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GENERALIZED HYERS-ULAM STABILITY OF A 2-VARIABLE RECIPROCAL FUNCTIONAL EQUATION

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Abstract. In this paper, we obtain the general solution and generalized Hyers-Ulam stability of 2-variable reciprocal functional equation

$$F(x+u,y+v) = \frac{F(x,y)F(u,v)}{F(x,y)+F(u,v)}.$$

1. Introduction

In 1940, S.M. Ulam [8], while he was giving a talk before the Mathematics Club of Wisconsin, he listed a number of important unsolved problems. One of the problem is the stability of functional equation. Over the last four or five decades, the above problem was tackled by numerous authors and its solutions via various forms of functional equation like additive, quadratic, cubic, quartic functional equations were discussed. To know more about the various forms of functional equations and its solutions, one can refer to the interesting monographs ([1] ,[2]).

Recently, K. Ravi and B.V. Senthil Kumar [4] investigated the generalized Hyers-Ulam stability for a new 2-dimensional reciprocal functional mapping $f : X \to Y$ satisfying the Rassias reciprocal functional equation

$$f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)}. \tag{1.1}$$

where X and Y are the spaces of non-zero real numbers. The reciprocal function $f(x) = \frac{c}{x}$ is the solution of the functional equation (1.1).

Definition 1.1. A mapping $f : X \to Y$ is called reciprocal if f satisfies the functional equation (1.1).

Later, K. Ravi, J.M. Rassias and B.V. Senthil Kumar [5] introduced the Reciprocal Difference Functional equation (RDF equation)

$$f\left(\frac{x+y}{2}\right) - f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)} \tag{1.2}$$

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and the Reciprocal Adjoint Functional equation (RAF equation)

$$f\left(\frac{x+y}{2}\right)+f(x+y)=\frac{3f(x)f(y)}{f(x)+f(y)} \tag{1.3}$$

and investigated the Hyers-Ulam stability, Generalized Ulam stability and the extended Ulam stability for the above two functional equations (1.2) and (1.3).

S.M. Jung [3] applied fixed point method and investigated its Hyers-Ulam stability for the reciprocal functional equation (1.1).

Very recently, K. Ravi, J.M. Rassias and B.V. Senthil Kumar [6] discussed the Hyers-Ulam stability, Generalized Hyers-Ulam stability, the Extended Ulam stability and Refined Ulam stability for the generalized reciprocal functional equation (or GRF equation)

$$f\left(\sum_{i=1}^{m}\alpha_i x_i\right)=\frac{\prod_{i=1}^{m} f(x_i)}{\sum_{i=1}^{m}\left[\alpha_i\left(\prod_{j=1,j\neq i}^{m} f(x_i)\right)\right]} \tag{1.4}$$

for arbitrary but fixed real numbers $(\alpha_1,\alpha_2,\ldots,\alpha_m)\neq(0,0,\ldots,0)$, so that $0<\alpha=\alpha_1+\alpha_2+\cdots+\alpha_m=\sum_{i=1}^{m}\alpha_i\neq 1$ and $f:X\to Y$ with X and Y are the sets of non-zero real numbers.

The stability of new reciprocal functional equations

$$f\left[(k_1-k_2)x+(k_1-k_2)y\right]=\frac{f(k_1x-k_2y)f(k_1y-k_2x)}{f(k_1x-k_2y)+f(k_1y-k_2x)} \tag{1.5}$$

where k_1 and k_2 are any integers with $k_1\neq k_2$ and

$$f\left[(k_1+k_2)x+(k_1+k_2)y\right]=\frac{f(k_1x+k_2y)f(k_1y+k_2x)}{f(k_1x+k_2y)+f(k_1y+k_2x)} \tag{1.6}$$

where k_1 and k_2 are any integers with $k_1\neq -k_2$ was discussed by K. Ravi, J.M. Rassias and B.V. Senthil Kumar in [7].

