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INVESTIGATING n-QUASI-(m, q)-ISOMETRIES IN THE CONTEXT OF METRIC SPACES

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Abstract. In this study, we introduce a new class of transformations on metric spaces by defining the concept of an n-quasi-(m, q)-isometric mapping. For a positive integers n, m and $q \in (0, \infty)$, a map $\mathcal{U} : (\mathcal{E}, d) \longrightarrow (\mathcal{E}, d)$ is said to be an n-quasi-(m, q)-isometric mapping if $\mathcal{U}$ satisfies

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} d\big(\mathcal{U}^{n+k}\omega, \mathcal{U}^{n+k}\psi\big)^q = 0, \quad \forall\, \omega, \psi \in \mathcal{E}.$$

We give some of their properties, studying the products and the power of such operators and we discuss their impact on the structure of metric spaces, paving the way for further mathematical applications in this field.

1. INTRODUCTION AND TERMINOLOGIES

The class of m-isometric operators on Hilbert spaces was introduced by Agler ([1]) and subsequently studied by Alger and Stankus ([2, 3, 4]). An operator $\mathcal{U}$ acting on a Hilbert space $\mathcal{K}$ is called an m-isometry if

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \mathcal{U}^{*k}\mathcal{U}^k = 0, \tag{1.1}$$

or equivalently

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \|\mathcal{U}^k\psi\|^2 = 0,\ \psi \in \mathcal{K}. \tag{1.2}$$

Sid Ahmed [18] and Botelho [13] employed equation (1.2) to introduce the concept of m-isometries on a Banach space. Bayart [5] replaced the exponent 2 in equation (1.2) with $q \in [1, \infty)$ and introduced the following definition: a bounded linear operator $\mathcal{U}$:acting on a Banach space $\mathcal{X}$ is an (m, q)-isometry if

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \|\mathcal{U}^k\psi\|^q = 0 \qquad (\psi \in \mathcal{X}). \tag{1.3}$$

Hoffmann and Mackey [17] explored the above definition for $p > 0$.

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In the paper [9], T. Bermdez et al., introduced and studied the concept of (m, q)-isometric maps on metric spaces. Let $\mathcal{E}$ be a metric space and $m \geq 1$ be integer and $q > 0$. A map $\mathcal{U} : \mathcal{E} \to \mathcal{E}$ is called an (m, q)-isometry if for all $\omega, \psi \in \mathcal{E}$

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} d\big(\mathcal{U}^k \omega, \mathcal{U}^k \psi\big)^q = 0. \tag{1.4}$$

Several authors have worked on the extension of m-isometries to n-quasi-m-isometries in the broader context of Hilbert and Banach spaces. An operator $\mathcal{U}$ acting on a Hilbert space $\mathcal{K}$ is refereed as n-quasi-m-isometric operator ([21])if it satisfies

$$\mathcal{U}^{*n} \bigg(\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \mathcal{U}^{*k} \mathcal{U}^k \bigg) \mathcal{U}^n = 0, \tag{1.5}$$

or

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \|\mathcal{U}^{n+k} \psi\|^2 = 0, \ \forall\ \psi \in \mathcal{K}. \tag{1.6}$$

In the context of Banach spaces, research on the extension of m-isometries to n-quasi-m-isometries is complex, particularly because Banach spaces do not have as rich a structure as Hilbert spaces. This implies that certain notions, such as isometries or quasi-isometries, require a more general definition and sometimes a relaxation of conditions. This leads the authors in [15] to propose the following definition. spaces. An operator $\mathcal{U}$ acting on a Banach space $\mathcal{X}$ is refereed as n-quasi-(m, q)-isometric operator if it satisfies

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \|\mathcal{U}^{n+k} \psi\|^q = 0, \ \forall\ \psi \in \mathcal{K}. \tag{1.7}$$

These operators have been the subject of extensive research by numerous authors, as detailed in references [6, 7, 8, 10, 11, 12, 14, 16, 19, 20, 22].

Following the works concerning the extensions of m-isometries in Hilbert and Banach spaces to n-quasi-m-isometries, this paper introduces a novel concept of mappings in metric spaces, which builds upon and generalizes the notion of m-isometries on metric spaces. Specifically, we extend the class of m-isometries to a more generalized framework of mappings, which we term n-quasi-(m, q)-isometries in context of metric spaces. A mapping $\mathcal{U} : (\mathcal{E}, d) \longrightarrow (\mathcal{E}, d)$ is called an n-quasi-(m, q)-isometric mapping if $\mathcal{U}$ satisfies

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} d\big(\mathcal{U}^{n+k} \omega, \mathcal{U}^{n+k} \psi\big)^q = 0, \quad \forall\ \omega, \psi \in \mathcal{E}. \tag{1.8}$$

We examine several key properties of n-quasi-(m, q)-isometries, focusing on their behavior under products and powers.

