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EXISTENCE OF FIXED POINT BY USING F-CONTRACTION AND F-SUZUKI CONTRACTION IN MODULAR FUNCTION SPACES

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Abstract. The purpose of this paper is to study the notions of F-contraction and F-Suzuki contraction in context of modular function spaces and to prove some fixed point results. Further we provide some examples to support our main results.

1. Introduction

In 1950, Nakano [9] introduced the concept of the modular spaces that was further generalized and redefined by Musielak and Orlicz [8] in 1959. Modular function spaces are the generalization of some class of Banach spaces which attracts many analysts to work in this field. The study of fixed point in modular function spaces was initiated by Khamsi et al. [7] in 1990. On the basis of their results, many work has been done in these spaces. Dhomopongsa et al. [3] proved that every ρ-contraction $T : C \to F_\rho(C)$ has a fixed point where ρ is a convex function modular satisfying Δ_2-type condition, C is a nonempty ρ-bounded, ρ-closed subset of L_ρ and $F_\rho(C)$ is the collection of ρ-closed subset of C. In 2011, Khamsi and Kozlowski [6] proved the existence of fixed points of asymptotic pointwise ρ-nonexpansive mappings in modular function spaces.

In 2012, Wardowski [15] introduced a new type of contraction $F : \mathbb{R}^+ \to \mathbb{R}$ called F-contraction and gave a fixed point result that generalized Banach contraction principle in metric spaces. In 2014, Piri and Kumam [11] transformed the result of Wardowski by applying some weaker conditions on the self map of a complete metric space and on the mapping F, concerning the contraction defined by Wardowski and with these weaker conditions, proved a fixed point result for F-Suzuki contraction which generalizes the result of Wardowski. R. Jain [4] in 2018, proved the existence of a fixed point for a nondecreasing mapping in partially ordered complete b-metric space using sequential monotone property of the space. In 2020, R. Jain [5] introduced the concept of generalized weak contraction mapping in setting of generating space of b-dislocated metric space endowed with partial order and proved

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some fixed-point theorems for the mappings in space satisfying the generalized weak contraction. Recently, Panwar and Pinki [10] transformed M iteration process in CAT(0) spaces to approximate fixed point of generalized α-nonexpansive mappings.

In our paper, we study the concepts of F-contraction and F-Suzuki contraction in context of modular function spaces and establish some fixed point existence results in these spaces. Further we construct some examples to support our results.

2. Preliminaries

To finish our paper, we collect some basic definitions and important results. Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a nontrivial δ-ring of subsets of Ω which means that $\mathcal{P}$ is closed under countable intersection, finite union and differences. Suppose that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \cup K_n$. By ε we denote the linear space of all simple functions with support from $\mathcal{P}$. Also $\mathcal{M}_\infty$ denotes the space of all extended measurable functions, i.e., all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence

$$\{g_n\} \subset \varepsilon,\ |g_n| \leq |f| \text{ and } g_n(w) \to f(w) \text{ for all } w \in \Omega.$$

We define

$$\mathcal{M} = \{f \in \mathcal{M}_\infty : |f(w)| < \infty \ \ \rho - a.e.\}.$$

Now, we recall the definition of modular function.

Definition 2.1. [14] Let X be a vector space (R or C). A functional $\rho : \mathcal{M} \to [0, \infty]$ is called a modular if for any arbitrary elements $f, g \in X$, the following conditions hold:

(i) $\rho(f) = 0 \Longleftrightarrow f = 0$
(ii) $\rho(\alpha f) = \rho(f)$ whenever $|\alpha| = 1$
(iii) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ whenever $\alpha, \beta \geq 0, \ \ \alpha + \beta = 1$.

If we replace (iii) by

(iv) $\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \beta\rho(g)$ whenever $\alpha, \beta \geq 0, \ \ \alpha + \beta = 1$.

Then modular ρ is called convex.

Definition 2.2. [14] If ρ is convex modular in X, then the set defined by

$$L_\rho = \{f \in \mathcal{M} : \rho(\lambda f) \to 0 \ as \ \lambda \to 0\}$$

is called modular function space.

Definition 2.3. [14] Let $\rho : \mathcal{M}_\infty \to [0, \infty]$ be a nontrivial, convex and even function. Then ρ is a regular convex function pseudo modular if

(i) $\rho(0) = 0$;
(ii) ρ is monotone, i.e., $|f(w)| \leq |g(w)|$ for any $w \in \Omega$ implies $\rho(f) \leq \rho(g)$, where $f, g \in \mathcal{M}_\infty$;
(iii) ρ is orthogonally sub-additive, i.e., $\rho(f\chi_{A\cup B}) \leq \rho(f\chi_A) + \rho(f\chi_B)$ for any $A, B \in \sum$ such that $A \cap B \neq \phi$, $f \in \mathcal{M}_\infty$;
(iv) ρ has Fatou property, i.e., $|f_n(w)| \uparrow |f(w)|$ for $w \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$;
(v) ρ is order continuous in ε, i.e., $g_n \in \varepsilon$ and $|g_n(w)| \downarrow 0$ and $\rho(g_n) \downarrow 0$.

ρ is regular convex function modular if $\rho(f) = 0$ implies $f = 0$ a.e. The class of all nonzero regular convex function modular on Ω is denoted by $\Re$.