In this paper, the authors discuss the general solution and generalized Hyers-Ulam stability for a new 2-variable reciprocal functional equation of the form

$$F(x+u,y+v)=\frac{F(x,y)F(u,v)}{F(x,y)+F(u,v)}. \tag{1.7}$$

Definition 1.2. A mapping $F: X\times X\to Y$ is called a 2-variable reciprocal mapping if there exist $a(\neq 0), b(\neq 0)\in\mathbb{R}$ such that

$$F(x,y)=\frac{ab}{bx+ay}.$$

Throughout this paper, we assume that X and Y are the sets of non-zero real numbers. In Section 2 of this paper, we show the relationship between the reciprocal functional equation (1.1) and (1.7). In Section 3, we establish that the function $F(x,y)=\frac{ab}{bx+ay}$ is the general solution of the 2-variable reciprocal functional equation (1.7). In Section 4, we obtain the generalized Hyers-Ulam stability for the functional equation (1.7). In Section 5, we compare the stability results obtained for the 2-variable reciprocal functional equation (1.7) with the stability results obtained for the 1-variable reciprocal functional equation (1.1) in [4].

2. RELATION BETWEEN (1.1) AND (1.7)

The 2-variable reciprocal functional equation (1.7) induces the reciprocal functional equation (1.1).

Theorem 2.1. *Let $F : X \times X \to Y$ be a mapping satisfying (1.7) and let $r : x \to Y$ be a mapping given by*

$$r(x) = F(x, x) \tag{2.1}$$

for all $x \in X$, then r satisfies (1.1).

Proof. From (1.7) and (2.1), we get

$$\begin{aligned} r(x+y) &= F(x+y, x+y) \\ &= \frac{F(x,x)F(y,y)}{F(x,x)+F(y,y)} \\ &= \frac{r(x)r(y)}{r(x)+r(y)} \end{aligned}$$

for all $x, y \in X$. □

Theorem 2.2. *Let $a(\neq 0), b(\neq 0) \in \mathbb{R}$ and $r : X \to Y$ be a mapping satisfying (1.1). If $F : X \times X \to Y$ is a mapping given by*

$$F(x,y) = \frac{abr(x)r(y)}{ar(x)+br(y)} \tag{2.2}$$

for all $x, y \in X$, then F satisfies (1.7).

Proof. From (1.1) and (2.2), we have

$$\begin{aligned} F(x+u, y+v) &= \frac{abr(x+u)r(y+v)}{ar(x+u)+br(y+v)} \\ &= \frac{ab\frac{r(x)r(u)}{r(x)+r(u)}\frac{r(y)r(v)}{r(y)+r(v)}}{a\frac{r(x)r(u)}{r(x)+r(u)}+b\frac{r(y)r(v)}{r(y)+r(v)}} \\ &= \frac{abr(x)r(u)r(y)r(v)}{ar(x)r(u)[r(y)+r(v)]+br(y)r(v)[r(x)+r(u)]} \\ &= \frac{abr(x)r(u)r(y)r(v)}{r(x)r(y)[ar(u)+br(v)]+r(u)r(v)[ar(x)+br(y)]} \\ &= \frac{\frac{abr(x)r(y)}{ar(x)+br(y)}\frac{abr(u)r(v)}{ar(u)+br(v)}}{\frac{abr(x)r(y)}{ar(x)+br(y)}+\frac{abr(u)r(v)}{ar(u)+br(v)}} \\ &= \frac{F(x,y)F(u,v)}{F(x,y)+F(u,v)} \end{aligned}$$

for all $x, u, y, v \in X$. □

3. GENERAL SOLUTION OF EQUATION (1.7)

Theorem 3.1. *A mapping $F : X \times X \to Y$ satisfies (1.7) if and only if there exist two reciprocal mappings $r_1, r_2 : X \to Y$ such that*

$$F(x,y) = \frac{r_1(x)r_2(y)}{r_1(x)+r_2(y)}$$

for all $x, y \in X$.

