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SUBORDINATION AND SUPERORDINATION FOR FUNCTIONS BASED ON DZIOK-SRIVASTAVA LINEAR OPERATOR

RABHA W. IBRAHIM AND MASLINA DARUS

Abstract. In this article, we obtain some subordination and superordination results involving Dziok-Srivastava linear operator and fractional integral operator for certain normalized analytic functions in the open unit disk.

1. Introduction and Preliminaries.

Let $\mathcal{H}$(U) denote the class of analytic functions in the unit disk

$$U := \{z \in \mathbb{C}, |z| < 1\}.$$

For n positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] := \{f \in H(U) : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + ..., z \in U\},$$

and $\mathcal{A}_n = \{f \in H(U) : f(z) = z + a_n z^n + a_{n+1}z^{n+1} + ..., z \in U\}$ with $\mathcal{A}_1 = \mathcal{A}$. A function $f \in \mathcal{H}[a,n]$ is convex in U if it is univalent and $f(U)$ is convex. It is well-known that f is convex if and only if $f(0) \neq 0$ and

$$\Re\{1 + \frac{zf''(z)}{f'(z)}\} > 0, \ \ z \in U.$$

Definition 1.1. [1] Denote by $\mathbf{Q}$ the set of all functions $f(z)$ that are analytic and injective on $\overline{U} - E(f)$ where

$$E(f) := \{\zeta \in \partial U : lim_{z\to\zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U - E(f)$.

Given two functions F and G in the unit disk U, the function F is *subordinate* to G, written $F \prec G$, if G is univalent, $F(0) = G(0)$ and $F(U) \subset G(U)$. Alternatively, given two functions F and G, which are analytic in U, the function F is said to be subordinate to G in U if there exists a function h, analytic in U with

$$h(0) = 0 \quad and \quad |h(z)| < 1 \ \ for \ all \ z \in U$$

such that

$$F(z) = G(h(z)) \ \ for \ all \ z \in U.$$

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Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the differential subordination $\phi(p(z)), zp'(z)) \prec h(z)$, then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, if $p \prec q$. If p and $\phi(p(z)), zp'(z))$ are univalent in U and satisfy the differential superordination $h(z) \prec \phi(p(z)), zp'(z))$, then p is called a solution of the differential superordination. An analytic function q is called subordinant of the solution of the differential superordination if $q \prec p$.

We shall need the following results:

Lemma 1.1. [2] Let q be univalent in the unit disk U, and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) := zq'(z)\phi(q(z))$, $h(z) := \theta(q(z)) + Q(z)$. Suppose that
1. $Q(z)$ is starlike univalent in U, and
2. $\Re\frac{zh'(z)}{Q(z)} > 0$ for $z \in U$.
If $\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$, then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 1.2. [3] Let q be convex univalent in the unit disk U and ψ and $\gamma \in \mathbb{C}$ with $\Re\{1+\frac{zq''(z)}{q'(z)}+\frac{\psi}{\gamma}\} > 0$. If $p(z)$ is analytic in U and $\psi p(z)+\gamma zp'(z) \prec \psi q(z)+\gamma zq'(z)$, then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 1.3. [4] Let q be convex univalent in the unit disk U and ϑ and φ be analytic in a domain D containing $q(U)$. Suppose that
1. $zq'(z)\varphi(q(z))$ is starlike univalent in U, and
2. $\Re\{\frac{\vartheta'(q(z))}{\varphi(q(z))}\} > 0$ for $z \in U$.
If $p(z) \in \mathcal{H}[q(0), 1] \cap \mathbf{Q}$, with $p(U) \subseteq D$ and $\vartheta(p(z)) + zp'(z)\varphi(z)$ is univalent in U and $\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z))$ then $q(z) \prec p(z)$ and q is the best subordinant.

Lemma 1.4. [1] Let q be convex univalent in the unit disk U and $\gamma \in \mathbb{C}$. Further, assume that $\Re\{\overline{\gamma}\} > 0$. If $p(z) \in \mathcal{H}[q(0), 1] \cap \mathbf{Q}$, with $p(z) + \gamma zp'(z)$ is univalent in U then $q(z) + \gamma zq'(z) \prec p(z) + \gamma zp'(z)$ implies $q(z) \prec p(z)$ and q is the best subordinant.