Definition 2.4. [14] Let $\rho \in \mathfrak{R}$. Then ρ satisfies Δ_2-property if $\rho(2f_n) \to 0$ whenever $\rho(f_n) \to 0 \;\; as \;\; n \to \infty$.

Definition 2.5. [15] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:

(F1) F is strictly increasing, i.e., for all $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta, F(\alpha) < F(\beta)$;
(F2) For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

The set of all functions satisfying the conditions (F1)-(F3) is denoted by $\mathcal{F}$.

3. Fixed point result for F-contraction

In the beginning of this section, we define F-contraction in modular function spaces and then some examples of F-contraction are provided. In the end, we prove a theorem for the existence of fixed point for F-contraction.

Definition 3.1. Let $\rho \in \mathfrak{R}$. Let D_ρ be a nonempty, ρ-closed and ρ-bounded subset of L_ρ. Then a mapping $T : D_\rho \to D_\rho$ is said to be F-contraction if there exists $\tau > 0$ such that for all $f, g \in D_\rho$

$$\rho(Tf - Tg) > 0 \Longrightarrow \tau + F(\rho(Tf - Tg)) \leq F(\rho(f - g)) \tag{3.1}$$

Now, we provide some examples of F-contraction.

Example 3.2. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be defined by $F(\alpha) = \ln \alpha + \sqrt{\alpha}$. It can be easily shown that F satisfies all the conditions of definition 2.5 for any $k \in (0, 1)$. Let $T : D_\rho \to D_\rho$ be a mapping defined as:

$$\frac{\rho(Tf - Tg)}{\rho(f - g)} e^{\left[\sqrt{\rho(Tf-Tg)} - \sqrt{\rho(f-g)}\right]} \leq e^{-\tau}$$

satisfying (3.1) is F-contraction.

Example 3.3. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be defined by $F(\alpha) = \ln(\alpha + \sqrt{\alpha})$. It can be easily shown that F satisfies all the conditions of definition 2.5 for any $k \in (0, 1)$. Let $T : D_\rho \to D_\rho$ be a mapping defined as:

$$\frac{\rho(Tf - Tg) \left[1 + (\rho(Tf - Tg))^{-\frac{1}{2}}\right]}{\rho(f - g) \left[1 + (\rho(Tf - Tg))^{-\frac{1}{2}}\right]} \leq e^{-\tau}$$

satisfying (3.1) is F-contraction.

Example 3.4. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be defined by $F(\alpha) = \frac{1}{3} \ln \alpha$. It can be easily shown that F satisfies all the conditions of definition 2.5 for any $k \in (0, 1)$. Let $T : D_\rho \to D_\rho$ be a mapping defined as:

$$\frac{\rho(Tf - Tg)}{\rho(f - g)} \leq e^{-3\tau}$$

satisfying (3.1) is F-contraction.

Example 3.5. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be defined by $F(\alpha) = \frac{1}{2} \ln \alpha + \alpha$. It can be easily shown that F satisfies all the conditions of definition 2.5 for any $k \in (0, 1)$. Let $T : D_\rho \to D_\rho$ be a mapping defined as:

$$\frac{\rho(Tf - Tg)}{\rho(f - g)} e^{2[\rho(Tf-Tg) - \rho(f-g)]} \leq e^{-2\tau}$$

satisfying (3.1) is F-contraction.

Now, we prove the main result of the paper.

Theorem 3.6. *Let $\rho \in \mathfrak{R}$ satisfying Δ_2-type condition. If D_ρ is a non-empty, ρ-closed and ρ-bounded subset of L_ρ and $T : D_\rho \to D_\rho$ is an F-contraction then T has a unique fixed point f^* and for every $f_0 \in D_\rho$, the sequence $\{T^n f_0\}_{n\in\mathbb{N}}$ converges to f^*.*

Proof. We define a sequence $\{f_n\}_{n\in\mathbb{N}} \subset D_\rho$, $f_{n+1} = Tf_n$, $n = 1, 2, 3, ...$
Let $\alpha_n = \rho(f_{n+1} - f_n)$. If there exists $n_0 \in \mathbb{N}$ for which $Tf_{n_0} = f_{n_0}$, then nothing to prove. Suppose that $f_{n+1} \neq f_n$ for every $n \in \mathbb{N}$. Then $\alpha_n > 0$ for all $n \in \mathbb{N}$.

$$\begin{aligned} F(\rho(f_{n+1} - f_n)) &= F(\rho(Tf_n - Tf_{n-1})) \\ &\leq F(\rho(f_n - f_{n-1})) - \tau \\ \text{or } F(\alpha_n) &\leq F(\alpha_{n-1}) - \tau \end{aligned}$$

$$F(\alpha_n) \leq F(\alpha_{n-1}) - \tau \leq F(\alpha_{n-2}) - 2\tau \leq ... \leq F(\alpha_0) - n\tau \tag{3.2}$$

From inequality (3.2), we get $\lim_{n\to\infty} F(\alpha_n) = -\infty$ that together with (F2) gives

$$\lim_{n\to\infty} \alpha_n = 0 \tag{3.3}$$

From (F3), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} \alpha_n^k F(\alpha_n) = 0 \tag{3.4}$$

By inequality (3.2), the following inequality holds for all $n \in \mathbb{N}$

$$\alpha_n^k F(\alpha_n) - \alpha_n^k F(\alpha_0) \leq \alpha_n^k (F(\alpha_0) - n\tau) - \alpha_n^k F(\alpha_0) = -n\alpha_n^k \tau \leq 0 \tag{3.5}$$