2. BASIC PROPERTIES OF n-QUASI-(m, q)-ISOMETRIES ON METRIC SPACES

In this section we introduce and study the of n-quasi (m, q)-isometries on metric spaces and prove results that generalizes the existing ones corresponding to n-quasi (m, q)-isometries on Banach spaces.

Proposition 2.1. *Let $\mathcal{U} : \mathcal{E} \to \mathcal{E}$ be an n-quasi (m, q)-isometry if and only if is an (m, q)-isometry on $\overline{\mathcal{R}(\mathcal{U}^n)}$.*

Proof. Let $\mathcal{U}:\mathcal{E}\to\mathcal{E}$ is an n-quasi (m,q)-isometry on $\mathcal{E}$, if and only if, for all $\omega,\psi\in\mathcal{E}$ we have

$$\begin{aligned} 0 &= \sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k} d\big(\mathcal{U}^{n+k}\omega,\mathcal{U}^{n+k}\psi\big)^q \\ &= \sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k} d\big(\mathcal{U}^{k}(\mathcal{U}^n\omega),\mathcal{U}^{k}(\mathcal{U}^n\psi)\big)^q \\ &= \sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k} d\big(\mathcal{U}^{k}\omega,\mathcal{U}^{k}\psi\big)^q \;\forall\omega,\psi\in\overline{\mathcal{R}(\mathcal{U}^n)}. \end{aligned}$$

Equivalently, $\mathcal{U}$ is an (m,q)-isometry on $\overline{\mathcal{R}(\mathcal{U}^n)}$. □

Remark. (1) *When $m=1$, Equation (1.8) is equivalent to*

$$d\big(\mathcal{U}^{n+1}\omega,\mathcal{U}^{n+1}\psi\big)^q = d\big(\mathcal{U}^{n}\omega,\mathcal{U}^{n}\psi\big)^q,\quad\forall\,\omega,\psi\in\mathcal{E};$$

(2) *When $m=2$, Equation (1.8) is equivalent to*

$$d\big(\mathcal{U}^{n}\omega,\mathcal{U}^{n}\psi\big)^q - 2d\big(\mathcal{U}^{n+1}\omega,\mathcal{U}^{n+1}\psi\big)^q + d\big(\mathcal{U}^{n+2}\omega,\mathcal{U}^{n+2}\psi\big)^q = 0,\quad\forall\,\omega,\psi\in\mathcal{E};$$

(3) *When $m=3$, Equation (1.8) is equivalent to*

$$d\big(\mathcal{U}^{n}\omega,\mathcal{U}^{n}\psi\big)^q - 3d\big(\mathcal{U}^{n+1}\omega,\mathcal{U}^{n+1}\psi\big)^q + 3d\big(\mathcal{U}^{n+2}\omega,\mathcal{U}^{n+2}\psi\big)^q - d\big(\mathcal{U}^{n+3}\omega,\mathcal{U}^{n+3}\psi\big)^q = 0,$$
$$\forall\,\omega,\psi\in\mathcal{E}.$$

Remark. *In the previous proposition, we have shown that $\mathcal{U}$ is a n-quasi-(m,q)-isometry on $\mathcal{E}$ if and only if $\mathcal{U}$ is an (m,q)-isometry on $\overline{\mathcal{R}(\mathcal{U}^n)}$. In particular, if $\mathcal{R}(\mathcal{U}^n)$ is dense in $\mathcal{E}$, then $\mathcal{U}$ is an n-quasi-(m,q)-isometry on $\mathcal{E}$ if and only if $\mathcal{U}$ is an (m,q)-isometry on $\mathcal{E}$. For this reason, we will assume throughout this paper that $\mathcal{R}(\mathcal{U}^n)$ is not dense on $\mathcal{E}$.*

Proposition 2.2. *Let $\mathcal{U}:\mathcal{E}\to\mathcal{E}$ be an n-quasi (m,q)-isometry, then $\mathcal{U}$ is an n_1-quasi (m,q)-isometry for all $n_1\ge n$.*

Proof. Assume that $\mathcal{U}$ is an n-quasi (m,q)-isometry on $\mathcal{E}$. Referring to Proposition 2.1 we get that $\mathcal{U}$ is an (m,q)-isometry on $\overline{\mathcal{R}(\mathcal{U}^n)}$. On the other hand, it is easily seen that

$$\overline{\mathcal{R}(\mathcal{U}^n)}\supset\overline{\mathcal{R}(\mathcal{U}^{n_1})}\;\;\forall n_1\ge n.$$

