambda S^*)^m S^{*n}A\xi\| \geq C\|S^{*n}A\xi\|.$$

We find $S^{*n}A\xi = 0$. But then

$$0 = \langle S^{*n}A\xi \mid \xi\rangle = \langle A\xi \mid S^n\xi\rangle = \overline{\lambda}^n\langle A\xi \mid \xi\rangle.$$

Since $0 \notin \sigma_p(A)$ it follows that $\langle A\xi \mid \xi\rangle = \|A^{\frac{1}{2}}\xi\| \neq 0$ and hence $\lambda = 0$, contradiction. This shows that $(I_{\mathcal{K}} - \lambda S^*)$ is not bounded from below. From Theorem 3.1 we have $|\lambda| = 1$, and then $(I_{\mathcal{K}} - \lambda S^*) = \lambda\big(\overline{\lambda}I_{\mathcal{K}} - S^*\big)$. We conclude that $\overline{\lambda}I_{\mathcal{K}} - S^*$ is not bounded from below. This proves the statement in (i).

(ii) Let $\lambda \in \sigma_{ap}(S)$. If $\lambda = 0$, then as argued above, one can show that $0 \in \sigma_{ap}(S^*)$. Assume that λ is non-zero. Choose a sequence $(\xi_p)_p$ of unit vectors of $\mathcal{K}$ such that

$\|S-\lambda I_{\mathcal{K}})\xi_p\| \to 0$ as $p \to \infty$, and we can choose $\gamma > 0$ such that $\|A\xi_p\| \geq \gamma\|\xi_p\|$. for all p. Since $\sigma_{ap}(S) - \{0\} \subseteq \partial\mathbb{D}$ (by Theorem 3.1), we have

$$\begin{aligned}
0 &= S^{*n}\Big(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}AS^k\Big)S^n\xi_p \\
&= S^{*n}\Big(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}A\big(S^{n+k}-\lambda^{n+k}I_{\mathcal{K}}+\lambda^{n+k}I_{\mathcal{K}}\big)\xi_p\Big) \\
&= S^{*n}\Big(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}A\big(S^{n+k}-\lambda^{n+k}I_{\mathcal{K}}\big)\xi_p\Big) \\
&\quad +S^{*n}\Big(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}A\lambda^{n+k}I_{\mathcal{K}}\xi_p\Big) \\
&= S^{*n}\Big(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}A\big(S^{n+k}-\lambda^{n+k}I_{\mathcal{K}}\big)\xi_p\Big)+\lambda^nS^{n*}\big(I_{\mathcal{K}}-\lambda S^*\big)^mA\xi_p.
\end{aligned}$$

Since, $\lim_{p\to\infty}\|S^{*k}A\big(S^{n+k}-\lambda^{n+k}\big)\xi_p\| = 0$ for $j = 0, 1, \cdots, m$ we get

$$\|S^{n*}\big(I_{\mathcal{K}}-\lambda S^*\big)^mA\xi_p\| = \frac{1}{|\lambda|^n}\|S^{*n}\Big(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}A\big(S^{n+k}-\lambda^{n+k}I_{\mathcal{K}}\big)\xi_p\Big)\| \to 0, \quad \text{as } p\to\infty.$$

Hence, $\|\big(I-\lambda S^*\big)^mS^{n*}A\xi_p\| \to 0$, as $p \to \infty$.

If $(I-\lambda S^*)$ is bounded from below, then so is $(I-\lambda S^*)^m$ and hence there exists a positive constant $C > 0$ such that

$$\|(I_{\mathcal{K}}-\lambda S^*)^m\xi\| \geq C\|\xi\|, \ \ \forall\, \xi \in \mathcal{K}.$$

In particular

$$\|(I_{\mathcal{K}}-\lambda S^*)^mS^{*n}A\xi_p\| \geq C\|S^{*n}A\xi_p\|.$$

We find $\|S^{*n}A\xi_p\| \to 0$, as $p \to \infty$. Thus we have

$$\begin{aligned}
0 &= \lim_{p\to\infty}\langle S^{*n}A\xi_p \mid \xi_p\rangle \\
&= \lim_{p\to\infty}\langle A\xi_p \mid S^n\xi_p\rangle \\
&= \lim_{p\to\infty}\langle A\xi_p \mid \big(S^n-\lambda^n\big)\xi_p\rangle + \lim_{p\to\infty}\langle A\xi_p \mid \lambda^n\xi_p\rangle \\
&= \overline{\lambda}^n\lim_{p\to\infty}\langle A\xi_p \mid \xi_p\rangle.
\end{aligned}$$

So $\overline{\lambda}^n = 0$ or $\lambda = 0$, a contradiction. We conclude that $\overline{\lambda}I_{\mathcal{K}} - S^*$ is not bounded from below. This proves the statement in (ii).