Letting $n \to \infty$ in inequality (3.5), and using equations (3.3) and (3.4), we get

$$\lim_{n\to\infty} n\alpha_n^k = 0 \tag{3.6}$$

From equation (3.6), there exists $n_1 \in \mathbb{N}$ such that $n\alpha_n^k \leq 1$ for all $n \geq n_1$. Consequently, we have

$$\alpha_n \leq \frac{1}{n^{\frac{1}{k}}} \ for \ all \ n \geq n_1 \tag{3.7}$$

We show that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Consider $p, q \in \mathbb{N}$ such that $p > q \geq n_1$. We get

$$\begin{aligned} \rho(f_p - f_q) &\leq \frac{\omega(p-q)}{p-q}[\rho(f_p - f_{p-1}) + \rho(f_{p-1} - f_{p-2}) + ... + \rho(f_{q+1} - f_q)] \\ &\leq \omega(p-q)[\rho(f_p - f_{p-1}) + \rho(f_{p-1} - f_{p-2}) + ... + \rho(f_{q+1} - f_q)] \\ &= \omega(p-q)[\alpha_{p-1} + \alpha_{p-2} + ... + \alpha_q] \\ &= \omega(p-q)\sum_{i=q}^{p-1} \alpha_i < \sum_{i=q}^{\infty} \alpha_i \\ &\leq \omega(p-q)\sum_{i=q}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

Since $\sum_{i=q}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is convergent, so $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. By the completeness of D_ρ, there exists $f^* \in D_\rho$ such that $\lim_{n\to\infty} f_n = f^*$.

$$\begin{aligned}\rho(Tf^* - f^*) &= \lim_{n\to\infty} \rho(Tf_n - f_n)\\ &= \lim_{n\to\infty} \rho(f_{n+1} - f_n) = 0.\end{aligned}$$

This shows that f^* is the fixed point of T. Now, we show that T has a unique fixed point. If $f_1, f_2 \in D_\rho$ such that $Tf_1 = f_1 \neq Tf_2 = f_2$,

$$\begin{aligned}&\tau \leq F(\rho(f_1 - f_2)) - F(\rho(Tf_1 - Tf_2)) = 0\\ \Longrightarrow\ &\tau \leq 0\end{aligned}$$

which contradicts to the fact that $\tau > 0$. Hence, T has a unique fixed point. □

Example 3.7. Let the real number system $\mathbb{R}$ be the space modulared as

$$\rho(f) = \mid f \mid$$

Consider the sequence $\{S_n\}_{n\in\mathbb{N}}$ as defined below:

$$\begin{cases} S_1 = 1\\ S_2 = 1 + 2\\ \vdots\\ S_n = \frac{n(n+1)}{2}, \quad n \in \mathbb{N}\end{cases}$$

Let $D_\rho = \{S_n : n \in \mathbb{N}\}$. Let $T : D_\rho \to D_\rho$ be a mapping defined as:

$$\begin{cases} T(S_n) = S_{n-1} \ for\ n > 1\\ T(S_1) = S_1.\end{cases}$$

Consider the mappings $F_1(\alpha) = \frac{1}{3}\ln\alpha$, $F_2(\alpha) = \frac{1}{2}\ln\alpha + \alpha$ and $F_3(\alpha) = \ln\alpha + \sqrt{\alpha}$. Let us first consider F_1 defined in example 3.4, we have

$$\lim_{n\to\infty} \frac{\rho(TS_n - TS_1)}{\rho(S_n - S_1)} = \lim_{n\to\infty} \frac{S_{n-1} - S_1}{S_n - S_1} = 1,$$

which is a contradiction. So, T is not F_1-contraction.
Now, we take F_2 defined in example 3.5, we observe that T is F_2-contraction having $\tau = 1$. For all $m, n \in \mathbb{N}$

$$T(S_n) \neq T(S_m) \Leftrightarrow m > 2 \ and\ n = 1 \ or\ m > n > 1.$$

For all $m > 2, m \in \mathbb{N}$ and n=1, we get

$$\begin{aligned}\frac{\rho(T(S_m) - T(S_1))}{\rho(S_m - S_1)} e^{2[\rho(T(S_m)-T(S_1))-\rho(S_m-S_1)]} &= \frac{S_{m-1} - S_1}{S_m - S_1} e^{2[(S_{m-1}-S_1)-(S_m-S_1)]}\\ &= \frac{m^2 - m - 2}{m^2 + m - 2} e^{-2m}\\ &< e^{-2m} < e^{-2}\end{aligned}$$

For all $m, n \in \mathbb{N}, m > n > 1$, we have

$$\begin{aligned}\frac{\rho(T(S_m) - T(S_n))}{\rho(S_m - S_n)} e^{2[\rho(T(S_m)-T(S_n))-\rho(S_m-S_n)]} &= \frac{S_{m-1} - S_{n-1}}{S_m - S_n} e^{2[(S_{m-1}-S_{n-1})-(S_m-S_n)]} \\ &= \frac{m+n-1}{m+n+1} e^{2(n-m)} \\ &< e^{2(n-m)} \le e^{-2}\end{aligned}$$

