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ON THE DRAZIN INVERSE FOR UPPER TRIANGULAR OPERATOR MATRICES

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Abstract. In this paper we investigate the stability of Drazin spectrum $\sigma_D(.)$ for upper triangular operator matrices $M_C = \left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right]$ using tools from local spectral theory. We show that $\sigma_D(M_C) \cup [\mathcal{S}(A^*) \cap \mathcal{S}(B)] = \sigma_D(A) \cup \sigma_D(B)$ where $\mathcal{S}(.)$ is the set where an operator fails to have the SVEP. As application we explore how the generalized Weyl's theorem survives for M_C.

1. Introduction

Let X and Y be Banach spaces and let $\mathcal{L}(X, Y)$ denote the space of all bounded linear operators from X to Y. For $Y = X$ we write $\mathcal{L}(X, Y) = \mathcal{L}(X)$. For $T \in \mathcal{L}(X)$, let $N(T)$, $R(T)$, $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$ and $\sigma_s(T)$, denote the null space, the range, the spectrum, the point spectrum, the approximate point spectrum and the surjectivity spectrum of T, respectively.

A bounded linear operator T is called an *upper semi-Fredholm* (resp. *lower semi-Fredholm*) if $R(T)$ is closed and $\alpha(T) := dim\, N(T) < \infty$ (resp. $\beta(T) := codim\, R(T) < \infty$). If T is either upper or lower semi-Fredholm then T is called a *semi-Fredholm* operator. The *index* of a semi-Fredholm operator T is defined by $ind(T) = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite then T is a *Fredholm* operator. The *essential spectrum* $\sigma_e(T)$ of T is defined as the set of all λ in $\mathbb{C}$ for which $T - \lambda$ is not a Fredholm operator. An operator T is called a *Weyl* operator if it is a Fredholm operator of index zero. We denote by $\sigma_W(T)$ the *Weyl spectrum* of T defined as the set of all λ in $\mathbb{C}$ for which $T - \lambda$ is not a Weyl operator.

For each nonnegative integer n define $T_{[n]}$ to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_{[0]} = T$). If for some n, $R(T^n)$ is closed and $T_{[n]}$ is an upper (resp. lower) semi-Fredholm operator then T is called an *upper* (resp. *lower*) *semi-B-Fredholm* operator. A *semi-B-Fredholm* operator is an upper or lower semi-B-Fredholm operator. If moreover, $T_{[n]}$ is a Fredholm operator then T is called a *B-Fredholm* operator. From [6, Proposition 2.1] if $T_{[n]}$ is a semi-Fredholm operator then $T_{[m]}$ is also a semi-Fredholm operator for each $m \geq n$, and $ind(T_{[m]}) = ind(T_{[n]})$. Then the *index* of a semi-B-Fredholm operator is defined as the index of the semi-Fredholm operator $T_{[n]}$ (see [5, 6]).

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$T \in \mathcal{L}(X)$ is said to be a *B-Weyl* operator if it is a B-Fredholm operator of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a B-Weyl operator}\}.$$

From [2, Lemma 4.1], T is a B-Weyl operator if and only if $T = F \oplus N$, where F is a Fredholm operator of index zero and N is a nilpotent operator.

Recall that the *ascent*, $a(T)$, of an operator T is the smallest non-negative integer p such that $N(T^p) = N(T^{p+1})$. If such integer does not exist we put $a(T) = \infty$. Analogously, the *descent*, $d(T)$, of an operator T is the smallest non-negative integer q such that $R(T^q) = R(T^{q+1})$, and if such integer does not exist we put $d(T) = \infty$. It is well-known that if $a(T)$ and $d(T)$ are both finite then they are equal, see [17, Corollary 20.5].

An operator $T \in \mathcal{L}(X)$ is said to be a *Drazin invertible* if there exists a positive integer k and an operator $S \in \mathcal{L}(X)$ such that

$$T^k ST = T^k, \quad STS = S \text{ and } TS = ST.$$

The *Drazin spectrum* is defined by

$$\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\}.$$

It is well known that T is Drazin invertible if and only if T is of finite ascent and descent, which is also equivalent to the fact that $T = R \oplus N$ where R is invertible and N nilpotent (see [16, Corollary 2.2]). Clearly, T is Drazin invertible if and only if T^* is Drazin invertible.

