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SUBORDINATION AND SUPERORDINATION FOR CERTAIN ANALYTIC FUNCTIONS

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Abstract. In this article, we investigate results on subordination and superordination given by some authors. Motivated by earlier work and by using a method based upon the Briot-Bouquet differential subordination, we prove several subordination results related to the class $\mathcal{B}(\alpha)$. For this purpose, a class denoted by $\mathcal{B}_b^*$ is defined and some properties are obtained in the open unit disk.

1. Introduction and Definition

Let F and G be analytic functions in the open unit disk $\mathbb{D} = \{z : z \in \mathbb{C}, \, |z| < 1\}$. If $f, F \in H(\mathbb{D})$ and F is univalent in $\mathbb{D}$ we say that the function f is *subordinate* to F, or F is *superordinate* to f, written $f(z) \prec F(z)$, if $f(0) = F(0)$ and $f(\mathbb{D}) \subseteq F(\mathbb{D})$. In general, given two functions F and G, which are analytic in $\mathbb{D}$, the function F is said to be subordinate to G in $\mathbb{D}$ if there exists a function h, analytic in $\mathbb{D}$ with

$$h(0) = 0 \quad and \quad |h(z)| < 1 \quad for \;\; all \quad z \in \mathbb{D}$$

such that

$$F(z) = G(h(z)) \;\; for \;\; all \;\; z \in \mathbb{D}.$$

Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in $\mathbb{D}$. If p is analytic and satisfies the differential subordination $\varphi(p(z), zp'(z)) \prec h(z)$ then p is called a solution of the differential subordination.

The univalent function q is called a dominant of the solutions of the differential subordination, if $p \prec q$. If p and $\varphi(p(z), zp'(z))$ are univalent in $\mathbb{D}$ and satisfy the differential superordination $h(z) \prec \varphi(p(z), zp'(z))$ then p is called a solution of the differential superordination. An analytic function q is called subordinant of the solution of the differential superordination if $q \prec p$.

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Let H be the class of functions analytic in $\mathbb{D}$ and $H[a,n]$ be the subclass of H. For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we denote

$$H[a,n] = \{f \in H(\mathbb{D}) : f(z) = a + a_n z^n + \cdots\}.$$

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

and normalized by $f(0) = f'(0) - 1 = 0$, which are analytic in $\mathbb{D}$.

Also, denote

$$S^*(\alpha) = \left\{ f : f \in \mathcal{A}, \mathrm{Re}\left(\frac{zf'(z)}{f(z)} \right) > \alpha, \quad z \in \mathbb{D}; \quad 0 \le \alpha < 1 \right\}, \tag{1.2}$$

and

$$S^*_{st}(\alpha) = \left\{ f : f \in \mathcal{A}, \left| \arg\left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{2}\alpha \quad z \in \mathbb{D}; \quad 0 < \alpha \le 1 \right\} \tag{1.3}$$

be the the familiar classes starlike function of order α in $\mathbb{D}$ and strongly starlike functions of order α in $\mathbb{D}$, respectively.

We note that

$$S^*_{st}(\alpha) \subset S^*, \qquad (0 < \alpha \le 1), \quad \text{and} \quad S^*_{st}(1) = S^*.$$

A function $f \in \mathcal{A}$ is said to be a member of the class $\mathcal{B}(\alpha)$ if only if

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 - \alpha \qquad (z \in \mathbb{D}). \tag{1.4}$$

Note that the condition (1.4) is equivalent to

$$\mathrm{Re}\left\{ \frac{z^2 f'(z)}{f^2(z)} \right\} > \alpha \tag{1.5}$$

for some α $(0 \le \alpha < 1)$ and for all $z \in \mathbb{D}$.

Frasin and Darus [1] have defined the class $\mathcal{B}(\alpha)$ and investigate some interesting properties for this class. In this paper we shall give new additional results for functions of the class $\mathcal{B}(\alpha)$.

