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EXCHANGE FORMULA FOR GENERALIZED LAMBERT TRANSFORM AND ITS EXTENSION TO BOEHMIANS

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Abstract. We derive the exchange formula for the generalized Lambert transform, by defining a suitable product in the range of generalized Lambert transform on $\mathscr{E}'((0,\infty))$. We prove that the generalized Lambert transform from $\mathscr{E}'((0,\infty))$ into $\mathscr{E}((0,\infty))$ is continuous. Applying the exchange formula and continuity of generalized Lambert transform, we construct a new Boehmian space which will be the range of generalized Lambert transform on $\mathscr{B}(\mathscr{E}'((0,\infty)),\mathscr{D}((0,\infty)),*,\Delta_+)$. We establish that the generalized Lambert transform on Boehmians is consistent with that on $\mathscr{E}'((0,\infty))$, linear, one-to-one, onto and continuous with respect to δ-convergence and Δ-convergence. We also obtain the exchange formula for generalized Lambert transform in the context of Boehmians.

1. Introduction

We denote the set of all natural numbers and non-negative integers, respectively by $\mathbb{N}$ and $\mathbb{N}_0$. Let $\mathscr{E}((0,\infty))$ be the space of all infinitely differentiable complex valued functions on $(0,\infty)$ equipped with the Fréchet space topology given by the family of semi-norms [23, p. 36],

$$\gamma_{K,k}(\phi) = \sup_{x\in K} |\phi^{(k)}(x)|, \quad \text{where } K \subset (0,\infty) \text{ is compact and } k \in \mathbb{N}_0. \tag{1.1}$$

The dual space $\mathscr{E}'((0,\infty))$ of $\mathscr{E}((0,\infty))$ is called the space of compactly supported distributions on $(0,\infty)$. Throughout this paper, we use strong convergence in the space $\mathscr{E}'((0,\infty))$ which is defined as follows: (f_n) converges to f in $\mathscr{E}'((0,\infty))$, if for each bounded subset B of $\mathscr{E}((0,\infty))$,

$$\sup_{\phi\in B} |\langle f_n - f, \phi\rangle| \to 0 \text{ as } n \to \infty.$$

We recall the usual convolution defined on $\mathscr{E}'((0,\infty))$ by

$$\langle f * g, \phi\rangle = \langle f(t), \langle g(s), \phi(s+t)\rangle\rangle, \ \forall \phi \in \mathscr{E}((0,\infty)). \tag{1.2}$$

By $\mathscr{D}((0,\infty))$ and $\mathscr{D}'((0,\infty))$, as usual we mean the Schwartz testing function space of all smooth functions with compact support and the space of Schwartz

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distributions respectively. It is well known that $\mathscr{D}((0,\infty))$ is a subset of $\mathscr{E}'((0,\infty))$ by the canonical identification.

Now we recall the theory of Lambert transform from the literature. Widder [22] introduced the Lambert transform of a suitable function by

$$F(x) = \int_0^\infty f(t) \frac{1}{e^{xt}-1} dt,\ x > 0, \tag{1.3}$$

and R. R. Goldberg [3] generalized the Lambert transform by

$$F(x) = \int_0^\infty f(t) \sum_{k=1}^{\infty} a_k e^{-kxt}\, dt,\ x > 0, \tag{1.4}$$

where $\{a_k\}$ is a sequence of class C_r, for $r > 0$. That is, $A = \{a_k\}$ satisfies the following conditions.

(1) $a_k \geq 0$, $k = 1, 2, \ldots$.
(2) $a_k = O(k^{r-1}), k \to \infty$.
(3) $a_1 > 0$.
(4) $b_m = O(m^{r-1}), m \to \infty$ where $\sum_{d/p} a_d b_{p/d} = \begin{cases} 1, & p = 1 \\ 0, & p = 2, 3, \ldots. \end{cases}$

Observe that for $a_k = 1$, for all $k = 1, 2, \ldots$, the generalized Lambert transform (1.4) agrees with the Lambert transform (1.3).