A bounded linear operator $T \in \mathcal{L}(X)$ is said to have the *single-valued extension property* (SVEP, for short) at $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ, the constant function $f \equiv 0$ is the only analytic solution of the equation

$$(T - \mu)f(\mu) = 0 \quad \forall \mu \in U_\lambda.$$

We use $\mathcal{S}(T)$ to denote the open set where T fails to have the SVEP and we say that T has the SVEP if $\mathcal{S}(T)$ is the empty set, [12]. It is easy to see that T has the SVEP at every point $\lambda \in iso\,\sigma(T)$, where $iso\,\sigma(T)$ denotes the set of all isolated points of $\sigma(T)$. Note that (see [12])

$$\mathcal{S}(T) \subseteq \sigma_p(T) \text{ and } \sigma(T) = \mathcal{S}(T) \cup \sigma_s(T). \tag{1.1}$$

Also it follows from [15] if T is of finite ascent and descent then T and T^* have the SVEP. Hence

$$\mathcal{S}(T) \cup \mathcal{S}(T^*) \subseteq \sigma_D(T). \tag{1.2}$$

For $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and $C \in \mathcal{L}(Y, X)$ we denote by M_C the operator defined on $X \oplus Y$ by

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}.$$

In [11] it is proved that

$$\sigma(M_C) \cup [\mathcal{S}(A^*) \cap \mathcal{S}(B)] = \sigma(A) \cup \sigma(B).$$

Numerus mathematicians were interested by the defect set $[\sigma_*(A) \cup \sigma_*(B)] \setminus \sigma_*(M_C)$ where $\sigma_* \in \{\sigma, \sigma_e, \sigma_w, \ldots\}$. See for instance [11, 13, 14] for the spectrum and the essential spectrum, [19] for the Weyl spectrum, [10] for the Browder spectrum and [9, 10] for the essential approximate point spectrum and the Browder essential approximate point spectrum. See also the references therein.

For the Drazin spectrum, Campbell and Meyer [7] were the first studied the Drazin invertibility of 2×2 upper triangular operator matrices M_C where A, B and C are $n \times n$ complex matrices. They proved that

$$\sigma_D(M_C) \subseteq \sigma_D(A) \cup \sigma_D(B). \tag{1.3}$$

D. S. Djordjević and P. S. Stanimirović generalized the inclusion (1.3) to arbitrary Banach spaces [8].

Inclusion (1.3) may be strict. Indeed, let A be the unilateral shift operator defined on l_2 by $A(x_0, x_1, \cdots) = (0, x_0, x_1, \cdots)$. For $B = A^*$ and $C = I - AA^*$, we have M_C is unitary and then $\sigma_D(M_C) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ but $\sigma_D(A) \cup \sigma_D(B) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

So it is naturally to ask the following question: what is exactly the defect set $[\sigma_D(A) \cup \sigma_D(B)] \setminus \sigma_D(M_C)$?

Recently, Zhang et al. [21] proved that

$$\sigma_D(M_C) \cup \mathcal{W} = \sigma_D(A) \cup \sigma_D(B)$$

where $\mathcal{W}$ is the union of certain holes in $\sigma_D(M_C)$ which happen to be subsets of $\sigma_D(A) \cap \sigma_D(B)$. But without any explicit description of the set $\mathcal{W}$.

The main objective of this paper is to give an explicit description of $\mathcal{W}$ using tools from local spectral theory. We also obtain the main results of [21]. As application we give sufficient conditions under which $\sigma_*(M_C) = \sigma_*(A) \cup \sigma_*(B)$ for $\sigma_* \in \{\sigma_D, \sigma_{BW}\}$ and we explore how the generalized Weyl's theorem survives for M_C.

2. Main results

Our main result is the following.

Theorem 2.1. *For $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and $C \in \mathcal{L}(Y, X)$ we have*

$$\sigma_D(M_C) \cup [\mathcal{S}(A^*) \cap \mathcal{S}(B)] = \sigma_D(A) \cup \sigma_D(B). \tag{2.1}$$

Proof. Since the inclusion $\sigma_D(M_C) \cup (\mathcal{S}(A^*) \cap \mathcal{S}(B)) \subseteq \sigma_D(A) \cup \sigma_D(B)$ always holds, it suffices to prove the reverse inclusion. Let $\lambda \in (\sigma_D(A) \cup \sigma_D(B)) \setminus \sigma_D(M_C)$. Without loss of generality, we can assume that $\lambda = 0$. Then M_C is of finite ascent and descent. Hence from [9, Lemma 2.1] we have A is of finite ascent and B is of finite descent. Also by duality A^* is of finite descent and B^* is of finite ascent. For the sake of contradiction assume that $0 \notin \mathcal{S}(A^*) \cap \mathcal{S}(B)$.