We denote by $\mathcal{B}^*$ the class of $\mathcal{A}$ define by

$$\mathcal{B}^*_b = \left\{ \mathrm{Re}\left\{ \frac{z^2 f'(z)}{bf^2(z)} + \frac{1}{b} \left(\left(1 + \frac{zf''(z)}{f'(z)} \right) - \left(\frac{2zf'(z)}{f(z)} - 1 \right) \right) \right\} > 0; \quad z \in \mathbb{D} \right\} \tag{1.6}$$

and $\mathcal{B}^*_b(\alpha)$ the class of $\mathcal{A}$ define by

$$\mathcal{B}^*_b(\alpha) = \left\{ \mathrm{Re}\left\{ \frac{z^2 f'(z)}{bf^2(z)} + \frac{1}{b} \left(\left(1 + \frac{zf''(z)}{f'(z)} \right) - \left(\frac{2zf'(z)}{f(z)} - 1 \right) \right) \right\} > \alpha; \quad z \in \mathbb{D} \right\} \tag{1.7}$$

where $b \in \mathbb{C} = \mathbb{C} \setminus \{0\}$.

Srivastava and Lashin [3] investigated the starlike and convex functions of complex order.

The main objective of the present paper to the aforementioned works is to apply a method based upon the Briot-Bouquet differential subordination and superordination in order to derive several subordination and superordination results involving analytic functions.

2. Preliminaries

In order to prove our main subordination results, we shall make use of the following known results.

Lemma 2.1. *(see [4]) Let the (nonconstant) function w be analytic in $\mathbb{D}$ and such that $w(0) = 0$. If $|w(z)|$ attains its maximum value on circle $|z| = r < 1$ at a point $z_o \in \mathbb{D}$, we have*

$$z_o w'(z) = kw(z_o),$$

where $k \geq 1$ is a real number.

Lemma 2.2. *(Miller and Mocanu [6].) Let the functions F and G be analytic in the unit disk $\mathbb{D}$ and let*

$$F(0) = G(0).$$

If the function $H(z) := zG'(z)$ is starlike in $\mathbb{D}$ and

$$zF'(z) \prec zG'(z),$$

then

$$F(z) \prec G(z) = G(0) + \int_0^z \frac{H(t)}{t} dt, \tag{2.1}$$

The function G is convex and is the best dominant in (2.1).

Lemma 2.3. *(Eenigenburg et. al [5]). Let β and γ be complex constants. Also let the function h be convex (univalent) in $\mathbb{D}$ with*

$$h(0) = 1 \quad \text{and} \quad \text{Re}(\beta h(z) + \gamma) > 0, \qquad (z \in \mathbb{D}).$$

Suppose that the function

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots.$$

is analytic in $\mathbb{D}$ and satisfies the following differential subordination:

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \tag{2.2}$$

If the differential equation:

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \qquad (q(0) := 1), \tag{2.3}$$

has a univalent solution q, then

$$p(z) \prec q(z) \prec h(z)$$

and q is the best dominant in (2.2) (that is, $p(z) \prec q(z)$) for all $p(z)$ satisfying (2.2) and if $p(z) \prec \hat{q}(z)$ for all $p(z)$ satisfying (2.2), then $q(z) \prec \hat{q}(z)$).

Lemma 2.4. *(Miller and Mocanu [7]) Let $q(z)$ be convex univalent in the unit disk $\mathbb{D}$ and $\gamma \in \mathbb{C}$. Further, assume that $\text{Re}\{\overline{\gamma}\} > 0$. If $p(z) \in H[q(0), 1] \cap Q$, with $p(z) + \gamma zp'(z)$ is univalent in $\mathbb{D}$ then $q(z) + \gamma zq'(z) \prec p(z) + \gamma zp'(z)$ implies $q(z) \prec p(z)$, and q is the best subordinate.*

Remark 2.5. *The conclusion of Lemma 2.3 can be written in the following form:*

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}\Rightarrow p(z)\prec q(z)$$

Remark 2.6. *The differential equation (2.3) has its formal solution given by*

$$q(z)=\frac{zF'(z)}{F(z)}=\frac{\beta+\gamma}{\beta}\left(\frac{H(z)}{F(z)}\right)^{\beta}-\frac{\gamma}{\beta},$$

where

$$F(z)=\left(\frac{\beta+\gamma}{z^{\gamma}}\int_0^z\{H(t)\}^{\beta}t^{\gamma-1}dt\right)^{\frac{1}{\beta}},$$

and

$$H(z)=z.\exp\left(\int_0^z\frac{h(t)-1}{t}dt\right).$$

3. Main Result

We begin with the following theorem.