Negrin [11] extended the Lambert transform to the space $\mathscr{E}'((0,\infty))$ of compactly supported distributions on $(0,\infty)$ by

$$F(x) = \left\langle f(t), \frac{1}{e^{xt}-1} \right\rangle,\ x > 0. \tag{1.5}$$

N. Hayek, B. J. González and E. R. Negrin [4] extended the generalized Lambert transform to the context of compactly supported distributions on $(0,\infty)$ by

$$F(x) = \left\langle f(t), \sum_{k=1}^{\infty} a_k e^{-kxt} \right\rangle,\ x > 0. \tag{1.6}$$

It is proved that F is infinitely differentiable and the inversion formula is obtained as follows.

$$\langle f, \phi \rangle = \lim_{n\to\infty} \left\langle \frac{(-1)^n}{n!} \cdot \left(\frac{n}{t}\right)^{n+1} \cdot \sum_{m=1}^{\infty} b_m m^n \cdot D^n F\left(\frac{mn}{t}\right), \phi(t) \right\rangle, \tag{1.7}$$

for every $\phi \in \mathscr{D}((0,\infty))$.

For our convenience, we denote the generalized Lambert transform of f and the Laplace transform of f, respectively by $\mathcal{L}_A f$, $\hat{f}$, where Laplace transform of f is defined by

$$\hat{f}(s) = \left\langle f(t), e^{-st} \right\rangle,\ s > 0. \tag{1.8}$$

By equation (1.7), if $\mathcal{L}_A f = \mathcal{L}_A g$ then $f = g$ as members of $\mathscr{D}'((0,\infty))$. Since $\mathscr{D}((0,\infty))$ is dense in $\mathscr{E}((0,\infty))$, the equality holds in $\mathscr{E}'((0,\infty))$. In other words

$$\mathcal{L}_A : \mathscr{E}'((0,\infty)) \to \mathscr{E}((0,\infty)) \text{ is one-to-one.} \tag{1.9}$$

2. Exchange formula and continuity

Now we define a new product $\otimes$ for $F \in \mathcal{L}_A(\mathscr{E}'((0,\infty)))$ and $G \in \mathcal{L}_A(\mathscr{E}'((0,\infty)))$ by

$$(F \otimes G)(x) = \sum_{k=1}^{\infty} a_k \hat{f}(kx) \cdot \hat{g}(kx), \tag{2.1}$$

where $f, g \in \mathscr{E}'((0,\infty))$ such that $\mathcal{L}_A f = F$ and $\mathcal{L}_A g = G$.
We note that the above definition is well defined. Indeed, by using (1.9), we can find unique $f, g \in \mathscr{E}'((0,\infty))$ such that $F = \mathcal{L}_A f$ and $G = \mathcal{L}_A g$. Therefore supp $f \subset [a,b]$, supp $g \subset [c,d]$, for some $a, b, c, d \in (0,\infty)$. As $\mathscr{E}'((0,\infty))$ is a subset of Laplace-transformable generalized functions we can apply Theorem 3.10-2 of [23] and we get

$$|\hat{f}(kx)| \le e^{-kxa} P_1(kx), \text{ and } |\hat{g}(kx)| \le e^{-kxc} P_2(kx), \tag{2.2}$$

for some polynomials P_1 and P_2.

Since $\{a_k\}$ is of class C_r from the relation (2.2) the series in equation (2.1) converges.

Before discussing the exchange formula for the generalized Lambert transform, it is necessary to show that $f * g \in \mathscr{E}'((0,\infty))$ whenever $f, g \in \mathscr{E}'((0,\infty))$. We note that every $f \in \mathscr{E}'((0,\infty))$ can be viewed as Schwartz distribution on $(0,\infty)$ with compact support. We also note that if $f, g \in \mathscr{E}'((0,\infty))$ with supp $f \subset [a,b]$ and supp $g \subset [c,d]$, for some $a, b, c, d \in (0,\infty)$, with $a < b$ and $c < d$ then the Schwartz distribution $f * g$ has compact support, in fact, supp $f * g \subset$ supp $f +$ supp $g \subset [a+b, c+d]$. Hence $f * g \in \mathscr{E}'((0,\infty))$.

Theorem 2.1 (Exchange formula)**.**
If $f, g \in \mathscr{E}'((0,\infty))$ then $\mathcal{L}_A(f * g) = \mathcal{L}_A f \otimes \mathcal{L}_A g$.