Proof. Assume that F is a solution of (1.7). Define $F_1(x) = F(x,0)$, $F_2(y) = F(0,y)$, for all $x, y \in X$. It is easy to verify that F_1, F_2 are reciprocal functions. Let $F_1(x) = r_1(x)$ and $F_2(x) = r_2(x)$, for all $x \in X$. Hence

$$\begin{aligned}\frac{r_1(x)r_2(y)}{r_1(x)+r_2(y)} &= \frac{\frac{a}{x}\frac{b}{y}}{\frac{a}{x}+\frac{b}{y}}\\ &= \frac{ab}{bx+ay}\\ &= F(x,y), \text{ for all } x, y \in X.\end{aligned}$$

Conversely, assume that there exist two reciprocal mappings $r_1, r_2 : X \to Y$ such that $F(x,y) = \frac{r_1(x)r_2(y)}{r_1(x)+r_2(y)}$, for all $x, y \in X$. Hence,

$$\begin{aligned}F(x+u, y+v) &= \frac{r_1(x+u)r_2(y+v)}{r_1(x+u)+r_2(y+v)}\\ &= \frac{\frac{r_1(x)r_1(u)}{r_1(x)+r_1(u)}\frac{r_2(y)r_2(v)}{r_2(y)+r_2(v)}}{\frac{r_1(x)r_1(u)}{r_1(x)+r_1(u)}+\frac{r_2(y)r_2(v)}{r_2(y)+r_2(v)}}\\ &= \frac{r_1(x)r_1(u)r_2(y)r_2(v)}{r_1(x)r_1(u)[r_2(y)+r_2(v)]+r_2(y)r_2(v)[r_1(x)+r_1(u)]}\\ &= \frac{r_1(x)r_1(u)r_2(y)r_2(v)}{r_1(x)r_2(y)[r_1(u)+r_2(v)]+r_1(u)r_2(v)[r_1(x)+r_2(y)]}\\ &= \frac{\frac{r_1(x)r_2(y)}{r_1(x)+r_2(y)}\frac{r_1(u)r_2(v)}{r_1(u)+r_2(v)}}{\frac{r_1(x)r_2(y)}{r_1(x)+r_2(y)}+\frac{r_1(u)r_2(v)}{r_1(u)+r_2(v)}}\\ &= \frac{F(x,y)F(u,v)}{F(x,y)+F(u,v)}\end{aligned}$$

for all $x, u, y, v \in X$. □

4. GENERALIZED HYERS-ULAM STABILITY OF EQUATION (1.7)

Theorem 4.1. *Let $F : X^2 \to Y$ be a mapping for which there exists a function $\phi : X^4 \to Y$ with the condition*

$$\lim_{n\to\infty} 2^{-n}\phi(2^{-n}x, 2^{-n}x, 2^{-n}y, 2^{-n}y) = 0 \tag{4.1}$$

such that the functional inequality

$$\left|F(x+u, y+v) - \frac{F(x,y)F(u,v)}{F(x,y)+F(u,v)}\right| \le \frac{1}{2}\phi(x,u,y,v) \tag{4.2}$$

holds for all $x, u, y, v \in X$. Then there exists a unique 2-variable reciprocal mapping $R : X^2 \to Y$ satisfying the functional equation (1.7) and

$$|F(x,y) - R(x,y)| \le \sum_{i=0}^{\infty} 2^{-i-1}\phi(2^{-i-1}x, 2^{-i-1}x, 2^{-i-1}y, 2^{-i-1}y) \tag{4.3}$$

for all $x, y \in X$. The mapping $R(x,y)$ is defined by

$$R(x,y) = \lim_{n\to\infty} 2^{-n}F(2^{-n}x, 2^{-n}y)$$

for all $x, y \in X$.

Proof. Replacing (x, u, y, v) by $(\frac{x}{2}, \frac{x}{2}, \frac{y}{2}, \frac{y}{2})$ in (4.2), we obtain

$$\Big|F(x,y) - \frac{1}{2}F\Big(\frac{x}{2}, \frac{y}{2}\Big)\Big| \leq \frac{1}{2}\phi\Big(\frac{x}{2}, \frac{x}{2}, \frac{y}{2}, \frac{y}{2}\Big). \tag{4.4}$$

Now, replacing (x, y) by $(\frac{x}{2}, \frac{y}{2})$ in (4.4), dividing by 2 and adding the resulting inequality with (4.4), we arrive

$$\Big|F(x,y) - \frac{1}{2^2}F\Big(\frac{x}{2^2}, \frac{y}{2^2}\Big)\Big| \leq \sum_{i=0}^{1} \frac{1}{2^{i+1}}\phi\Big(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}, \frac{y}{2^{i+1}}\Big).$$