(iii) Let λ and μ be two distinct eigenvalues of S and suppose that $S\xi = \lambda\xi$ and $S\eta = \mu\eta$. If λ or μ is zero the desired result is obvious. Now assume the $\lambda \neq 0$ and

$\mu \neq 0$. Since S is an n-quasi-(A,m)-isometry, then

$$\begin{aligned} 0 &= \langle S^{*n}\Big(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*k}AS^k\Big)S^n\xi \mid \eta\rangle \\ &= \sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\langle AS^{n+k}\xi \mid S^{n+k}\eta\rangle \\ &= \big(\lambda\overline{\mu}\big)^n\big(1-\lambda\overline{\mu}\big)^m\langle A\xi \mid \eta\rangle \\ &= \big(\lambda\overline{\mu}\big)^n\big(|\mu|^2-\lambda\overline{\mu}\big)^m\langle A\xi \mid \eta\rangle \quad \big(\text{by Theorem 3.1 }\big) \\ &= \big(\lambda\overline{\mu}\big)^n(\overline{\mu})^m\big(\mu-\lambda\big)^m\langle A\xi \mid \eta\rangle. \end{aligned}$$

As $\lambda \neq \mu$ it follows that $\langle A\xi \mid \eta\rangle = 0$ as required.

(iv) Let $\lambda, \mu \in \sigma_{ap}(S)$ such as $\lambda \neq \mu$. Consider $(\xi_p)_p \subset \mathcal{K}$ and $\eta_p \subset \mathcal{K}$ with $\|\xi_p\| = \|\eta_p\| = 1$ and

$$\|\big(S-\lambda I_{\mathcal{K}}\big)\xi_p\| \to 0 \ \text{ and } \ \|\big(S-\mu I_{\mathcal{K}}\big)\eta_p\| \to 0, \ \text{ as } \ p\to\infty.$$

If $\lambda = 0$ or $\mu = 0$, then clearly $\langle A\xi_p \mid \eta_p\rangle \to 0;$ as $p \to \infty$.

Assume that $\lambda \neq 0$ or $\mu \neq 0$. Since for all $j \in \{0, 1, \cdots, m\}$ we have

$$\|\big(S^{n+j}-\lambda^{n+j}I_{\mathcal{K}}\big)\xi_p\| \to 0 \ \text{ and } \ \|\big(S^{n+j}-\mu^{n+j}I_{\mathcal{K}}\big)\eta_p\| \to 0, \ \text{ as } \ p\to\infty.$$

An analogous calculation as in the statement (iii) gives

$$0 = \big(\lambda\overline{\mu}\big)^n(\overline{\mu})^m\big(\mu-\lambda\big)^m\lim_{p\to\infty}\langle A\xi \mid \eta\rangle.$$

Then clearly $\lim_{p\to\infty}\langle A\xi_p \mid \eta_p\rangle = 0$ as required. □

Theorem 3.3. *Under the same hypothesis as in Theorem 2.8, if S be a n-quasi-(A,m)-isometric operator such that $S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}$ on $\mathcal{H} = \overline{\mathcal{R}(S^n)} \oplus \mathcal{N}(S^{*n})$, then the following properties hold:*

(i) $\sigma(S) = \sigma(S_1) \cup \{0\}$

(ii) $\sigma_w(S) \cup \pi_0(S)\backslash\{0\} = \sigma_w(S_1) \cup \pi_0(S_1)\backslash\{0\}$.

Proof. (i) From [9, Corollary 7], it follows that $\sigma(S) \cup W = \sigma(S_1) \cup \sigma(S_3)$, where W is the union of certain of the holes in $\sigma(S)$ which is a subsets of $\sigma(S_1) \cap \sigma(S_3)$. Further $\sigma(S_3) = \{0\}$ and $\sigma(S_1) \cap \sigma(S_3)$ has no interior points. So we have by [9, Corollary 8]

$$\sigma(S) = \sigma(S_1) \cup \sigma(S_3) = \sigma(S_1) \cup \{0\}.$$

(ii) By Corollary 2.9 and proposition 2.9, it follows that

$$\sigma_w(S)\backslash\{0\} = \sigma_w(S_1)\backslash\{0\} \ \text{ and } \ \pi_0(S)\backslash\{0\} = \pi_0(S_1)\backslash\{0\}.$$

Consequently,

$$\sigma_w(S) \cup \pi_0(S)\backslash\{0\} = \sigma_w(S_1) \cup \pi_0(S_1)\backslash\{0\}.$$

□

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