Proof. First we observe from Proposition 2.2 of [4] that $(\mathcal{L}_A f)(x) = \sum_{k=1}^{\infty} a_k \hat{f}(kx)$. Now let $x \in (0,\infty)$ be arbitrary. Since $\sum_{k=1}^{\infty} a_k e^{-kxs}$ converges in $\mathscr{E}((0,\infty))$ and $\{e^{-kxt}\}$, $\{\hat{g}(kx)\}$ are bounded, we have the series $\sum_{k=1}^{\infty} a_k e^{-kxt} e^{-kxs}$ of functions of s and the series $\sum_{k=1}^{\infty} a_k \hat{g}(kx) e^{-kxt}$ of functions of t converge in $\mathscr{E}((0,\infty))$. Therefore

$$\begin{aligned}
(\mathcal{L}_A(f*g))(x) &= \left\langle f(t), \left\langle g(s), \sum_{k=1}^{\infty} a_k e^{-kx(s+t)} \right\rangle \right\rangle \\
&= \left\langle f(t), \left\langle g(s), \sum_{k=1}^{\infty} a_k e^{-kxs} e^{-kxt} \right\rangle \right\rangle \\
&= \left\langle f(t), \sum_{k=1}^{\infty} a_k \left\langle g(s), e^{-kxs} \right\rangle e^{-kxt} \right\rangle \\
&= \left\langle f(t), \sum_{k=1}^{\infty} a_k \hat{g}(kx) e^{-kxt} \right\rangle \\
&= \sum_{k=1}^{\infty} a_k \left\langle f(t), \hat{g}(kx) e^{-kxt} \right\rangle \\
&= \sum_{k=1}^{\infty} a_k \hat{f}(kx) \cdot \hat{g}(kx) \\
&= (\mathcal{L}_A f \otimes \mathcal{L}_A g)(x).
\end{aligned}$$

Hence the theorem follows. □

Theorem 2.2. *The generalized Lambert transform $\mathcal{L}_A : \mathscr{E}'((0,\infty)) \to \mathscr{E}((0,\infty))$ is continuous .*

Proof. Since $\mathscr{E}((0,\infty))$ is metrizable and the generalized Lambert transform is linear, it is enough to prove that $\mathcal{L}_A f_n \to 0$ as $n \to \infty$, whenever $f_n \to 0$ as $n \to \infty$ in $\mathscr{E}'((0,\infty))$. We observe that the functions $\psi_x(t) = \sum_{k=1}^{\infty} a_k e^{-kxt}$, $\forall t \in (0,\infty)$ constitute a bounded subset of $\mathscr{E}((0,\infty))$ if x ranges over a compact subset of $\mathbb{R}$. For given $k \in \mathbb{N}_0$ and $K \subset \mathbb{R}$ compact, we put $\mathcal{B} = \{\psi_x^{(k)} : x \in K\}$. Therefore from the equality

$$\sup_{x\in K}\left|(\mathcal{L}_A f_n)^{(k)}(x)\right| = \sup_{\phi\in\mathcal{B}} |\langle f_n, \phi\rangle|$$

and by the assumption $f_n \to 0$ as $n \to \infty$ in $\mathscr{E}'((0,\infty))$, we conclude that $\mathcal{L}_A f_n \to 0$ as $n \to \infty$ in $\mathscr{E}((0,\infty))$. □

3. Boehmian space

J. Mikusiński and P. Mikusiński [7] introduced Boehmians as a generalization of distributions. An abstract construction of Boehmian space was given in [8] with two notions of convergence. Thereafter various Boehmian spaces have been defined and also various integral transforms have been extended on them, see [1, 2, 6, 9, 10, 12, 13, 14, 15, 16, 17, 21].

First we recall the construction of an abstract Boehmian space from [8].