Theorem 3.1. *Let the function h be univalent in $\mathbb{D}$, let h and $\operatorname{Re}(bh(z))>0$, $z\in\mathbb{D}$, $b\in\mathbb{C}=\mathbb{C}\setminus\{0\}$, also let $f\in\mathcal{A}$.*
a) *If*

$$\frac{z^2f'(z)}{bf^2}+\frac{1}{b}\left(\left(1+\frac{zf''(z)}{f'(z)}\right)-\left(\frac{2zf'(z)}{f(z)}-1\right)\right)\prec h(z), \tag{3.1}$$

then

$$\frac{z^2f'(z)}{bf^2}\prec h(z). \tag{3.2}$$

b) *If the following differential equation:*

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}=h(z)\qquad (q(0):=1),$$

has a univalent solution $q(z)$, then

$$\frac{z^2f'(z)}{bf^2}+\frac{1}{b}\left(\left(1+\frac{zf''(z)}{f'(z)}\right)-\left(\frac{2zf'(z)}{f(z)}-1\right)\right)\prec h(z)\Rightarrow\frac{z^2f'(z)}{bf^2}\prec q(z)\prec h(z), \tag{3.3}$$

and q is the best dominant in (3.3).

Proof. a) We begin by setting

$$\frac{z^2f'(z)}{bf^2(z)}=:p(z), \tag{3.4}$$

so that p has the following expansion:

$$p(z)=1+p_1z+p_2z^2+\dots.$$

By differentiating logarithmically (3.4), we obtain

$$p(z)+\frac{zp'(z)}{bp(z)}=\frac{z^2f'(z)}{bf^2}+\frac{1}{b}\left(\left(1+\frac{zf''(z)}{f'(z)}\right)-\left(\frac{2zf'(z)}{f(z)}-1\right)\right)$$

and the subordination (3.1) can be written as follows:

$$p(z)+\frac{zp'(z)}{bp(z)}\prec h(z).$$

The conclusion of the theorem would follow from Lemma 2.3 by taking

$$\beta=b \qquad \gamma=0.$$

This evidently completes the proof of Theorem 3.1. □

Theorem 3.2. *Let f be analytic in $\mathbb{D}$ such that $f(0)=0$, h be convex univalent in $\mathbb{D}$ and $h\in H[0,1]\cap Q$. Assume that*

$$\frac{z^2f'(z)}{bf^2}+\frac{1}{b}\left(\left(1+\frac{zf''(z)}{f'(z)}\right)-\left(\frac{2zf'(z)}{f(z)}-1\right)\right)$$

is univalent function in $\mathbb{D}$, where $\mathrm{Re}\{\gamma\}>0$ and $b\in\mathbb{C}=\mathbb{C}\setminus\{0\}$. If $h\in\mathcal{A}$ and the subordination

$$h(z)=q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}\prec\frac{z^2f'(z)}{bf^2}+\frac{\gamma}{b}\left(\left(1+\frac{zf''(z)}{f'(z)}\right)-\left(\frac{2zf'(z)}{f(z)}-1\right)\right),$$

holds, then

$$h(z)\prec\frac{z^2f'(z)}{bf^2(z)}\quad implies\quad h(z)\prec q(z)\prec p(z),$$

where $p(z)=\dfrac{z^2f'(z)}{bf^2(z)}$ and h is the best subordinant.