Now, taking F_3 defined in example 3.2, we observe that T is F_3-contraction having $\tau = 0.37184$. For all $m, n \in \mathbb{N}$

$$T(S_n) \neq T(S_m) \Leftrightarrow m > 2 \; and \; n = 1 \; or \; m > n > 1.$$

For all $m > 2, m \in \mathbb{N}$ and n=1, we get

$$\begin{aligned}\frac{\rho(T(S_m) - T(S_1))}{\rho(S_m - S_1)} e^{\left[\sqrt{\rho(T(S_m)-T(S_1))}-\sqrt{\rho(S_m-S_1)}\right]} &= \frac{S_{m-1} - S_1}{S_m - S_1} e^{[\sqrt{S_{m-1}-S_1}-\sqrt{S_m-S_1}]} \\ &= \frac{m^2 - m - 2}{m^2 + m - 2} e^{\left[\sqrt{\frac{m^2-m-2}{2}}-\sqrt{\frac{m^2+m-2}{2}}\right]} \\ &\le e^{\left[\sqrt{\frac{m^2-m-2}{2}}-\sqrt{\frac{m^2+m-2}{2}}\right]} \\ &\le e^{\left[\sqrt{2}-\sqrt{5}\right]} = e^{-0.82185},\end{aligned}$$

if we take m=3.
For all $m, n \in \mathbb{N}, m > n > 1$, we obtain the following calculation

$$\begin{aligned}\frac{\rho(T(S_m) - T(S_n))}{\rho(S_m - S_n)} e^{\left[\sqrt{\rho(T(S_m)-T(S_n))}-\sqrt{\rho(S_m-S_n)}\right]} &= \frac{S_{m-1} - S_{n-1}}{S_m - S_n} e^{[\sqrt{S_{m-1}-S_{n-1}}-\sqrt{S_m-S_n}]} \\ &= \frac{m+n-1}{m+n+1} e^{\left[\sqrt{\frac{(m-n)(m+n-1)}{2}}-\sqrt{\frac{(m-n)(m+n+1)}{2}}\right]} \\ &\le e^{\left[\sqrt{\frac{(m-n)(m+n-1)}{2}}-\sqrt{\frac{(m-n)(m+n+1)}{2}}\right]} \\ &\le e^{\sqrt{2}-\sqrt{3}} = e^{-0.37184},\end{aligned}$$

if we take m=3, n=2.

From this example, we conclude that T is not F_1-contraction while it is F_2 and F_3-contraction. In the following table, we compare Banach contraction with F-contraction. The generated iteration start from a point $f_0 = S_{31} = 496$ and $C_F(S_n, S_1)$ denotes $F(\rho(S_n - S_1)) - F(\rho(T(S_n) - T(S_1)))$. From the table 3.7, we conclude that $S_1 = 1$ is the fixed point of T.

n	f_n	$C_{F_1}(S_n, S_1)$	$C_{F_2}(S_n, S_1)$	$C_{F_3}(S_n, S_1)$
3	406	0.30543	3.45814	1.73815
4	378	0.19592	4.29389	1.35172
5	351	0.14727	5.22091	1.18349
6	325	0.11889	6.17833	1.087153
7	300	0.10003	7.15005	1.02412
8	276	0.08650	8.12975	0.97943
9	253	0.07628	9.11442	0.94601
10	231	0.06826	10.10239	0.920014
11	210	0.06180	11.09270	0.89919
12	190	0.05647	12.08470	0.88212
13	171	0.05200	13.07800	0.86787
14	153	0.04819	14.07229	0.85578
15	136	0.04491	15.06737	0.84540
16	120	0.04205	16.06307	0.83638
17	105	0.03953	17.05930	0.82848
18	91	0.03731	18.05595	0.82149
19	78	0.03532	19.05297	0.81527
20	66	0.03353	20.05029	0.80969
21	55	0.031915	21.04787	0.80466
22	45	0.03045	22.04567	0.80010
23	36	0.029114	23.04367	0.79595
24	28	0.02789	24.04183	0.792164
25	21	0.02677	25.04015	0.78868
26	15	0.02573	26.03859	0.78547
27	10	0.02477	27.03716	0.78251
28	6	0.02388	28.03582	0.77976
29	3	0.02305	29.03458	0.77721
30	1	0.02228	30.03342	0.77483
31	1	0.02156	31.03234	0.77260
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n \to \infty$	1	$\tau \to 0$	$\geq \tau = 1$	$\geq \tau = 0.37184$

4. Fixed point result for F-Suzuki contraction

In 2013, Secelean [12] proved the following lemma.

Lemma 4.1. *Let $F : \mathbb{R}_+ \to \mathbb{R}$ be an increasing mapping and $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Then the following assertion.*

1.(a) *if $\lim_{n\to\infty} F(\alpha_n) = -\infty$, then $\lim_{n\to\infty} \alpha_n = 0$;*
2.(b) *if $\inf F = -\infty$, then $\lim_{n\to\infty} F(\alpha_n) = -\infty$.*

The condition (F2) in definition 2.5 is replaced by Secelean [12] by an equivalent but a more simple condition with the help of lemma 4.1,

$(F2')$ $\inf F = -\infty$
or also by
$(F2'')$ there exists a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive real numbers such that

$$\lim_{n\to\infty} F(\alpha_n) = -\infty.$$

The condition (F3) in definition 2.5 is replaced by Piri and Kumam [11] with the following condition:

$(F3')$ F is continuous on $(0, \infty)$.

The set of all functions satisfying the condition (F1), $(F2')$ and $(F3')$ is denoted by $\mathfrak{F}$.