To consider the Boehmian space we need $G, S, \star$ and Δ where G is a topological vector space, $S \subset G$ and $\star : G \times S \to G$ satisfying the following conditions.
Let $\alpha, \beta \in G$ and $\zeta, \xi \in S$ be arbitrary.
1. $\zeta \star \xi = \xi \star \zeta \in S$; 2. $(\alpha \star \zeta) \star \xi = \alpha \star (\zeta \star \xi)$; 3. $(\alpha + \beta) \star \zeta = \alpha \star \zeta + \beta \star \zeta$; 4. If $\alpha_n \to \alpha$ as $n \to \infty$ in G and $\xi \in S$ then $\alpha_n \star \xi \to \alpha \star \xi$ as $n \to \infty$,

and Δ is a collection of sequences from S satisfying
(a) If $(\xi_n), (\zeta_n) \in \Delta$ then $(\xi_n \star \zeta_n) \in \Delta$.
(b) If $\alpha \in G$ and $(\xi_n) \in \Delta$ then $\alpha \star \xi_n \to \alpha$ in G as $n \to \infty$.

Let $\mathscr{A}$ denote the collection of all pairs of sequences $((\alpha_n), (\xi_n))$ where $\alpha_n \in G$, $\forall n \in \mathbb{N}$ and $(\xi_n) \in \Delta$ satisfying the property

$$\alpha_n \star \xi_m = \alpha_m \star \xi_n, \ \forall \ \ m, n \in \mathbb{N}. \tag{3.1}$$

Each element of $\mathscr{A}$ is called a quotient and it is denoted by α_n/ξ_n. Define a relation $\sim$ on $\mathscr{A}$ by

$$\alpha_n/\xi_n \sim \beta_n/\zeta_n \quad \text{if} \quad \alpha_n \star \zeta_m = \beta_m \star \xi_n, \ \forall \ \ m, n \in \mathbb{N}. \tag{3.2}$$

It is easy to verify that $\sim$ is an equivalence relation on $\mathscr{A}$ and hence it decomposes $\mathscr{A}$ into disjoint equivalence classes. Each equivalence class is called a Boehmian and is denoted by $[\alpha_n/\xi_n]$. The collection of all Boehmians is denoted by $\mathscr{B} = \mathscr{B}(G, S, \star, \Delta)$. Every element α of G is identified uniquely as a member of $\mathscr{B}$ by $[(\alpha \star \xi_n)/\xi_n]$ where $(\xi_n) \in \Delta$ is arbitrary.

$\mathscr{B}$ is a vector space with addition and scalar multiplication defined as follows.

- $[\alpha_n/\xi_n] + [\beta_n/\zeta_n] = [(\alpha_n \star \zeta_n + \beta_n \star \xi_n)/(\xi_n \star \zeta_n)]$.
- $c\,[\alpha_n/\xi_n] = [(c\alpha_n)/\xi_n]$.

The operation $\star$ can be extended to $\mathscr{B} \times S$ by the following definition.

Definition 3.1. *If $x = [\alpha_n/\xi_n] \in \mathscr{B}$, and $\zeta \in S$ then $x \star \zeta = [(\alpha_n \star \zeta)/\xi_n]$.*

Now we recall the δ-convergence on $\mathscr{B}$.

Definition 3.2 (δ-Convergence)**.** *We say that $X_n \xrightarrow{\delta} X$ as $n \to \infty$ in $\mathscr{B}$ if there exists a delta sequence (ξ_n) such that $X_n \star \xi_k \in G$, $\forall n, k \in \mathbb{N}$, $X \star \xi_k \in G, \forall k \in \mathbb{N}$ and for each $k \in \mathbb{N}$,*

$$X_n \star \xi_k \to X \star \xi_k \text{ as } n \to \infty \text{ in } G.$$

The following lemma states an equivalent statement for δ-convergence.

Lemma 3.1. *$X_n \xrightarrow{\delta} X$ as $n \to \infty$ in $\mathscr{B}$ if and only if there exist $\alpha_{n,k}, \alpha_k \in G$ and $(\xi_k) \in \Delta$ such that $X_n = [\alpha_{n,k}/\xi_k]$, $X = [\alpha_k/\xi_k]$ and for each $k \in \mathbb{N}$,*

$$\alpha_{n,k} \to \alpha_k \text{ as } n \to \infty \text{ in } G.$$

Definition 3.3 (Δ-convergence)**.** *We say that $X_n \xrightarrow{\Delta} X$ as $n \to \infty$ in $\mathscr{B}$ if there exists $(\xi_k) \in \Delta$ such that $(X_n - X) \star \xi_n \in G$ for all $n \in \mathbb{N}$ and $(X_n - X) \star \xi_n \to 0$ as $n \to \infty$ in G.*

We construct a Boehmian space $\mathscr{B}_1 = \mathscr{B}(\mathscr{E}'((0,\infty)), \mathscr{D}((0,\infty)), *, \Delta_+)$ where $*$ is the usual convolution defined in (1.2) and Δ_+ is the collection of all sequences (δ_n) from $\mathscr{D}((0,\infty))$ satisfying the following properties.