Case 1. $0 \notin \mathcal{S}(A^*)$: Since M_C is Drazin invertible, then there exists $\varepsilon > 0$ such that for every λ, $0 < |\lambda| < \varepsilon$, $M_C - \lambda$ is invertible. Hence $A - \lambda$ is right invertible. Thus $0 \notin acc\, \sigma_{ap}(A) = acc\, \sigma_s(A^*)$. If $0 \notin \sigma(A^*)$ then A^* is Drazin invertible and so A is. Now if $0 \in \sigma(A^*)$, since $\sigma(A^*) = \mathcal{S}(A^*) \cup \sigma_s(A^*)$ (see (1.1)) then 0 is an isolated point of $\sigma(A^*)$. Now A^* is of finite descent and $0 \in iso\, \sigma(A^*)$ hence it follows from [18, Theorem 10.5] that A^* is Drazin invertible. Thus A is Drazin invertible. Since M_C is Drazin invertible it follows from [21, Lemma 2.7] that B is also Drazin invertible which contradicts our assumption.

Case 2. If $0 \notin \mathcal{S}(B)$, the proof goes similarly. □

Corollary 2.2. *If A^* or B has the SVEP, then for every $C \in \mathcal{L}(Y, X)$,*

$$\sigma_D(M_C) = \sigma_D(A) \cup \sigma_D(B). \tag{2.2}$$

Since $\mathcal{S}(T)$ is a subset of $\sigma_p(T)$ we have the following

Corollary 2.3. *If $\sigma_p(A^*)$ or $\sigma_p(B)$ has no interior point, in particular if A or B is a compact operator, then equality (2.2) holds for every $C \in \mathcal{L}(Y, X)$.*

Corollary 2.4. *If $\mathcal{S}(A^*) \cap \mathcal{S}(B) \subseteq \sigma(M_C)$ then $\mathcal{S}(A^*) \cap \mathcal{S}(B) \subseteq \sigma_D(M_C)$. In other words, if $\sigma(M_C) = \sigma(A) \cup \sigma(B)$ then $\sigma_D(M_C) = \sigma_D(A) \cup \sigma_D(B)$.*

Proof. Assume that $\mathcal{S}(A^*) \cap \mathcal{S}(B) \neq \emptyset$. Let $\lambda \in \mathcal{S}(A^*) \cap \mathcal{S}(B)$ then there exists $\varepsilon > 0$ such that for every $\mu \in \mathbb{C}$, $0 < |\lambda - \mu| < \varepsilon$, $M_C - \mu$ is not invertible. Thus $M_C - \lambda$ is not Drazin invertible. Therefore $\mathcal{S}(A^*) \cap \mathcal{S}(B) \subseteq \sigma_D(M_C)$. □

Let $\rho_D(T) = \mathbb{C} \setminus \sigma_D(T)$ be the Drazin resolvent set of T. Now we retrieve the main result of [21].

Corollary 2.5. [21, Theorem 3.1] *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$. Then*

a) $\displaystyle\bigcap_{C \in \mathcal{L}(Y,X)} \sigma_D(M_C) \subseteq \Big(\bigcap_{C \in \mathcal{L}(Y,X)} \sigma(M_C)\Big) \setminus [\rho_D(A) \cap \rho_D(B)]$.

b) *In particular, if one of the following conditions holds:*
i) $\sigma(A) \cap \sigma(B) = \emptyset$. ii) int $\sigma_p(B) = \emptyset$ iii) int $\sigma_p(A^*) = \emptyset$
iv) $\sigma_s(B) = \sigma(B)$ v) $\sigma_{ap}(A) = \sigma(A)$

then we have

$$\bigcap_{C \in \mathcal{L}(Y,X)} \sigma_D(M_C) = \Big(\bigcap_{C \in \mathcal{L}(Y,X)} \sigma(M_C)\Big) \setminus [\rho_D(A) \cap \rho_D(B)].$$

Proof. a) Follows directly from (1.3).
b) From *i*) we have $\sigma(M_C) = \sigma(A) \cup \sigma(B)$, see [13, Corollary 4]. Then $\sigma_D(M_C) = \sigma_D(A) \cup \sigma_D(B)$ for every C by Corollary 2.4. Also from *ii*) or *iii*) we get $\sigma_D(M_C) = \sigma_D(A) \cup \sigma_D(B)$ for every C by Corollary 2.3. Thus if $\lambda \notin \displaystyle\bigcap_{C \in \mathcal{L}(Y,X)} \sigma_D(M_C) = \sigma_D(A) \cup \sigma_D(B)$ then $\lambda \in \rho_D(A) \cap \rho_D(B)$.