This implies that $\mathcal{U}$ is an (m,q)-isometry on $\overline{\mathcal{R}(\mathcal{U}^{n_1})}$. Therefore, $\mathcal{U}$ is an n_1-quasi (m,q)-isometry for all $n_1\ge n$. □

Example 2.3. *Consider the metric space $(\mathcal{E},d)$, where $\mathcal{E}=\mathbb{R}^2$ and*

$$d\big((\omega,\psi),(u,v)\big)=|\omega-u|+|\psi-v|.$$

Define $\mathcal{U}:\mathbb{R}^2\to\mathbb{R}^2$ as follows $\mathcal{U}(\omega,\psi)=(\frac{\omega+\psi-1}{2},\frac{\omega+\psi+1}{2})$.
We see that

$$\begin{aligned} &\sum_{0\le k\le 2}(-1)^{2-k}\binom{2}{k} d\big(\mathcal{U}^{2+k}\omega,\mathcal{U}^{2+k}\psi\big)^q \\ =\;& d\big(\mathcal{U}^{2}\omega,\mathcal{U}^{2}\psi\big)^q - 2d\big(\mathcal{U}^{3}\omega,\mathcal{U}^{3}\psi\big)^q + d\big(\mathcal{U}^{4}\omega,\mathcal{U}^{4}\psi\big)^q \\ =0\;&\;. \end{aligned}$$

A simple computation shows that

$$\begin{aligned}
& \sum_{0\leq k\leq 2}(-1)^{2-k}\binom{2}{k}d\big(\mathcal{U}^k\omega,\mathcal{U}^k\psi\big)^q \\
= \;& d\big(\omega,\psi\big)^q-2d\big(\mathcal{U}\omega,\mathcal{U}\psi\big)^q+d\big(\mathcal{U}^2\omega,\mathcal{U}^2\psi\big)^q \\
\neq \;& 0.
\end{aligned}$$

It follows from that $\mathcal{U}$ is a 2-quasi $(2,q)$-isometry but is not a $(2,q)$-isometry.

Theorem 2.4. *Let $\mathcal{U}:\mathcal{E}\to\mathcal{E}$ be an n-quasi (m,q)-isometry.If $\overline{\mathcal{R}(\mathcal{U}^p)}=\overline{\mathcal{R}(\mathcal{U}^{p+1})}$, then $\mathcal{U}$ is an p-quasi (m,q)-isometry for all $1\leq p\leq n-1$.*

Proof. Under the assumption that $\overline{\mathcal{R}(\mathcal{U}^p)}=\overline{\mathcal{R}(\mathcal{U}^{p+1})}$, we have $\overline{\mathcal{R}(\mathcal{U}^p)}=\overline{\mathcal{R}(\mathcal{U}^n)}$, Indeed, we know that $\overline{\mathcal{R}(\mathcal{U}^{p+1})}\supset\overline{\mathcal{R}(\mathcal{U}^{p+2})}$. Let $z\in\mathcal{R}(\mathcal{U}^{p+1})$, then there exist $\omega\in\mathcal{E}$ such that

$$z=\mathcal{U}^{p+1}\omega=\mathcal{U}(\mathcal{U}^p\omega)$$

this implies that there exist $v\in\mathcal{R}(\mathcal{U}^p)=\mathcal{R}(\mathcal{U}^{p+1})$ such that $z=\mathcal{U}v$. So, we get the existence of $\psi\in\mathcal{E}$ such that

$$z=\mathcal{U}(\mathcal{U}^{p+1}\psi)=\mathcal{U}^{p+2}(\psi).$$

By applying the same procedure $(n-p)$ times, we obtain that $\mathcal{R}(\mathcal{U}^p)=\mathcal{R}(\mathcal{U}^n)$, It results that $\overline{\mathcal{R}(\mathcal{U}^p)}=\overline{\mathcal{R}(\mathcal{U}^n)}$. Else, we have $\mathcal{U}$ is an n-quasi (m,q)-isometry on $\mathcal{E}$ then $\mathcal{U}$ is an n-quasi (m,q)-isometry on $\overline{\mathcal{R}(\mathcal{U}^n)}=\overline{\mathcal{R}(\mathcal{U}^p)}$. Hence, $\mathcal{U}$ is a n-quasi (m,q)-isometry on $\mathcal{E}$ for some $1\leq p\leq n-1$. □

Theorem 2.5. *Let $\mathcal{U}:\mathcal{E}\to\mathcal{E}$ be an n-quasi (m,q)-isometry, then $\mathcal{U}$ is an n-quasi (l,q)-isometry for all $l\geq m$.*