Proof. Our aim is to apply Lemma 2.4. Setting $p(z):=\frac{z^2f'(z)}{bf^2(z)}$.
Now we must show that

$$q(z)+zq'(z)\prec p(z)+zp'(z).$$

By the assumption of the theorem we have

$$h(z)=q(z)+\gamma zq'(z)\prec\ \ \frac{z^2f'(z)}{bf^2}+\frac{\gamma}{b}\left(\left(1+\frac{zf''(z)}{f'(z)}\right)-\left(\frac{2zf'(z)}{f(z)}-1\right)\right)=p(z)+\gamma zp'(z).$$

Thus in view of Lemma 2.3 and Lemma 2.4, $h(z)\prec q(z)\prec p(z)$ and h is the best subordinate. □

If we combine Theorem 3.2 together with Theorem 3.1, then we obtain the differential *sandwich-type theorem.*

Next, applying Lemma 2.1, we prove the following:

Theorem 3.3. *Let $f\in\mathcal{A}$. If*

$$\left|\frac{z^2f'(z)}{bf^2}-1+\frac{1}{b}\left(\left(1+\frac{zf''(z)}{f'(z)}\right)-\left(\frac{2zf'(z)}{f(z)}-1\right)\right)\right|<\frac{1-\alpha}{2\alpha},\qquad (z\in\mathbb{D}),\ (3.5)$$

where $\frac{1}{2}\le\alpha<1$ and $b\in\mathbb{C}=\mathbb{C}\setminus\{0\}$, then $f\in\mathcal{B}_b^*(\alpha)$.

Proof. We define $w(z)$ by

$$\frac{z^2 f'(z)}{bf^2(z)} = \frac{1+(1-2\alpha)w(z)}{1-w(z)} \quad (w(z) \neq 1), \tag{3.6}$$

we see that w is regular in $\mathbb{D}$ and $w(0) = 0$. By the logarithmic differentiations, we get from (3.6) that

$$\frac{1}{b}\left(\Big(1+\frac{zf''(z)}{f'(z)}\Big) - \Big(\frac{2zf'(z)}{f(z)} - 1\Big)\right) = \frac{(1-2\alpha)zw'(z)}{1+(1-2\alpha)w(z)} + \frac{zw'(z)}{1-w(z)}. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\begin{aligned}
&\frac{z^2 f'(z)}{bf^2} + \frac{1}{b}\left(\Big(1+\frac{zf''(z)}{f'(z)}\Big) - \Big(\frac{2zf'(z)}{f(z)} - 1\Big)\right) \\
&= \frac{1+(1-2\alpha)w(z)}{1-w(z)} + \frac{(1-2\alpha)zw'(z)}{1+(1-2\alpha)w(z)} + \frac{zw'(z)}{1-w(z)}
\end{aligned}$$

or equivalently,

$$\begin{aligned}
&\frac{z^2 f'(z)}{bf^2} - 1 + \frac{1}{b}\left(\Big(1+\frac{zf''(z)}{f'(z)}\Big) - \Big(\frac{2zf'(z)}{f(z)} - 1\Big)\right) \\
&= \frac{2(1-\alpha)w(z)}{1-w(z)}\left(1 + \frac{zw'(z)}{[1+(1-2\alpha)w(z)]w(z)}\right).
\end{aligned} \tag{3.8}$$

Suppose there exist $z_o \in \mathbb{D}$ such that

$$\max_{|z|<|z_o|} |w(z)| = |w(z_o)| = 1, \quad (w(z_o) \neq -1),$$

and then from Lemma 2.1, we have

$$z_o w'(z) = kw(z_o),$$

where $k \geq 1$ is a real number. From (3.8), we have

$$\begin{aligned}
&\left|\frac{z_o^2 f'(z_o)}{bf^2(z_o)} - 1 + \frac{1}{b}\left(\Big(1+\frac{z_o f''(z_o)}{f'(z_o)}\Big) - \Big(\frac{2z_o f'(z_o)}{f(z_o)} - 1\Big)\right)\right| \\
= &\left|\frac{2(1-\alpha)w(z_o)}{1-w(z_o)}\left(1 + \frac{z_o w'(z_o)}{[1+(1-2\alpha)w(z_o)]w(z_o)}\right)\right| \\
\geq &\left|\frac{2(1-\alpha)w(z_o)}{1-w(z_o)}\right|\left|\frac{z_o w'(z_o)}{[1+(1-2\alpha)w(z_o)]w(z_o)}\right| \\
\geq &\frac{(1-\alpha)k}{2\alpha} \\
\geq &\frac{1-\alpha}{2\alpha}
\end{aligned}$$

which contradicts our assumption (3.5). Therefore $|w(z)| < 1$ holds for all $z \in \mathbb{D}$. We finally have $f \in \mathcal{B}_b^*(\alpha)$. □