Now assume *iv*). Let $\lambda \notin \displaystyle\bigcap_{C \in \mathcal{L}(Y,X)} \sigma_D(M_C)$. Then there exists C_0 such that $M_{C_0} - \lambda$ is Drazin invertible. Thus for $\varepsilon > 0$ small enough, we have for all μ, $0 < |\mu| < \varepsilon$, $M_{C_0} - \lambda - \mu$ is invertible. Hence it follows form [13, Theorem 2] that $B - \lambda - \mu$ is right invertible. Thus $\lambda \notin acc\, \sigma_s(B) = acc\, \sigma(B)$. Therefore $\lambda \notin \mathcal{S}(B)$. Now from Theorem 2.1 we get $\lambda \in \rho_D(A) \cap \rho_D(B)$.

For *v*) the proof goes by duality. □

Recall that if T is Drazin invertible then $T = R \oplus N$ where R is invertible and N is nilpotent, in particular R is Fredholm of index zero. Hence it follows from [2, Lemma 4.1] that T is a B-Weyl operator. Therefore the inclusion $\sigma_{BW}(T) \subseteq \sigma_D(T)$ always holds. In [4, Theorem 3.3] it is shown that the reverse inclusion holds under the assumption that T has the SVEP.

The defect set $\sigma_D(T) \setminus \sigma_{BW}(T)$ is characterized in the following.

Theorem 2.6. *Let $T \in \mathcal{L}(X)$. Then*

$$\sigma_{BW}(T) \cup [\mathcal{S}(T) \cap \mathcal{S}(T^*)] = \sigma_D(T). \tag{2.3}$$

Proof. Since $\sigma_{BW}(T)\cup(\mathcal{S}(T)\cap\mathcal{S}(T^*)) \subseteq \sigma_D(T)$ always holds, then let $\lambda \notin \sigma_{BW}(T)\cup(\mathcal{S}(T)\cap\mathcal{S}(T^*))$. Without loss of generality we assume that $\lambda = 0$. Then T is a B-Fredholm operator of index zero.

Case 1. If $0 \notin \mathcal{S}(T)$: Since T is a B-Fredholm operator of index zero, then it follows from [2, Lemma 4.1] that there exists a Fredholm operator F of index zero and a nilpotent operator N such that $T = F \oplus N$. If $0 \notin \sigma(F)$, then F is invertible and hence T is Drazin invertible. Now assume that $0 \in \sigma(F)$. Since T has the SVEP at 0, then F has also the SVEP at 0. Hence it follows from [1, Theorem 3.16] that $a(F)$ is finite. F is a Fredholm operator of index zero, then it follows from [1, Theorem 3.4] that $d(F)$ is also finite. Then $a(F) = d(F) < \infty$ which implies that 0 is a pole of F and hence an isolated point of $\sigma(F)$. N is nilpotent, then 0 is isolated point of $\sigma(T)$. From [2, Theorem 4.2] we get $0 \notin \sigma_D(T)$.

Case 2. If $0 \notin \mathcal{S}(T^*)$, then proof goes similarly. □

In [14, Proposition 3.1] it is proved that if A and B have the SVEP then for every $C \in \mathcal{L}(Y, X)$, M_C has the SVEP. Now the following result is an immediate consequence of Corollary 2.2 and Theorem 2.6.

Corollary 2.7. *If A and B (or A^* and B^*) have the SVEP, then for every $C \in \mathcal{L}(Y, X)$,*

$$\sigma_{BW}(M_C) = \sigma_{BW}(A) \cup \sigma_{BW}(B). \tag{2.4}$$

In [19] and under the same conditions of Corollary 2.7 we proved equality (2.4) for the Weyl spectrum.