Proof. It is enough to prove the result for $l=m+1$. So for all $\omega\in\mathcal{E}$, we have

$$\begin{aligned}
& \sum_{0\leq k\leq l}(-1)^{l-k}\binom{l}{k}d\big(\mathcal{U}^{n+k}\omega,\mathcal{U}^{n+k}\psi\big)^q \\
= \;& \sum_{0\leq k\leq m+1}(-1)^{m+1-k}\binom{m+1}{k}d\big(\mathcal{U}^{n+k}\omega,\mathcal{U}^{n+k}\psi\big)^q \\
= \;& \sum_{1\leq k\leq m}(-1)^{m-k}\binom{m+1}{k}d\big(\mathcal{U}^{n+k}\omega,\mathcal{U}^{n+k}\psi\big)^q \\
+ \;& (-1)^{m+1}d\big(\mathcal{U}^n\omega,\mathcal{U}^n\psi\big)^q+d\big(\mathcal{U}^{n+m+1}\omega,\mathcal{U}^{n+m+1}\psi\big)^q
\end{aligned}$$

$$\begin{aligned}
&= \sum_{1\le k\le m}(-1)^{m-k}\Big(\binom{m}{k}+\binom{m}{k-1}\Big)d\big(\mathcal{U}^{n+k}\omega,\mathcal{U}^{n+k}\psi\big)^q \\
&+ (-1)^{m+1}d\big(\mathcal{U}^{n}\omega,\mathcal{U}^{n}\psi\big)^q + d\big(\mathcal{U}^{n+m+1}\omega,\mathcal{U}^{n+m+1}\psi\big)^q \\
&= \sum_{1\le k\le m-1}(-1)^{m-(k+1)}\binom{m}{k}d\big(\mathcal{U}^{n+k}\omega,\mathcal{U}^{n+k}\psi\big)^q \\
&+ \sum_{1\le k\le m}(-1)^{m-(k+1)}\binom{m}{k-1}d\big(\mathcal{U}^{n+k}\omega,\mathcal{U}^{n+k}\psi\big)^q \\
&+ d\big(\mathcal{U}^{n+m+1}\omega,\mathcal{U}^{n+m+1}\psi\big)^q + (-1)^{m+1}d\big(\mathcal{U}^{n}\omega,\mathcal{U}^{n}\psi\big)^q \\
&= \sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}d\big(\mathcal{U}^{n+k}\omega,\mathcal{U}^{n+k}\psi\big)^q \\
&+ \sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}d\big(\mathcal{U}^{n+1+k}\omega,\mathcal{U}^{n+1+k}\psi\big)^q.
\end{aligned}$$

Since $\mathcal{U}$ is an n-quasi (m,q)-isometry, then

$$\sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}d\big(\mathcal{U}^{n+k}\omega,\mathcal{U}^{n+k}\psi\big)^q = 0$$

and by using Theorem 2.4 and put $n-1=p$ we get $\mathcal{U}$ is an p-quasi (m,q)-isometry, then

$$\sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}d\big(\mathcal{U}^{p+k}\omega,\mathcal{U}^{p+k}\psi\big)^q = 0.$$

So $\mathcal{U}$ is an n-quasi (l,q)-isometry for all $l\ge m$. □

Example 2.6. *For $q\ge 1$, Consider $l^q_{\mathbb{N}}(\mathbb{R}) = \{(\omega_n)_{k\ge 0}\ /\ \sum_{k\ge 1}|\omega_k|^q<\infty\}$ and define $d_q : l^q_{\mathbb{N}}(\mathbb{R})\times l^q_{\mathbb{N}}(\mathbb{R})\longrightarrow\mathbb{R}$ by*

$$d_q(\omega,\psi) = d_q((\omega_k)_k,(\psi_k)_k) = \Big(\sum_{k\ge 1}|\omega_k-\psi_k|^q\Big)^{\frac{1}{q}}.$$

Let $\mathcal{U} : \big(l^q_{\mathbb{N}}(\mathbb{R}),d_q\big)\to\big(l^q_{\mathbb{N}}(\mathbb{R}),d_q\big)$ given by $\mathcal{U}((\omega_k)_k) = \big(0,\omega_1,\omega_2,\ \cdots,\omega_k,\cdots\big)$. Direct computation shows that

$$\begin{aligned}
\sum_{0\le l\le 2}(-1)^{m-l}\binom{m}{l}d_q\big(\mathcal{U}^{n+l}\omega,\mathcal{U}^{m+l}\psi\big)^q &= \sum_{0\le l\le 2}(-1)^{m-l}\binom{m}{l}\Big(\sum_{k\ge 1}|\omega_k-\psi_k|^q\Big) \\
&= 0.
\end{aligned}$$

So, $\mathcal{U}$ is an n-quasi-(m,q)-isometry for all positive integers m and n.