Proceeding further and using induction on a positive integer n, we get

$$\begin{aligned} |F(x,y) - 2^{-n}F(2^{-n}x, 2^{-n}y)| &\leq \sum_{i=0}^{n-1} 2^{-i-1}\phi(2^{-i-1}x, 2^{-i-1}x, 2^{-i-1}y, 2^{-i-1}y) \\ &\leq \sum_{i=0}^{\infty} 2^{-i-1}\phi(2^{-i-1}x, 2^{-i-1}x, 2^{-i-1}y, 2^{-i-1}y) \end{aligned} \tag{4.5}$$

for all $x, y \in X$. In order to prove the convergence of the sequence $\{2^{-n}F(2^{-n}x, 2^{-n}y)\}$, replacing (x, y) by $(2^{-p}x, 2^{-p}y)$ in (4.5) and multiplying by 2^{-p}, we find that for $n > p > 0$

$$\begin{aligned} &|2^{-n-p}F(2^{-n-p}x, 2^{-n-p}y) - 2^{-p}F(2^{-p}x, 2^{-p}y)| \\ &\qquad = 2^{-p}|2^{-n}F(2^{-n-p}x, 2^{-n-p}y) - F(2^{-p}x, 2^{-p}y)| \\ &\qquad \leq \sum_{i=0}^{\infty} 2^{-p-i}\phi(2^{-p-i}x, 2^{-p-i}x, 2^{-p-i}y, 2^{-p-i}y) \\ &\qquad \to 0 \text{ as } p \to \infty. \end{aligned}$$

This shows that the sequence $\{2^{-n}F(2^{-n}x, 2^{-n}y)\}$ is a Cauchy sequence. Allow $n \to \infty$ in (4.5), we arrive (4.3). To show that R satisfies (1.4), replacing (x, u, y, v) by $(2^{-n}x, 2^{-n}u, 2^{-n}y, 2^{-n}v)$ in (4.2) and multiplying by 2^{-n}, we obtain

$$\begin{aligned} 2^{-n}\Big|F(2^{-n}(x+u), 2^{-n}(y+v)) - \frac{F(2^{-n}x, 2^{-n}y)F(2^{-n}u, 2^{-n}v)}{F(2^{-n}x, 2^{-n}y) + F(2^{-n}u, 2^{-n}v)}\Big| \\ \leq 2^{-n}\phi(2^{-n}x, 2^{-n}u, 2^{-n}y, 2^{-n}v). \end{aligned} \tag{4.6}$$

Allow $n \to \infty$ in (4.6), we see that R satisfies (1.7) for all $(x, u, y, v) \in X^4$. To prove R is unique 2-variable reciprocal function satisfying (1.7), let $S : X^2 \to Y$ be another 2-variable reciprocal function which satisfies (1.7) and the inequality (4.3). Clearly S and R satisfy (1.7) and using (4.3), we arrive

$$\begin{aligned} &|S(x,y) - R(x,y)| \\ &\quad = 2^{-n}|S(2^{-n}x, 2^{-n}y) - R(2^{-n}x, 2^{-n}y)| \\ &\quad \leq 2^{-n}\Big(|S(2^{-n}x, 2^{-n}y) - F(2^{-n}x, 2^{-n}y)| + |F(2^{-n}x, 2^{-n}y) - R(2^{-n}x, 2^{-n}y)|\Big) \\ &\quad \leq 2\sum_{i=0}^{\infty} 2^{-n-i-1}\phi(2^{-n-i-1}x, 2^{-n-i-1}x, 2^{-n-i-1}y, 2^{-n-i-1}y) \end{aligned} \tag{4.7}$$

for all $(x, y) \in X^2$. Allow $n \to \infty$ in (4.7) and using (4.1), we find that R is unique which completes the proof of Theorem 4.1. □

Corollary 4.2. *Let $F : X \times X \to Y$ be a mapping for which there exists a constant c (independent of x,y)≥ 0 such that the functional inequality*

$$\left|F(x+u,y+v) - \frac{F(x,y)F(u,v)}{F(x,y)+F(u,v)}\right| \leq \frac{c}{2} \tag{4.8}$$

holds for all $x,u,y,v \in X$. Then there exists a unique mapping $R : X^2 \to Y$ satisfying the functional equation (1.7) and

$$|F(x,y) - R(x,y)| \leq c \tag{4.9}$$

for all $x,y \in X$.