Example 4.2. [11] Let $F_1(\alpha) = -\frac{1}{\alpha}, F_2(\alpha) = -\frac{1}{\alpha} + \alpha, F_3(\alpha) = \frac{1}{1-e^{\alpha}}$, $F_4(\alpha) = \frac{1}{e^{\alpha}-e^{-\alpha}}$. Then $F_1, F_2, F_3, F_4 \in \mathfrak{F}$.

Remark. The condition (F3) and $(F3')$ are independent of each other. For example, $F(\alpha) = -\frac{1}{\alpha}$ satisfies the conditions (F1), (F2) and $(F3')$ but it does not satisfy (F3). Therefore, $\mathfrak{F} \notin \mathcal{F}$. Also, $F(\alpha) = -\frac{1}{\sqrt{\alpha+[\alpha]}}$, where $[\alpha]$ denotes the integral part of α, satisfies conditions (F1), (F2) and (F3) for any $k \in \left(\frac{1}{2}, 1\right)$ but it does not satisfy $(F3')$. Therefore, $\mathcal{F} \notin \mathfrak{F}$. But if we take $F(\alpha) = \frac{1}{3}\ln\alpha$, then it satisfies conditions of both $\mathfrak{F}$ and $\mathcal{F}$ and hence, $F \in \mathcal{F} \cap \mathfrak{F}$.

Definition 4.3. Let $\rho \in \mathfrak{R}$ and satisfy Δ_2-condition. Then the growth function $\omega : [0, \infty) \to [0, \infty)$ is defined as:

$$\omega = \sup\left\{\frac{\rho(tx)}{\rho(x)} : 0 < \rho(x) < \infty\right\}.$$

Then, $1 < \omega(2)$. In addition $\rho(tx) \leq \omega(t)\rho(t), \forall t \geq 0, \forall x \in X_\rho$ and also that, for each positive integer l and for arbitrary $x_1, x_2, ..., x_l \in X_\rho$

$$\rho(x_1 + x_2 + ... + x_l) \leq \frac{\omega(l)}{l}[\rho(x_1) + \rho(x_{2)} + ... + \rho(x_l)].$$

In 2008, Suzuki [13] introduced the condition (C). Motivated by his work, we transform this condition to modular structure resulting in the modular-(C_ρ) condition as follows:

Definition 4.4. Let $\rho \in \mathfrak{R}$. Assume that ρ satisfies Δ_2-type condition and D_ρ be a nonempty subset of L_ρ. A mapping $T : D_\rho \to D_\rho$ is said to satisfy condition (C_ρ) if

$$\frac{1}{\omega(2)}\rho(f - Tf) \leq \rho(f - g) \Longrightarrow \rho(Tf - Tg) \leq \rho(f - g), \forall f, g \in D_\rho.$$

Definition 4.5. Let $\rho \in \mathfrak{R}$. Assume that ρ satisfies Δ_2-type condition and D_ρ be a nonempty subset of L_ρ. A mapping $T : D_\rho \to D_\rho$ is said F-Suzuki contraction if there exists $\tau > 0$ such that for all $f, g \in D_\rho$ with $Tf \neq Tg$

$$\frac{1}{\omega(2)}\rho(f - Tf) \leq \rho(f - g) \Longrightarrow \tau + F(\rho(Tf - Tg)) \leq F(\rho(f - g)), \tag{4.1}$$

where $F \in \mathfrak{F}$.

Theorem 4.6. *Let $\rho \in \mathfrak{R}$. Assume that ρ satisfies Δ_2-type condition and D_ρ be a nonempty bounded, closed subset of L_ρ and $T : D_\rho \to D_\rho$ be an F-Suzuki contraction. Then T has a unique fixed point $\overline{f} \in D_\rho$ and for every $f_0 \in D_\rho$, the sequence $\{T^n f_0\}$ converges to $\overline{f}$.*

Proof. We define a sequence $\{f_n\}_{n\in\mathbb{N}} \subset D_\rho,\ f_n = Tf_{n-1},\ n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ for which $Tf_{n_0} = f_{n_0}$, then nothing to prove. Suppose that $f_{n+1} \neq f_n$ for every $n \in \mathbb{N}$. As $\rho(f_n - Tf_n) > 0$ for all $n \in \mathbb{N}$, therefore

$$\frac{1}{\omega(2)}\rho(f_n - Tf_n) < \rho(f_n - Tf_n), \forall n \in \mathbb{N} \tag{4.2}$$

For any $n \in \mathbb{N}$

$$\begin{aligned} F(\rho(f_{n+1} - Tf_{n+1})) &= F(\rho(Tf_n - T^2 f_n)) \\ &\leq F(\rho(f_n - Tf_n)) - \tau. \end{aligned}$$

Continuing this process, we get

$$\begin{aligned} F(\rho(f_n - Tf_n)) &\leq F(\rho(f_{n-1} - Tf_{n-1})) - \tau \\ &\leq F(\rho(f_{n-2} - Tf_{n-2})) - 2\tau \\ &\cdots \\ &\leq F(\rho(f_0 - Tf_0)) - n\tau \end{aligned} \tag{4.3}$$