Example 2.7. *Consider $\mathbb{R}^3$ the metric space with its Euclidean metric and define the map $\mathcal{U}_0 : \mathbb{R}^3\longrightarrow\mathbb{R}^3$ given by $\mathcal{U}(\omega,\psi,\varphi) = (\varphi,0,0)$. A Direct calculation shows that $\mathcal{U}_0$ is 2-quasi-$(2,q)$-isometry but $\mathcal{U}_0$ is not a quasi-$(2,q)$-isometry.*

In the following result, we obtain some properties of the approximate spectral of an n-quasi (m,q)-isometry.

Proposition 2.8. *Let $\mathcal{E}$ be normed space and $\mathcal{U} : \mathcal{E} \to \mathcal{E}$ be an n-quasi (m, q)-isometry. Then a nonzero approximate eigenvalue of $\mathcal{U}$ lies in the unit circle.*

Proof. Let $\lambda \neq 0$ be an approximate eigenvalue of $\mathcal{U}$. Then, there exist $(x_j) \subset \mathcal{E}$ with $\|x_j\| = 1$ and $(\mathcal{U} - \lambda)x_j \longrightarrow 0$, so for all integer $k \geq 1$, we get $(\mathcal{U}^{n+k} - \lambda^{n+k})x_j \longrightarrow 0$. Since $\mathcal{U}$ is an n-quasi (m, q)-isometry, we have

$$\begin{aligned} 0 &= \sum_{0\leq k\leq m} (-1)^{m-k} \binom{m}{k} \|\mathcal{U}^{n+k} x_j\|^q \\ &= \sum_{0\leq k\leq m} (-1)^{m-k} \binom{m}{k} |\lambda|^{q(n+k)} \\ &= |\lambda|^{qn}(|\lambda|^q - 1)^m. \end{aligned}$$

Since $\lambda \neq 0$, we get $|\lambda| = 1$ Hence the desired claim follows from that ☐

3. POWER AND PRODUCT OF n-QUASI-(m, q)-ISOMETRIES ON METRIC SPACES

In this section we introduce and study the stability of n-quasi (m, q)- isometry under products and ,particularity, under power.

Theorem 3.1. *Let $\mathcal{U} : \mathcal{E} \longrightarrow \mathcal{E}$ be an n-quasi (m, q)- isometry, then $\mathcal{U}^r$ is an n-quasi (m, q)- isometry for all positive integer r.*

Proof. Let $\mathcal{U}$ be an n-quasi (m, q)- isometry, then $\mathcal{U}$ is a (m, q)- isometry on $\overline{\mathcal{R}(\mathcal{U}^n)}$, then we obtain $\mathcal{U}^r$ is an (m, q)- isometry for all positive integer r on $\overline{\mathcal{R}(\mathcal{U}^n)}$. Since $nr > n$ for all $r \geq 1$, we get on $\overline{\mathcal{R}(\mathcal{U}^{nr})} \subset \overline{\mathcal{R}(\mathcal{U}^n)}$. This implies that$\mathcal{U}^r$ is an n-quasi (m, q)- isometry on $\overline{\mathcal{R}(\mathcal{U}^n)}$. Hence, $\mathcal{U}^r$ is an n-quasi (m, q)- isometry for all positive integer r. ☐

Proposition 3.2. *Let $\mathcal{U} : \mathcal{E} \longrightarrow \mathcal{E}$ be a map and n_1, n_2, r, s, m, l be positive integers. If $\mathcal{U}^r$ is an n_1-quasi (m, q)-isometry and $\mathcal{U}^s$ is an n_2-quasi (l, q)-isometry, then $\mathcal{U}^t$ is an n_0-quasi (p, q)-isometry, where t is the greatest common divisor of r and s, $n_0 = max(\frac{n_1 r}{t}, \frac{n_2 s}{t})$ and $p = min(m, l)$.*

Proof. Since $\mathcal{U}^r$ is an n_1-quasi (m, q)-isometry and $\mathcal{U}^s$ is an n_2-quasi (l, q)- isometry, we deduce that $\mathcal{U}^r$ is an (m, q)- isometry on $\overline{\mathcal{R}(\mathcal{U}^{n_1 r})}$ and $\mathcal{U}^s$ is an (l, q)-isometry on $\overline{\mathcal{R}(\mathcal{U}^{n_2 r})}$. On the other hand, if we define t as the greatest common divisor of r and s, then

$$\overline{\mathcal{R}(\mathcal{U}^{n_1 r})} = \overline{\mathcal{R}((\mathcal{U}^t)^{n_1 \frac{r}{t}})} \; \textit{and} \; \overline{\mathcal{R}(\mathcal{U}^{n_2 s})} = \overline{\mathcal{R}((\mathcal{U}^t)^{n_2 \frac{s}{t}})}.$$