Proof. Taking $\phi(x,u,y,v) = c$, for all $x,u,y,v \in X$ in Theorem 4.1, the proof of Corollary 4.2 follows immediately by similar arguments. □

Corollary 4.3. *For any fixed $\theta \geq 0$ and $p > -1$, if $F : X^2 \to Y$ satisfies*

$$\left|F(x+u,y+v) - \frac{F(x,y)F(u,v)}{F(x,y)+F(u,v)}\right| \leq \frac{1}{2}\theta\Big(|x|^p + |u|^p + |y|^p + |v|^p\Big) \tag{4.10}$$

for all $x,u,y,v \in X$, then there exists a 2-variable reciprocal function $R : X^2 \to Y$ such that

$$|F(x,y) - R(x,y)| \leq \frac{2\theta}{2^{p+1}-1}\Big(|x|^p + |y|^p\Big) \tag{4.11}$$

for all $x,y \in X$.

Proof. Letting $\phi(x,u,y,v) = \theta\Big(|x|^p + |u|^p + |y|^p + |v|^p\Big)$, for all $x,u,y,v \in X$ in Theorem 4.1, we obtain $\phi(x,x,y,y) = 2\theta\Big(|x|^p + |y|^p\Big)$. From (4.3), we get

$$\begin{aligned}
|F(x,y) - R(x,y)| &\leq \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} 2\theta\Big(\Big|\frac{x}{2^{i+1}}\Big|^p + \Big|\frac{y}{2^{i+1}}\Big|^p\Big) \\
&\leq \frac{\theta}{2^p}\Big(1 - \frac{1}{2^{p+1}}\Big)^{-1}\Big(|x|^p + |y|^p\Big) \\
&\leq \frac{2\theta}{2^{p+1}-1}\Big(|x|^p + |y|^p\Big)
\end{aligned}$$

for all $x,y \in X$. □

Corollary 4.4. *Let $F : X^2 \to Y$ be a mapping and there exist real numbers $a,b : \rho = a+b > -1$. If there exists c_1 such that*

$$\left|F(x+u,y+v) - \frac{F(x,y)F(u,v)}{F(x,y)+F(u,v)}\right| \leq c_1|x|^{\frac{a}{2}}|u|^{\frac{b}{2}}|y|^{\frac{a}{2}}|v|^{\frac{b}{2}} \tag{4.12}$$

for all $x,u,y,v \in X$, then there exists a unique 2-variable reciprocal function $R : X^2 \to Y$ satisfying the functional equation (1.7) and

$$|F(x,y) - R(x,y)| \leq \frac{2c_1}{2^{\rho+1}-1}\Big(|x|^{\frac{\rho}{2}}|y|^{\frac{\rho}{2}}\Big) \tag{4.13}$$

for all $x,y \in X$.

Proof. Considering $\phi(x,u,y,v) = 2c_1|x|^{\frac{a}{2}}|u|^{\frac{b}{2}}|y|^{\frac{a}{2}}|v|^{\frac{b}{2}}$, for all $x,u,y,v \in X$ in Theorem 4.1, we have $\phi(x,x,y,y) = 2c_1|x|^{\frac{\rho}{2}}|y|^{\frac{\rho}{2}}$. From (4.3), we obtain

$$\begin{aligned}|F(x,y)-R(x,y)| &\le 2c_1\sum_{i=0}^{\infty}\frac{1}{2^{i+1}}\left|\frac{x}{2^{i+1}}\right|^{\frac{\rho}{2}}\left|\frac{y}{2^{i+1}}\right|^{\frac{\rho}{2}}\\ &\le \frac{c_1}{2^{\rho}}\Big(1-\frac{1}{2^{\rho+1}}\Big)^{-1}|x|^{\frac{\rho}{2}}|y|^{\frac{\rho}{2}}\\ &\le \frac{2c_1}{2^{\rho+1}-1}\Big(|x|^{\frac{\rho}{2}}|y|^{\frac{\rho}{2}}\Big)\end{aligned}$$