From inequality (4.3), we get $\lim_{n\to\infty} F(\rho(f_n - Tf_n)) = -\infty$ that together with $(F2')$ gives

$$\lim_{n\to\infty} \rho(f_n - Tf_n) = 0 \tag{4.4}$$

Now, we show that $\{f_n\}$ is a Cauchy sequence. By contradiction, we suppose that there exists $\epsilon > 0$ and the sequences $\{u(n)\}_{n=1}^{\infty}$ and $\{v(n)\}_{n=1}^{\infty}$ of natural numbers such that

$$u(n) > v(n) > n, \rho(f_{u(n)} - f_{v(n)}) \geq \epsilon, \rho(f_{u(n)-1} - f_{v(n)}) < \frac{\epsilon}{\omega(2)}, \forall n \in \mathbb{N} \tag{4.5}$$

so we have

$$\begin{aligned} \epsilon &\leq \rho(f_{u(n)} - f_{v(n)}) \\ &\leq \omega(2)[\rho(f_{u(n)} - f_{u(n)-1}) + \rho(f_{u(n)-1} - f_{v(n)})] \\ &\leq \omega(2)\left[\rho(f_{u(n)} - f_{u(n)-1}) + \frac{\epsilon}{\omega(2)}\right] \end{aligned}$$

Using equation (4.4) and above inequality, we get

$$\lim_{n\to\infty} \rho(f_{u(n)} - f_{v(n)}) = \epsilon. \tag{4.6}$$

From equation (4.4) and inequality (4.5), we can choose a positive integer $n_1 \in \mathbb{N}$ such that

$$\begin{aligned} \frac{1}{\omega(2)}\rho(f_{u(n)} - Tf_{u(n)}) &< \frac{\epsilon}{\omega(2)} \\ &\leq \rho(f_{u(n)} - f_{v(n)}), \forall n \geq n_1. \end{aligned}$$

So, by definition of F-Suzuki contraction

$$\tau + F(\rho(Tf_{u(n)} - Tf_{v(n)})) \leq F(\rho(f_{u(n)} - f_{v(n)})), \forall n \geq n_1$$

$$\tau + F(\rho(f_{u(n)+1} - f_{v(n)+1})) \leq F(\rho(f_{u(n)} - f_{v(n)})), \forall n \geq n_1 \tag{4.7}$$

From $(F3')$, inequalities (4.6) and (4.7), we get

$$\tau + F(\epsilon) \leq F(\epsilon),$$

which is a contradiction. Therefore, $\{f_n\}$ is a Cauchy sequence. Since D_ρ is complete, so there exists $\overline{f} \in D_\rho$ such that

$$\lim_{n\to\infty} \rho(f_n - \overline{f}) = 0 \tag{4.8}$$

We claim that

$$\frac{1}{\omega(2)}\rho(f_n - Tf_n) < \rho(f_n - \overline{f}) \text{ or } \frac{1}{\omega(2)}\rho(Tf_n - T^2 f_n) < \rho(Tf_n - \overline{f}), \forall n \in \mathbb{N} \tag{4.9}$$

But we suppose that there exists $m \in \mathbb{N}$ such that

$$\frac{1}{\omega(2)}\rho(f_m - Tf_m) \geq \rho(f_m - \overline{f}) \text{ or } \frac{1}{\omega(2)}\rho(Tf_m - T^2 f_m) \geq \rho(Tf_m - \overline{f}), \forall m \in \mathbb{N}. \tag{4.10}$$

From first part of inequality (4.10)

$$\begin{aligned}\rho(f_m - \overline{f}) &\leq \frac{1}{\omega(2)}\rho(f_m - Tf_m)\\ &\leq \frac{1}{\omega(2)}\frac{\omega(2)}{2}[\rho(f_m - \overline{f}) + \rho(\overline{f} - Tf_m)]\\ &\leq \frac{1}{2}[\rho(f_m - \overline{f}) + \rho(\overline{f} - Tf_m)]\\ \rho(f_m - \overline{f}) &\leq \rho(\overline{f} - Tf_m) \end{aligned} \tag{4.11}$$

From inequalities (4.10) and (4.11), we obtain

$$\rho(f_m - \overline{f}) \leq \rho(\overline{f} - Tf_m) \leq \frac{1}{\omega(2)}\rho(Tf_m - T^2 f_m) \tag{4.12}$$

Since, $\frac{1}{\omega(2)}\rho(f_m - Tf_m) < \rho(f_m - Tf_m)$, therefore by definition 4.6,

$$\tau + F(\rho(Tf_m - T^2 f_m)) \leq F(\rho(f_m - Tf_m))$$

Since $\tau > 0$, $F(\rho(Tf_m - T^2 f_m)) < F(\rho(f_m - Tf_m))$. Using (F1), we get

$$\rho(Tf_m - T^2 f_m) < \rho(f_m - Tf_m) \tag{4.13}$$

From inequalities (4.10), (4.12) and (4.13), we get

$$\begin{aligned}\rho(Tf_m - T^2 f_m) &< \rho(f_m - Tf_m)\\ &\leq \frac{\omega(2)}{2}[\rho(f_m - \overline{f}) + \rho(\overline{f} - Tf_m)]\\ &\leq \frac{\omega(2)}{2}\left[\frac{1}{\omega(2)}\rho(Tf_m - T^2 f_m) + \frac{1}{\omega(2)}\rho(Tf_m - T^2 f_m)\right]\\ &= \rho(Tf_m - T^2 f_m),\end{aligned}$$

which is a contradiction. Hence, the inequality (4.9) holds. So, from inequality (4.9) for all $n \in \mathbb{N}$, we get

$$\begin{aligned}&\textit{either } \tau + F(\rho(Tf_n - T\overline{f})) \leq F(\rho(f_n - \overline{f}))\\ &\textit{or } \tau + F(\rho(T^2 f_n - T\overline{f})) \leq F(\rho(Tf_n - \overline{f}))\\ &\textit{or } \tau + F(\rho(f_{n+2} - T\overline{f})) \leq F(\rho(f_{n+1} - \overline{f})).\end{aligned}$$