. Let $n_0 = max(\frac{n_1 r}{t}, \frac{n_2 s}{t})$, then $\overline{\mathcal{R}(\mathcal{U}^{n_0 t})} \subset \overline{\mathcal{R}(\mathcal{U}^{n_1 r})}$ and $\overline{\mathcal{R}(\mathcal{U}^{n_0 t})} \subset \overline{\mathcal{R}(\mathcal{U}^{n_2 s})}$. It follows from that $\mathcal{U}^r$ is an (m, q)- isometry and $\mathcal{U}^s$ is an (l, q)-isometry on $\overline{\mathcal{R}(\mathcal{U}^{n_0 t})}$ we can easily show that $\mathcal{U}^t$ is an (p, q)- isometry on on $\overline{\mathcal{R}(\mathcal{U}^{n_0 t})}$ where $p = min(m, l)$ and t as the greatest common divisor of r and s. According to Proposition 2.1, we get that $\mathcal{U}^t$ is an n_0-quasi (p, q)-isometry on $\mathcal{E}$. ☐

As an immediate consequence of Proposition 3.2, we have the following result.

Corollary 3.3. *Let $\mathcal{U} : \mathcal{E} \longrightarrow \mathcal{E}$ be a map and n_1, n_2, r, s, m, l be positive integers. Then, the following properties hold*

(1) *If $\mathcal{U}$ is an n-quasi (m,q)-isometry such that $\mathcal{U}^s$ is an n-quasi (l,q)-isometry then $\mathcal{U}$ is an n-quasi (l,q)-isometry,*
(2) *If $\mathcal{U}$ and $\mathcal{U}^{r+1}$ are n-quasi (m,q)-isometries, then $\mathcal{U}$ is a $n(r+1)$-quasi (m,q)-isometry,*
(3) *If $\mathcal{U}^r$ is an n-quasi (m,q)-isometry and $\mathcal{U}^{r+1}$ is an n-quasi (l,q)-isometry with $m<l$, then $\mathcal{U}$ is a $n(r+1)$-quasi (m,q)-isometry.*

T. Bermúdez, A. Martinón and J. A. Noda [11] have proved that if $\mathcal{U}$ is an (m,q)-isometry and $\mathcal{V}$ is an (l,q)-isometry with $\mathcal{UV}=\mathcal{VU}$, then $\mathcal{UV}$ is an $(m+l-1,q)$-isometry. In following theorem we will generalize this result for the class of an n-quasi (l,q)-isometry operators.

Theorem 3.4. *Let $\mathcal{U},\mathcal{V}:\mathcal{E}\longrightarrow\mathcal{E}$ such that $\mathcal{UV}=\mathcal{VU}$. If $\mathcal{U}$ is an n_1-quasi (m,q)-isometry and $\mathcal{V}$ is an n_2-quasi (l,q)-isometry, then $\mathcal{UV}$ is an n-quasi $(m+l-1,q)$-isometry, where $n=max\{n_1,n_2\}$.*

Proof. For all $\omega,\psi\in\mathcal{E}$ we have

$$\begin{aligned}
&\sum_{0\le k\le m+l-1}(-1)^{m+l-1-k}\binom{m+l-1}{k}d\big((\mathcal{UV})^{n+k}\omega,(\mathcal{UV})^{n+k}\psi\big)^q\\
=&\sum_{0\le k\le m+l-1}(-1)^{m+l-1-k}\binom{m+l-1}{k}d\big((\mathcal{UV})^{k}(\mathcal{UV})^{n}\omega,(\mathcal{UV})^{k}(\mathcal{UV})^{n}\psi\big)^q\\
=&\sum_{0\le k\le m+l-1}(-1)^{m+l-1-k}\binom{m+l-1}{k}d\big((\mathcal{UV})^{k}\omega,(\mathcal{UV})^{k}\psi\big)^q.\,\forall\omega,\psi\in\overline{\mathcal{R}((\mathcal{UV})^n)}\\
=&\ 0.
\end{aligned}$$

Equivalently, $\mathcal{UV}$ is an (m,q)-isometry by referring to [9, Theorem 2.16], Proposition 2.1 and Proposition 2.2 we get that $\mathcal{UV}$ is an n-quasi $(m+l-1,q)$-isometry, where $n=max\{n_1,n_2\}$. □

Corollary 3.5. *Let $\mathcal{U},\mathcal{V}:\mathcal{E}\longrightarrow\mathcal{E}$ be commuting operators. If $\mathcal{U}$ is an n_1-quasi (m,q)-isometry and $\mathcal{V}$ is an n_2-quasi (l,q)-isometry, then $\mathcal{U}^t\mathcal{V}^s$ is an n-quasi $(m+l-1,q)$-isometry, where $n=max\{n_1,n_2\}$ and for all positive integers t,s.*