(1) $\int_0^\infty \delta_n(t)\, dt = 1,\ \forall n \in \mathbb{N}$.
(2) $\int_0^\infty |\delta_n(t)|\, dt \leq M,\ \forall n \in \mathbb{N}$ for some $M > 0$.
(3) If $s(\delta_n) = \sup\{t \in (0,\infty) : \delta_n(t) \neq 0\}$ then $s(\delta_n) \to 0$ as $n \to \infty$.

It is well known that $\mathscr{E}'((0,\infty))$ is contained in $\mathscr{B}_1$ and one can prove that $\mathscr{B}_1$ is properly larger than $\mathscr{E}'(\mathbb{R})$, by modifying the example given in [8].

Another Boehmian space is given by

$$\mathscr{B}_2 = \mathscr{B}(\mathcal{L}_A(\mathscr{E}'((0,\infty))), \mathcal{L}_A(\mathscr{E}'((0,\infty))), \otimes, \mathcal{L}_A(\Delta_+))$$

where $\mathcal{L}_A(\Delta_+) = \{(\mathcal{L}_A(\delta_n)) : (\delta_n) \in \Delta_+\}$.

4. Extended Lambert transform

Definition 4.1. *The extended Lambert transform $\mathfrak{L}_A : \mathscr{B}_1 \to \mathscr{B}_2$ is defined by*

$$\mathfrak{L}_A([f_n/\delta_n]) = [\mathcal{L}_A f_n/\mathcal{L}_A \delta_n]. \tag{4.1}$$

It is necessary to verify that $[\mathcal{L}_A f_n/\mathcal{L}_A \delta_n] \in \mathscr{B}_2$ and this definition is independent of the representatives. Indeed, using $[f_n/\delta_n] \in \mathscr{B}_1$, we get

$$f_n * \delta_m = f_m * \delta_n,\ \forall m, n \in \mathbb{N}. \tag{4.2}$$

Since $f_n * \delta_m \in \mathscr{E}'((0,\infty))$, we can apply the generalized Lambert transform on both sides and we get, in light of Theorem 2.1

$$\mathcal{L}_A f_n \otimes \mathcal{L}_A \delta_m = \mathcal{L}_A f_m \otimes \mathcal{L}_A \delta_n,\ \forall m, n \in \mathbb{N}. \tag{4.3}$$

Hence $\mathcal{L}_A f_n/\mathcal{L}_A \delta_n$ is a quotient and hence $[\mathcal{L}_A f_n/\mathcal{L}_A \delta_n] \in \mathscr{B}_2$. Moreover, if $[f_n/\delta_n] = [g_n/\epsilon_n]$ then

$$f_n * \epsilon_m = g_m * \delta_n,\ \forall m, n \in \mathbb{N}. \tag{4.4}$$

Again by the same reason, we get

$$\mathcal{L}_A f_n \otimes \mathcal{L}_A \epsilon_m = \mathcal{L}_A g_m \otimes \mathcal{L}_A \delta_n, \ \forall m, n \in \mathbb{N}, \tag{4.5}$$

and hence $[\mathcal{L}_A f_n / \mathcal{L}_A \delta_n] = [\mathcal{L}_A g_n / \mathcal{L}_A \epsilon_n]$.