In first case, from inequality (4.9), $(F2')$ and lemma 4.1, we obtain

$$\lim_{n\to\infty} F(\rho(Tf_n - T\overline{f})) = -\infty.$$

From $(F2')$ and lemma 4.1, $\lim_{n\to\infty}\rho(Tf_n - T\overline{f}) = 0$, therefore

$$\begin{aligned}\rho(\overline{f} - T\overline{f}) &= \lim_{n\to\infty}\rho(f_{n+1} - T\overline{f})\\ &= \lim_{n\to\infty}\rho(Tf_n - T\overline{f}) = 0.\end{aligned}$$

In second case, from inequality (4.9), $(F2')$ and lemma 4.1, we get

$$\lim_{n\to\infty} F(\rho(T^2 f_n - T\overline{f})) = -\infty.$$

From $(F2')$ and lemma 4.1, $\lim_{n\to\infty}\rho(T^2 f_n - T\overline{f}) = 0$, therefore

$$\begin{aligned}\rho(\overline{f} - T\overline{f}) &= \lim_{n\to\infty}\rho(f_{n+2} - T\overline{f})\\ &= \lim_{n\to\infty}\rho(T^2 f_n - T\overline{f}) = 0.\end{aligned}$$

Hence, $\overline{f}$ is a fixed point of T. Now, we show that T has atmost one fixed point. If $f_1, f_2 \in D_\rho$ such that $Tf_1 = f_1 \neq f_2 = Tf_2$, therefore $\rho(Tf_1 - Tf_2) > 0$, then we have

$$\frac{1}{\omega(2)}\rho(f_1 - Tf_1) < \rho(f_1 - f_2),$$

therefore, $\tau \leq F(\rho(f_1 - f_2)) - F(\rho(Tf_1 - Tf_2)) = 0$ which implies that $\tau \leq 0$, which contradicts to the fact that $\tau > 0$. This shows that T has a unique fixed point. □

Example 4.7. Let the real number system $\mathbb{R}$ be the space modulared as

$$\rho(f) = \mid f \mid .$$

The corresponding growth function $\omega(t) = t, \forall t \geq 0$. Consider the sequence $\{S_n\}_{n\in\mathbb{N}}$ as defined below:

$$\begin{cases} S_1 = 1^2 \\ S_2 = 1^2 + 2^2 \\ \vdots \\ S_n = \frac{n(n+1)(2n+1)}{6}, \quad n \in \mathbb{N} \end{cases}$$

Let $D_\rho = \{S_n : n \in \mathbb{N}\}$. Let $T : D_\rho \to D_\rho$ be a mapping defined as:
$T(S_n) = S_{n-1}$ for $n > 1$ and $T(S_1) = S_1$.
Since

$$\begin{aligned}\lim_{n\to\infty}\frac{\rho(TS_n - TS_1)}{\rho(S_n - S_1)} &= \lim_{n\to\infty}\frac{\rho(S_{n-1} - S_1)}{\rho(S_n - S_1)}\\ &= \lim_{n\to\infty}\frac{2^2 + 3^2 + ... + (n-1)^2}{2^2 + 3^2 + ... + n^2} = 1\end{aligned}$$

T is neither Banach contraction nor Suzuki contraction. Taking $F(\alpha) = -\frac{1}{\alpha} + \alpha \in \mathfrak{F}$, we observe that T is an F-Suzuki contraction with $\tau = 4$. To see this, let us consider the following calculations. We observe that

$$\frac{1}{\omega(2)}\rho(S_n - TS_n) < \rho(S_n - S_m) \Leftrightarrow [(1 = n < m) \vee (1 \leq m < n) \vee (1 < m < n)].$$

For $1 = n < m$, we get

$$\begin{aligned} \mid TS_m - TS_1 \mid &= \mid S_{m-1} - S_1 \mid \\ &= 2^2 + 3^2 + ... + (m-1)^2 \\ \mid S_m - S_1 \mid &= 2^2 + 3^2 + ... + m^2 \end{aligned}$$

Since $m > 1$ and

$$-\frac{1}{2^2 + 3^2 + ... + (m-1)^2} < -\frac{1}{2^2 + 3^2 + ... + m^2}$$

$$4 - \frac{1}{2^2 + 3^2 + ... + (m-1)^2} < 4 - \frac{1}{2^2 + 3^2 + ... + m^2}$$

$$4 - \frac{1}{2^2 + 3^2 + ... + (m-1)^2} + [2^2 + 3^2 + ... + (m-1)^2] < 4 - \frac{1}{2^2 + 3^2 + ... + m^2} + [2^2 + 3^2 + ... + (m-1)^2]$$

$$4 - \frac{1}{2^2 + 3^2 + ... + (m-1)^2} + [2^2 + 3^2 + ... + (m-1)^2] < \frac{1}{2^2 + 3^2 + ... + m^2} + [2^2 + 3^2 + ... + (m-1)^2 + m^2]$$

$$4 - \frac{1}{\mid TS_m - TS_1 \mid} + \mid TS_m - TS_1 \mid < -\frac{1}{\mid S_m - S_1 \mid} + \mid S_m - S_1 \mid$$