Putting $\alpha = \frac{1}{2}$ in Theorem 3.3, we have the following corollary:

Corollary 3.4. *Let $f \in \mathcal{A}$. If*

$$\left| \frac{z^2 f'(z)}{bf^2} - 1 + \frac{1}{b}\left(\left(1 + \frac{zf''(z)}{f'(z)}\right) - \left(\frac{2zf'(z)}{f(z)} - 1\right)\right)\right| < \frac{1}{2}, \qquad (z \in \mathbb{D}). \tag{3.9}$$

Then $f \in \mathcal{B}_b^*(\frac{1}{2})$.

Remark 3.5. *Setting $b = 1$ in Theorem 3.3, we arrive to Theorem 2.5 obtained by Frasin el. al.,* [2].

Theorem 3.6. *If $f \in \mathcal{B}_b^*(\alpha)$, $(0 \le \alpha < 1)$ and $\mathrm{Re}(bz + b) > 0$; $z \in \mathbb{D}$, then*

$$\frac{z^2 f'(z)}{bf^2(z)} \prec q(z)$$

where q is the best dominant given by

$$q(z) = \frac{1}{b}\left[\frac{e^{bz}}{(-bz)^{-b}\Big(\Gamma(b) + \Gamma(b, -bz)\Big)} - 1\right],$$

and

$$\Gamma(b, -bz) = \Gamma(b) + z^b \cdot {}_1F_1(b, 1 + b, bz).$$

Proof. First of all, we observe that (1.4) is equivalent to the inequality:

$$\left| \frac{z^2 f'(z)}{bf^2} + \frac{1}{b}\left(\left(1 + \frac{zf''(z)}{f'(z)}\right) - \left(\frac{2zf'(z)}{f(z)} - 1\right)\right) - 1\right| < 1, \qquad (z \in \mathbb{D}),$$

which implies that

$$\frac{z^2 f'(z)}{bf^2} + \frac{1}{b}\left(\left(1 + \frac{zf''(z)}{f'(z)}\right) - \left(\frac{2zf'(z)}{f(z)} - 1\right)\right) \prec 1 + z.$$

Thus, in Theorem 3.1, we choose

$$h(z) = 1 + z,$$

and note that

$$\mathrm{Re}(bh(z)) > 0,$$

when $z \in \mathbb{D}$, and h satisfies the hypotheses of Lemma 2.3. Consequently, in the view of Lemma 2.3 and Remark 2.6, we have

$$H(z) = z \cdot \exp\left(\int_0^z \frac{h(t) - 1}{t} dt\right),$$

which, for $h(z) = 1 + t$, yields

$$H(z) = z\mathrm{e}^z, \tag{3.10}$$

and

$$\begin{aligned} F(z) &= \left[b \int_0^z \left(\frac{\{H(t)\}^b}{t}\right) dt\right]^{1/b} \\ &= \left[b \int_0^z \left(\frac{\mathrm{e}^{bt}}{t^{1-b}}\right) dt\right]^{1/b}. \end{aligned}$$

By using the software MAPLE, F can be simplified to the following form:

$$F(z) = \Big(- z^{b-1}(-bz)^{-b}\Big(- zb\Gamma(b) + zb\Gamma(b, -bz)\Big)\Big)^{1/b}. \tag{3.11}$$

From (3.10) and (3.11), we obtain

$$q(z) = \frac{1}{b}\left[\frac{\mathrm{e}^{bz}}{(-bz)^{-b}(\Gamma(b) + \Gamma(b, -bz))} - 1\right].$$

The proof of Theorem 3.6 is complete. □

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