for all $x,y \in X$. □

Corollary 4.5. *Let $k > 0$ and $\alpha > -\frac{1}{2}$ be real numbers, and $F : X^2 \to Y$ be a mapping satisfying the functional inequality*

$$\begin{aligned}&\left|F(x+u,y+v)-\frac{F(x,y)F(u,v)}{F(x,y)+F(u,v)}\right|\\ &\qquad \le k\Big(|x|^{\frac{\alpha}{2}}|u|^{\frac{\alpha}{2}}|y|^{\frac{\alpha}{2}}|v|^{\frac{\alpha}{2}}+\frac{1}{2}\Big(|x|^{2\alpha}+|u|^{2\alpha}+|y|^{2\alpha}+|v|^{2\alpha}\Big)\Big)\end{aligned} \tag{4.14}$$

for all $x,u,y,v \in X$. Then there exists a unique 2-variable reciprocal mapping $R : X^2 \to Y$ satisfying the functional equation (1.7) and

$$|F(x,y)-R(x,y)| \le \frac{2k}{2^{2\alpha+1}-1}\Big(|x|^{\alpha}|y|^{\alpha}+\Big(|x|^{2\alpha}+|y|^{2\alpha}\Big)\Big) \tag{4.15}$$

for all $x,y \in X$.

Proof. Choosing $\phi(x,u,y,v) = 2k\Big(|x|^{\frac{\alpha}{2}}|u|^{\frac{\alpha}{2}}|y|^{\frac{\alpha}{2}}|v|^{\frac{\alpha}{2}}+\frac{1}{2}\Big(|x|^{2\alpha}+|u|^{2\alpha}+|y|^{2\alpha}+|v|^{2\alpha}\Big)\Big)$, for all $x,u,y,v \in X$ in Theorem 4.1, we have $\phi(x,x,y,y) = 2k\Big(|x|^{\alpha}|y|^{\alpha}+\Big(|x|^{2\alpha}+|y|^{2\alpha}\Big)\Big)$. From (4.3), we get

$$\begin{aligned}|F(x,y)-R(x,y)| &\le 2k\sum_{i=0}^{\infty}\frac{1}{2^{i+1}}\Big(\left|\frac{x}{2^{i+1}}\right|^{\alpha}\left|\frac{y}{2^{i+1}}\right|^{\alpha}+\Big(\left|\frac{x}{2^{i+1}}\right|^{2\alpha}+\left|\frac{y}{2^{i+1}}\right|^{2\alpha}\Big)\Big)\\ &\le \frac{k}{2^{2\alpha}}\Big(1-\frac{1}{2^{2\alpha+1}}\Big)^{-1}\Big(|x|^{\alpha}|y|^{\alpha}+\Big(|x|^{2\alpha}+|y|^{2\alpha}\Big)\Big)\\ &\le \frac{2k}{2^{2\alpha+1}-1}\Big(|x|^{\alpha}|y|^{\alpha}+\Big(|x|^{2\alpha}+|y|^{2\alpha}\Big)\Big)\end{aligned}$$

for all $x,y \in X$. □

5. COMPARISON OF RECIPROCAL FUNCTIONAL EQUATIONS (1.1) AND (1.7)

In this section, we compare the stability results obtained in Corollaries 4.2, 4.3, 4.4 and 4.5 for the 2-variable reciprocal functional equation (1.7) with the 1-variable reciprocal functional equation (1.1).

Result 5.1. *The stability result obtained in Corollary* 4.2 *for* 2*-variable reciprocal functional equation (1.7) coincides with the stability result obtained in Theorem* 2.1

[4] *for* 1*-variable reciprocal functional equation (1.1).*
Proof. Replacing (y, v) by (x, u) in (4.8), we arrive

$$\Big|F(x+u, x+u) - \frac{F(x,x)F(u,u)}{F(x,x)+F(u,u)}\Big| \leq \frac{c}{2}. \tag{5.1}$$

Now, replacing u by y in (5.1), we obtain

$$\Big|F(x+y, x+y) - \frac{F(x,x)F(y,y)}{F(x,x)+F(y,y)}\Big| \leq \frac{c}{2}. \tag{5.2}$$

Taking $f(x) = F(x, x)$ (where f is a reciprocal function) in the equation (5.2), we arrive the inequality (2.1) in [4]. Now, replacing y by x in equation (4.9), we obtain

$$|F(x, x) - R(x, x)| \leq c.$$

Taking $r(x) = R(x, x)$ (where r is a reciprocal function) in the above inequality, we get

$$|f(x) - r(x)| \leq c$$

which satisfies the result (2.3) in paper [4].