Lemma 4.1. *The extended Lambert transform on $\mathscr{B}_1$ is consistent with the generalized Lambert transform on $\mathscr{E}'((0,\infty))$.*

Proof. Let $f \in \mathscr{E}'((0,\infty))$ be arbitrary. For any $(\delta_n) \in \Delta_+$, f is represented by $[(f * \delta_n)/\delta_n] \in \mathscr{B}_1$. Now

$$\mathfrak{L}_A [(f * \delta_n)/\delta_n] = [\mathcal{L}_A (f * \delta_n)/\mathcal{L}_A \delta_n] = [(\mathcal{L}_A f \otimes \mathcal{L}_A \delta_n)/\mathcal{L}_A \delta_n]$$

which represents $\mathcal{L}_A f$ in $\mathscr{B}_2$. Thus we have proved the lemma. □

Lemma 4.2. *The extended Lambert transform $\mathfrak{L}_A : \mathscr{B}_1 \to \mathscr{B}_2$ is a linear map.*

Proof. Let $[f_n/\delta_n], [g_n/\epsilon_n] \in \mathscr{B}_1$ and $\alpha, \beta \in \mathbb{C}$. By using the linearity of $\mathcal{L}_A : \mathscr{E}'((0,\infty)) \to \mathscr{E}((0,\infty))$ and by Theorem 2.1, we get

$$\begin{aligned}
&\mathfrak{L}_A (\alpha [f_n/\delta_n] + \beta [g_n/\epsilon_n]) \\
&\quad = \mathfrak{L}_A [((\alpha f_n) * \epsilon_n + (\beta g_n) * \delta_n)/(\delta_n * \epsilon_n)] \\
&\quad = \mathfrak{L}_A [(\alpha(f_n * \epsilon_n) + \beta(g_n * \delta_n))/(\delta_n * \epsilon_n)] \\
&\quad = [\mathcal{L}_A(\alpha(f_n * \epsilon_n) + \beta(g_n * \delta_n))/\mathcal{L}_A(\delta_n * \epsilon_n)] \\
&\quad = [(\alpha\mathcal{L}_A(f_n * \epsilon_n) + \beta\mathcal{L}_A(g_n * \delta_n))/\mathcal{L}_A(\delta_n * \epsilon_n)] \\
&\quad = [(\alpha(\mathcal{L}_A f_n) \otimes (\mathcal{L}_A \epsilon_n) + \beta(\mathcal{L}_A g_n) \otimes (\mathcal{L}_A \delta_n))/(\mathcal{L}_A \delta_n) \otimes (\mathcal{L}_A \epsilon_n)] \\
&\quad = [\alpha(\mathcal{L}_A f_n)/\mathcal{L}_A \delta_n] + [\beta(\mathcal{L}_A g_n)/\mathcal{L}_A \epsilon_n] \\
&\quad = \alpha [\mathcal{L}_A f_n/\mathcal{L}_A \delta_n] + \beta [\mathcal{L}_A g_n/\mathcal{L}_A \epsilon_n] \\
&\quad = \alpha \mathfrak{L}_A [f_n/\delta_n] + \beta \mathfrak{L}_A [g_n/\epsilon_n]
\end{aligned}$$

Hence the lemma follows. □

Lemma 4.3. *The extended Lambert transform $\mathfrak{L}_A : \mathscr{B}_1 \to \mathscr{B}_2$ is one-to-one.*

Proof. Let $X = [f_n/\delta_n], Y = [g_n/\epsilon_n] \in \mathscr{B}_1$. If $\mathfrak{L}_A X = \mathfrak{L}_A Y$ then we have

$$\mathcal{L}_A f_n \otimes \mathcal{L}_A \epsilon_m = \mathcal{L}_A g_m \otimes \mathcal{L}_A \delta_n, \forall m, n \in \mathbb{N}. \tag{4.6}$$

Theorem 2.1 enables us to obtain

$$\mathcal{L}_A(f_n * \epsilon_m) = \mathcal{L}_A(g_m * \delta_n), \forall m, n \in \mathbb{N}. \tag{4.7}$$

By virtue of (1.9) it follows that

$$f_n * \epsilon_m = g_m * \delta_n, \text{ as members of } \mathscr{E}'((0,\infty)) \, \forall m, n \in \mathbb{N}. \tag{4.8}$$

Thus we get $X = Y$. □

Lemma 4.4. *The extended Lambert transform $\mathfrak{L}_A : \mathscr{B}_1 \to \mathscr{B}_2$ is onto.*

The proof is straightforward.