Proof. Since $\mathcal{U}$ is an n_1-quasi (m,q)-isometry and $\mathcal{V}$ is an n_2-quasi (l,q)-isometry, it follows from Corollary 3.3 that $\mathcal{U}^t$ is an n_1-quasi (m,q)-isometry and $\mathcal{V}^s$ is an n_2-quasi (l,q)-isometry for all positive integers t,s. Moreover, since $\mathcal{UV}=\mathcal{VU}$ we deduce that $\mathcal{U}^t\mathcal{V}^s=\mathcal{V}^s\mathcal{U}^t$. Referring to Theorem 3.4, it holds that $\mathcal{U}^t\mathcal{V}^s$ is an n-quasi $(m+l-1,q)$-isometry, where $n=max\{n_1,n_2\}$. □

4. DISTANCES ASSOCIATED TO n-QUASI-(m,q)-ISOMETRIES ON METRIC SPACES

In this section we introduce some distances related to n-quasi (m,q)-isometry. Let $\mathcal{U}:\mathcal{E}\longrightarrow\mathcal{E}$ be an n-quasi (m,q)-isometry, we set

$$\rho_{\mathcal{U}}(\omega,\psi):=(m-1)!^{\frac{1}{q}}\lim_{r\to\infty}\frac{d(\mathcal{U}^{n+r}\omega,\mathcal{U}^{n+r}\psi)}{r^{\frac{m-1}{q}}}\ \ for\omega,\psi\in\mathcal{E}.$$

Proposition 4.1. *Let $\mathcal{U}$ be an n-quasi (m,q)-isometry, then*

$$\rho_{\mathcal{U}}(\omega,\psi)=(m-1)!^{\frac{1}{q}}\lim_{r\to\infty}\frac{d(\mathcal{U}^{n+r}\omega,\mathcal{U}^{n+r}\psi)}{r^{\frac{m-1}{q}}}\quad for\ \ \omega,\psi\in\mathcal{E}$$

is a semi-distance on $\mathcal{E}$.

Proof. Set $\mathcal{Q}_{m,q,n}(\omega,\psi):=\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}d(\mathcal{U}^{n+k}\omega,\mathcal{U}^{n+k}\psi)^q$. Under the condition that $\mathcal{U}$ is an n-quasi (m,q)-isometry it follows that

$$\begin{aligned}\mathcal{Q}_{m-1,q,n}(\omega,\psi) &= \lim_{r\longrightarrow\infty}\frac{1}{\binom{r}{m-1}}d(\mathcal{U}^{n+r}\omega,\mathcal{U}^{n+r}\psi)^q\\ &= (m-1)!\lim_{r\longrightarrow\infty}\frac{1}{r^{m-1}}d(\mathcal{U}^{n+r}\omega,\mathcal{U}^{n+r}\psi)^q.\end{aligned}$$

From which , we deduce that $\rho_{\mathcal{U}}(\omega,\psi)=\big(\mathcal{Q}_{m-1,q,n}(\omega,\psi)\big)^{\frac{1}{q}}$.

To show that $\rho_{\mathcal{U}}(\omega,\psi)$ is a semi-metric, firstly, we observe that $\rho_{\mathcal{U}}(\omega,\psi)\geq 0$, clearly $\rho_{\mathcal{U}}(\omega,\omega)=0$ and $\rho_{\mathcal{U}}(\omega,\psi)=\rho_{\mathcal{U}}(\psi,\omega)$ for all $\omega,\psi\in\mathcal{E}$. Next to prove the triangle inequality, we have for $\omega,\psi,\psi\in\mathcal{E}$ we have

$$\begin{aligned}\rho_{\mathcal{U}}(\omega,\psi) &= (m-1)!^{\frac{1}{q}}\lim_{r\to\infty}\frac{d(\mathcal{U}^{n+r}\omega,\mathcal{U}^{n+r}\psi)}{r^{\frac{m-1}{q}}}\\ &\leq (m-1)!^{\frac{1}{q}}\lim_{r\to\infty}\frac{d(\mathcal{U}^{n+r}\omega,\mathcal{U}^{n+r}\varphi)}{r^{\frac{m-1}{q}}}\\ &\quad +(m-1)!^{\frac{1}{q}}\lim_{r\to\infty}\frac{d(\mathcal{U}^{n+r}\varphi,\mathcal{U}^{n+r}\psi)}{r^{\frac{m-1}{q}}}\\ &= \rho_{\mathcal{U}}(\omega,\psi)+\rho_{\mathcal{U}}(\psi,\psi).\end{aligned}$$