For $1 \leq m < n$, similar to $1 = n < m$.
And now, for $1 < m < n$, we have

$$\begin{aligned} \mid TS_m - TS_n \mid &= \mid S_{m-1} - S_{n-1} \mid \\ &= n^2 + (n+1)^2 + ... + (m-1)^2 \\ \mid S_m - S_n \mid &= \mid S_m - S_n \mid \\ &= (n+1)^2 + (n+1)^2 + ... + m^2. \end{aligned}$$

Since $m > 1$ and

$$-\frac{1}{n^2 + (n+1)^2 + ... + (m-1)^2} < -\frac{1}{(n+1)^2 + (n+2)^2 + ... + m^2}$$

$$4 - \frac{1}{n^2 + (n+1)^2 + ... + (m-1)^2} < 4 - \frac{1}{2^2 + 3^2 + ... + m^2}$$

$$4 - \tfrac{1}{n^2+(n+1)^2+...+(m-1)^2} + [n^2 + (n+1)^2 + ... + (m-1)^2] < 4 - \tfrac{1}{(n+1)^2+(n+2)^2+...+m^2} + [n^2 + (n+1)^2 + ... + (m-1)^2]$$

$$4 - \tfrac{1}{n^2+(n+1)^2+...+(m-1)^2} + [n^2 + (n+1)^2 + ... + (m-1)^2] < \tfrac{1}{(n+1)^2+(n+2)^2+...+m^2} + [(4+n^2)+(n+1)^2+(n+2)^2+...+(m-1)^2]$$

$$4 - \tfrac{1}{n^2+(n+1)^2+...+(m-1)^2} + [n^2 + (n+1)^2 + ... + (m-1)^2] < \tfrac{1}{(n+1)^2+(n+2)^2+...+m^2} + [(n+1)^2 + (n+2)^2 + ... + (m-1)^2 + m^2]$$

$$4 - \frac{1}{\mid TS_m - TS_n \mid} + \mid TS_m - TS_n \mid < -\frac{1}{\mid S_m - S_n \mid} + \mid S_m - S_n \mid$$

Therefore, $\tau + F(\rho(TS_m - TS_n)) \leq F(\rho(S_m - S_n))$, for all $m, n \in \mathbb{N}$. Hence T is an F-Suzuki contraction. The following table shows the comparision of Banach contraction with F-contraction for $F_1(\alpha) = \ln \alpha, F_2(\alpha) = \ln \alpha + \sqrt{\alpha}, F_3(\alpha) = -\frac{1}{\sqrt{\alpha}} +$

n	f_n	$C_{F_1}(S_n, S_1)$	$C_{F_2}(S_n, S_1)$	$C_{F_3}(S_n, S_1)$	$C_{F_4}(S_n, S_1)$
3	6930	1.178654	9.178654	9.173076	9.157437
4	6201	0.802346	16.802346	16.424403	16.064809
5	5525	0.621688	25.621688	25.015964	25.035081
6	4900	0.510825	36.510825	36.007407	36.021689
7	4324	0.434664	49.434664	49.003916	49.014559
8	3795	0.378732	64.378732	64.002268	64.010346
9	3311	0.335768	81.335768	81.001404	81.007670
10	2870	0.301668	100.301668	100.000916	100.005874
11	2470	0.275894	121.275894	121.000623	121.004618
12	2109	0.248896	144.248896	144.000439	144.003709
13	1785	0.231429	169.231429	169.000318	169.003032
14	1496	0.214795	196.214795	196.000236	196.002517
15	1240	0.200401	225.200401	225.000179	225.002117
16	1015	0.187821	256.187821	256.000138	256.001801
17	819	0.176731	289.176731	289.000108	289.001546
18	650	0.166881	324.166881	324.000086	324.001340
19	506	0.158073	361.158073	361.000069	361.001170
20	385	0.150150	400.150150	400.000056	400.001029
21	285	0.142984	441.142984	441.000046	441.000911
22	204	0.136472	484.136472	484.000038	484.000811
23	140	0.130528	529.130528	529.130528	529.000725
24	91	0.125081	576.125081	576.000027	576.000651
25	55	0.120071	625.120071	625.000023	625.000588
26	30	0.115447	676.115447	676.000019	676.000534
27	14	0.111166	729.111166	729.000016	729.000485
28	5	0.107197	784.107197	784.000014	784.000443
29	1	0.103491	841.103491	841.000012	841.000406
30	1	0.100038	900.100038	900.000011	900.000373
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
3×10^3	1	0.001	9000000.001	9000000	9000000
$n \to \infty$	1	$\tau \to 0$	$\geq \tau = 1$	$\geq \tau = 1$	$\geq \tau = 1$

α and $F_4(\alpha) = -\frac{1}{\sqrt{\alpha + [\alpha]}}$ where $F_1, F_2 \in \mathbb{F} \cap \mathfrak{F}, F_3 \in \mathfrak{F} - \mathbb{F}$ and $F_3 \in \mathbb{F} - \mathfrak{F}$. The generated iteration start from a point $f_0 = S_{29} = 8555$ and $C_F(S_n, S_1)$ denotes $F(\rho(S_n - S_1)) - F(\rho(T(S_n) - T(S_1)))$. From the table 4.7, we conclude that $S_1 = 1$ is the fixed point of T.

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