3. Applications

Berkani [2, Theorem 4.5] has shown that every normal operator T acting on Hilbert space H satisfies

$$\sigma(T) \setminus \sigma_{BW}(T) = E(T), \tag{3.1}$$

where $E(T)$ is the set of all isolated eigenvalues of T. We say that *generalized Weyl's theorem* holds for T if equality (3.1) holds. This gives a generalization of the classical Weyl's theorem. Recall that $T \in \mathcal{L}(X)$ obeys *Weyl's theorem* if

$$\sigma(T) \setminus \sigma_W(T) = E_0(T), \tag{3.2}$$

where $E_0(T)$ denotes the set of all the isolated points of $\sigma(T)$ which are eigenvalues of finite multiplicity. From [5, Theorem 3.9] generalized Weyl's theorem implies Weyl's theorem and generally the reverse is not true.

In general the fact that generalized Weyl's theorem holds for A and B does not imply that generalized Weyl's theorem holds for $M_0 = \left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$. Indeed, let I_1 and I_2 be the identity operators on $\mathbb{C}$ and l_2, respectively. Let S_1 and S_2 be defined on l_2 by

$$S_1(x_1, x_2, \ldots) = (0, \frac{1}{3}x_1, \frac{1}{3}x_2, \ldots) \text{ and } S_2(x_1, x_2, \ldots) = (0, \frac{1}{2}x_1, \frac{1}{3}x_2, \ldots).$$

Let $T_1 = I_1 \oplus S_1$ and $T_2 = S_2 - I_2$. Let $A = T_1^2$ and $B = T_2^2$, then from [20, Example 1] we have A and B obey generalized Weyl's theorem but M_0 does not obey it.

It also may happen that M_C obeys generalized Weyl's theorem while M_0 does not obey it. If we take A, B and C as in the example given in the introduction, we have M_C is unitary without eigenvalues. Then M_C satisfies generalized Weyl's theorem (see [3, Remark 3.5]). But $\sigma_W(M_0) = \{\lambda : |\lambda| = 1\}$ and $\sigma(M_0)\setminus E_0(M_0) = \{\lambda : |\lambda| \leq 1\}$. Then M_0 does not satisfy Weyl's theorem and so by [5, Theorem 3.9] it does not satisfy generalized Weyls theorem either.

A bounded linear operator T is said to be an *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of T.

Theorem 3.1. *Let A be an isoloid. Assume that A and B (or A^* and B^*) have the SVEP. If A and $M_0 = \left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$ satisfy generalized Weyl's theorem then M_C satisfies generalized Weyl's theorem for every $C \in \mathcal{L}(Y, X)$.*

Proof. Let $\lambda \in \sigma(M_C)\setminus\sigma_{BW}(M_C)$. From [14, Theorem 2.1] $\sigma(M_C) = \sigma(M_0)$. Then by Corollary 2.7, $\sigma(M_C) \setminus \sigma_{BW}(M_C) = \sigma(M_0) \setminus \sigma_{BW}(M_0)$ which equals to $E(M_0)$ since M_0 satisfies generalized Weyl's theorem. Thus $\lambda \in iso\,\sigma(M_0) = iso\,\sigma(M_c)$. If $\lambda \in iso\,\sigma(A)$, since A is isoloid then $\lambda \in \sigma_p(A)$. Hence $\lambda \in \sigma_p(M_C)$. Then $\lambda \in E(M_C)$. Now assume that $\lambda \in iso\,\sigma(B) \setminus iso\,\sigma(A)$. If $\lambda \notin \sigma(A)$ then it is not difficult to see that $\lambda \in \sigma_p(M_C)$. Also if $\lambda \in \sigma_p(A)$ then $\lambda \in \sigma_p(M_C)$, so assume that $\lambda \in \sigma_p(B) \setminus \sigma_p(A)$. Then $\lambda \notin E(A)$. Since A satisfies generalized Weyl's theorem, then $\lambda \in \sigma_{BW}(A)$. This is impossible. Therefore $\lambda \in E(M_C)$.

Conversely assume that $\lambda \in E(M_C)$. Then $\lambda \in iso\,\sigma(M_C) = iso\,\sigma(M_0)$. On the other hand, $\lambda \in \sigma_p(M_C) \subseteq \sigma_p(A) \cup \sigma_p(B)$. Hence $\lambda \in \sigma_p(M_0)$. Thus $\lambda \in E(M_0) = \sigma(M_0) \setminus \sigma_{BW}(M_0)$ which equals to $\sigma(M_C) \setminus \sigma_{BW}(M_C)$. Therefore $\lambda \in \sigma(M_C) \setminus \sigma_{BW}(M_C)$. ☐

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