□

Remark. *If $\mathcal{U}$ be an n-quasi (m,q)-isometry, it follows that*

$$\mathcal{Q}_{m-1,q,n}(\omega,\psi)=\mathcal{Q}_{m-1,q,n}(\mathcal{U}\omega,\mathcal{U}\psi).$$

Hence, $\rho_{\mathcal{U}}(\omega,\psi)=\rho_{\mathcal{U}}(\mathcal{U}\omega,\mathcal{U}\psi)$, and therefore, $\mathcal{U}:(\mathcal{E},\rho_{\mathcal{U}})\to(\mathcal{E},\rho_{\mathcal{U}})$ is an isometry.

References

[1] J. Agler, *A disconjugacy theorem for Toeplitz operators.* Am J. Math. 1990; 112(1) : 1-14.

[2] J. Agler and M. Stankus, *m- isometric transformations on Hilbert space, I.* Integral Equ. Oper. Theory. 1995; 21(4): 383-429.

[3] J. Agler and M. Stankus, *m- isometric transformations on Hilbert space, II.* Integral Equ. Oper. Theory. 1995; 23(1): 1-48.

[4] J. Agler and M. Stankus, *m- isometric transformations on Hilbert space, III.* Integral Equ. Oper. Theory. 1996; 24(4): 379-421.

[5] F. Bayart, *m- isometries on Banach space.* Math. Nachr. 2011; 284: 2141-2147.

[6] O. Beinane Sid Ahmed, J. H. Sidi, and S. Mahmoud, *$\psi(m,q)$-isometric mappings on metric spaces.* Aust. J. Math. Anal. Appl. (AJMAA) 2020;17: 18 pp.

[7] T. Bermúdez, A. Mertinon and E. Negrin, *Weighted shift operators which are m- isometries* . Intergral Equ. Oper. Theory; 2010 ,86: 301- 312.

[8] T. Bermúdez, A. Bonilla and H. Zaway, *C_0- semigroups of m-isometries on Hilbert spaces.* J. Math. Anal. Appl.; 2019 ,472(1): 879- 893.

[9] T. Bermúdez, A. Martinón and V. Muller, *(m,q)-isometries on metric spaces.* J. Operator Theory; 2014 ,72(2): 313- 329.

[10] T. Bermúdez, C. D. Mendoza and A. Mertinón, *Powers of m-isometries.* Studia Mathematica; 2012 ,208(3).
[11] T. Bermúdez, A. Martinón and J. A. Noda, *Products of m-isometries.* Linear Algebra and its Applications; 2013 ,86: 301- 312.
[12] T. Bermúdez, A. Mertinón, V. Muller and J. A. Noda, *Perturbation of m-isometries by Nilpotent Operators.*J. Oerator Theory 2014 ,6 pages.
[13] F. Botelho and J. Jamison, *Isometric properties of elementary operators*, Linear Algebra Appl. 432 (2010), 357–365.
[14] B. P. Duggal, *Tensor product of n-isometries III.* Funct. Anal. Approx. Comput. 2012; 4, 61-67
[15] K. Gherairi, Z. Hajjej, H. Li and H. Regeiba, *n-quasi-A-(m, q)-isometry on a Banach space.*Aims Mathematics 2023;8(12): 28308- 28321.
[16] M. Guesba, E. M. O. Beiba and O. A. Mahmoud Sid Ahmed, *n-quasi- (A, m)-isometric operators on a Hilbert space.* Bull. Math. Anal. Appl. 2020; 12, 8-26.
[17] P. H. W. Hoffmann and M. Mackey, *(m, p)-and (m, ∞)-isometric operator tuples on normed spaces*, Asian-Eur. J. Math., Vol. **8**(2)(2015).
[18] O. A. Mahmoud Sid Ahmed, *m-isometric operators on Bannach spaces.*Asian-Eur. J. Math. 2010,3 (1) 119.
[19] O. A. Mahmoud Sid Ahmed, A. Saddi and K. Gherairi, *Some results on Higher orders quasi-isomtries Hacet.* J.Math.Stat. 2020; 49(4).
[20] O. A. Mahmoud Sid Ahmed and A. Saddi, *$A - m$-isometric operators in semi- Hilbertian spaces.* Linear Algebra Appl. 2012 ,436: 3930- 3942.
[21] S. Mecheri and T. Prasad, *On n-quasi -m-isometric operators* .Asian-Eur. J. Math. 2016,9 (3).
[22] J. Shen and F. Zuo, *Spectral properties of k-quasi-2-isometric operators.*J. Korean. Soc. Math. Edu. Ser. B: Pure Appl. Math. 2015,22 (3), 275-283.

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