Theorem 4.5. *The extended Lambert transform $\mathfrak{L}_A : \mathscr{B}_1 \to \mathscr{B}_2$ is continuous with respect to the δ-convergence.*

Proof. Let $X_n \xrightarrow{\delta} X$ as $n \to \infty$ in $\mathscr{B}_1$. Then by Lemma 3.1, there exists $f_{n,k}, f_k \in \mathscr{E}'((0,\infty))$, $\forall n, k \in \mathbb{N}$ and $(\delta_k) \in \Delta_+$ such that $X_n = [f_{n,k}/\delta_k]$, $X = [f_k/\delta_k]$ and for each $k \in \mathbb{N}$,

$$f_{n,k} \to f_k \text{ as } n \to \infty \text{ in } \mathscr{E}'((0,\infty)). \tag{4.9}$$

Applying Theorem 2.2, we get that for each $k \in \mathbb{N}$,

$$\mathcal{L}_A f_{n,k} \to \mathcal{L}_A f_k \text{ as } n \to \infty \text{ in } \mathscr{E}((0,\infty)). \tag{4.10}$$

Being for each $n \in \mathbb{N}$, $\mathfrak{L}_A X_n = [\mathcal{L}_A f_{n,k}/\mathcal{L}_A \delta_k]$ and $\mathfrak{L}_A X = [\mathcal{L}_A f_k/\mathcal{L}_A \delta_k]$, again by Lemma 3.1, it follows that $\mathfrak{L}_A X_n \to \mathfrak{L}_A X$ as $n \to \infty$ in $\mathscr{B}_2$. □

It is interesting to note that the operations $*$ and $\otimes$ can be extended as binary operations on $\mathscr{B}_1$ and $\mathscr{B}_2$ by

$$\begin{aligned} [f_n/\delta_n] * [g_n/\epsilon_n] &= [(f_n * g_n)/\delta_n * \epsilon_n]\,, \\ [F_n/\mathcal{L}_A\delta_n] \otimes [G_n/\mathcal{L}_A\epsilon_n] &= [(F_n \otimes G_n)/(\mathcal{L}_A\delta_n \otimes \mathcal{L}_A\epsilon_n)]\,. \end{aligned}$$

As a consequence of Theorem 2.1, the exchange formula of generalized Lambert transform holds in the context of Boehmians as follows.

Theorem 4.6. *If $X, Y \in \mathscr{B}_1$ and $f \in \mathscr{D}((0,\infty))$ then (1) $\mathfrak{L}_A(X*Y) = \mathfrak{L}_A X \otimes \mathfrak{L}_A Y$; (2) $\mathfrak{L}_A(X * f) = \mathfrak{L}_A X \otimes \mathcal{L}_A f$.*

Theorem 4.7. *The extended Lambert transform $\mathfrak{L}_A : \mathscr{B}_1 \to \mathscr{B}_2$ is continuous with respect to the Δ-convergence.*

Proof. Let $X_n \xrightarrow{\Delta} X$ as $n \to \infty$ in $\mathscr{B}_1$. Then there exist $(\delta_n) \in \Delta_+$ and $f_n \in \mathscr{E}'((0,\infty))$ such that $(X_n - X) * \delta_n = [(f_n * \delta_k)/\delta_k]$, $\forall n \in \mathbb{N}$ and $f_n \to 0$ as $n \to \infty$ in $\mathscr{E}'((0,\infty))$. Using the continuity of the generalized Lambert transform on $\mathscr{E}'((0,\infty))$, we get $\mathcal{L}_A f_n \to 0$ as $n \to \infty$ in $\mathscr{E}((0,\infty))$. Using Theorems 2.1, 4.2, 4.6, for each $n \in \mathbb{N}$ we obtain

$$\begin{aligned} (\mathfrak{L}_A X_n - \mathfrak{L}_A X) \otimes \mathcal{L}_A \delta_n &= \mathfrak{L}_A((X_n - X) * \delta_n) \\ &= [\mathcal{L}_A(f_n * \delta_k)/\mathcal{L}_A \delta_k] \\ &= [(\mathcal{L}_A f_n \otimes \mathcal{L}_A \delta_k)/\delta_k]\,, \end{aligned}$$

and hence $\mathfrak{L}_A X_n \xrightarrow{\Delta} \mathfrak{L}_A X$ as $n \to \infty$ in $\mathscr{B}_2$. □

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