Result 5.2. *The stability result obtained in Corollary* 4.3 *for* 2*-variable reciprocal functional equation (1.7) coincides with the stability result obtained in Theorem* 3.4 [4] *for* 1*-variable reciprocal functional equation (1.1).*
Proof. Replacing (y, v) by (x, u) in (4.10), we obtain

$$\Big|F(x+u, x+u) - \frac{F(x,x)F(u,u)}{F(x,x)+F(u,u)}\Big| \leq \theta\Big(|x|^p + |u|^p\Big). \tag{5.3}$$

Now, substituting u bu y in (5.3), we get

$$\Big|F(x+y, x+y) - \frac{F(x,x)F(y,y)}{F(x,x)+F(y,y)}\Big| \leq \theta\Big(|x|^p + |y|^p\Big). \tag{5.4}$$

Next, replacing y by x in (4.11), we arrive

$$|F(x, x) - R(x, x)| \leq \frac{4\theta}{2^{p+1}-1}|x|^p. \tag{5.5}$$

Taking $F(x, x) = f(x)$, $R(x, x) = r(x)$ (where f and r are reciprocal functions) in (5.4) and (5.5), we obtain the inequalities (3.40) and (3.41) respectively in [4].

Result 5.3. *The stability result obtained in Corollary* 4.4 *for* 2*-variable reciprocal functional equation (1.7) coincides with the stability result obtained in Theorem* 3.1 [4] *for* 1*-variable reciprocal functional equation (1.1).*
Proof. Replacing (y, v) by (x, u) in (4.12), we arrive

$$\Big|F(x+u, x+u) - \frac{F(x,x)F(u,u)}{F(x,x)+F(u,u)}\Big| \leq c_1|x|^a|u|^b. \tag{5.6}$$

Now, replacing u by y in (5.6), we obtain

$$\Big|F(x+y, x+y) - \frac{F(x,x)F(y,y)}{F(x,x)+F(y,y)}\Big| \leq c_1|x|^a|y|^b. \tag{5.7}$$

Replacing y by x in (4.13), we have

$$|F(x, x) - R(x, x)| \leq \frac{2c_1}{2^{\rho+1}-1}|x|^\rho. \tag{5.8}$$

Taking $F(x,x) = f(x)$, $R(x,x) = r(x)$ (where f and r are reciprocal functions) in (5.7) and (5.8), we obtain the inequalities (3.1) and (3.2) respectively in [4].

Result 5.4. *The stability result obtained in Corollary* 4.5 *for* 2*-variable reciprocal functional equation (1.7) coincides with the stability result obtained in Theorem* 4.1 [4] *for* 1*-variable reciprocal functional equation (1.1).*

Proof. Replacing (y,v) by (x,u) in (4.14), we arrive

$$\left|F(x+u,x+u) - \frac{F(x,x)F(u,u)}{F(x,x)+F(u,u)}\right| \leq k\Big(|x|^{\alpha}|u|^{\alpha} + \Big(|x|^{2\alpha} + |u|^{2\alpha}\Big)\Big). \tag{5.9}$$

Substituting u by y in (5.9) to get

$$\left|F(x+y,x+y) - \frac{F(x,x)F(y,y)}{F(x,x)+F(y,y)}\right| \leq k\Big(|x|^{\alpha}|y|^{\alpha} + \Big(|x|^{2\alpha} + |y|^{2\alpha}\Big)\Big). \tag{5.10}$$

Next, replacing y by x in (4.15), we obtain

$$|F(x,x) - R(x,x)| \leq \frac{6k}{2^{2\alpha+1}-1}|x|^{2\alpha}. \tag{5.11}$$

Taking $F(x,x) = f(x)$, $R(x,x) = r(x)$ (where f and r are reciprocal functions) in (5.10) and (5.11), we obtain the inequalities (4.1) and (4.2) respectively in [4].

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