For two functions $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z+\sum_{n=2}^{\infty} b_n z^n$, the Hadamard product (or convolution) of f and g defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z).$$

For $\alpha_j \in \mathbb{C}$ $(j = 1, 2, ..., l)$ and $\beta_j \in \mathbb{C}\backslash\{0, -1, -2, ...\}$ $(j = 1, 2, ..., m)$, the generalized hypergeometric function ${}_lF_m(\alpha_1, ..., \alpha_l; \beta_1, ...\beta_m; z)$ is defined by the infinite series

$${}_lF_m(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n ... (\alpha_l)_n}{(\beta_1)_n ... (\beta_m)_n} \frac{z^n}{n!}$$

$$(l \le m + 1 : l, m \in \mathbb{N}_0 := \{0, 1, 2, ...\})$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a + n)}{\Gamma(a)} = \begin{cases} 1, & (n = 0); \\ a(a + 1)(a + 2)...(a + n - 1), & (n \in \mathbb{N}). \end{cases}$$

Corresponding to the function

$$h(\alpha_1, ..., \alpha_l; \beta_1, ...\beta_m; z) := z_l F_m(\alpha_1, ..., \alpha_l; \beta_1, ...\beta_m; z),$$

the Dziok-Srivastava operator (see [5-7]) $H^l_m(\alpha_1, ..., \alpha_l; \beta_1, ...\beta_m)$ is defined by the Hadamard product

$$\begin{aligned} H^l_m(\alpha_1, ..., \alpha_l; \beta_1, ...\beta_m) f(z) &:= h(\alpha_1, ..., \alpha_l; \beta_1, ...\beta_m; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} ... (\alpha_l)_{n-1}}{(\beta_1)_{n-1} ... (\beta_m)_{n-1}} \frac{a_n z^n}{(n-1)!} \\ &:= H^l_m[\alpha_1] f(z). \end{aligned}$$

We can verify that

$$z(H^l_m[\alpha_1] f(z))' = \alpha_1 H^l_m[\alpha_1 + 1] f(z) - (\alpha_1 - 1) H^l_m[\alpha_1] f(z).$$

Special cases of the Dziok-Srivastava linear operator include the Hohlov linear operator [8], the Carlson-Shaffer linear operator $L(a, c)$ [9], the Ruscheweyh derivative operator $\mathcal{D}^n$[10], the generalized Bernardi-Libera-Livingston linear integral operator [11] and the Srivastava-Owa fractional derivative operator [12]:

Definition 1.2. The fractional derivative of order α is defined, for a function f by

$$D^\alpha_z f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta; \quad 0 \le \alpha < 1,$$

where the function f is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $log(z-\zeta)$ to be real when$(z-\zeta) > 0$.

Definition 1.3. The fractional integral of order α is defined, for a function f, by

$$I^\alpha_z f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta; \quad \alpha > 0,$$

where the function f is analytic in simply-connected region of the complex z-plane ($\mathbb{C}$) containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $log(z-\zeta)$ to be real when$(z-\zeta) > 0$.

Remark 1.1. [12]

$$D^\alpha_z \{z^\mu\} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} \{z^{\mu-\alpha}\}, \ \ \mu > -1; \ 0 \le \alpha < 1$$

and

$$I^\alpha_z \{z^\mu\} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} \{z^{\mu+\alpha}\}, \ \ \mu > -1; \ \alpha > 0.$$

The main object of the present paper is to find the sufficient conditions for certain normalized analytic functions f, g to satisfy

$$[\frac{I^\alpha_z H^l_m[\alpha_1] g_1(z)}{\rho_\alpha(z)}]^\mu \prec [\frac{I^\alpha_z H^l_m[\alpha_1] f(z)}{\rho_\alpha(z)}]^\mu \prec [\frac{I^\alpha_z H^l_m[\alpha_1] g_2(z)}{\rho_\alpha(z)}]^\mu$$

and

$$q_1(z) \prec [\frac{I^\alpha_z H^l_m[\alpha_1] f(z)}{\rho_\alpha(z)}]^\mu \prec q_2(z), \ \rho_\alpha(z) \neq 0, \ \ z \in U$$

where $\mu \geq 1$, q_1 and q_2 are given univalent functions in U. Also, we obtain the results as special cases. Further, in this paper, we study the existence of univalent solution for the fractional differential equation

$$D_z^\alpha \rho_\alpha(z)u(z) = H_m^l[\alpha_1]f(z), \tag{1.1}$$

subject to the initial condition $u(0) = 0$, where $u : U \to \mathbb{C}$ is an analytic function for all $z \in U$, $\rho : U \to \mathbb{C}\backslash\{0\}$ is an analytic functions in $z \in U$ and $f : U \to \mathbb{C}$ is a univalent function in U. The existence is obtained by applying Schauder fixed point theorem. Moreover, we discuss some properties of this solution involving fractional differential subordination. The following results are used in the sequel.

Theorem 1.1. (Arzela-Ascoli) (see [13]) Let E be a compact metric space and $\mathcal{C}(E)$ be the Banach space of real or complex valued continuous functions normed by

$$\|f\| := sup_{t\in E}|f(t)|.$$

If $A = \{f_n\}$ is a sequence in $\mathcal{C}(E)$ such that f_n is uniformly bounded and equi-continuous, then $\overline{A}$ is compact.

Let M be a subset of Banach space X and $A : M \to M$ an operator. The operator A is called *compact* on the set M if it carries every bounded subset of M into a compact set. If A is continuous on M (that is, it maps bounded sets into bounded sets) then it is said to be *completely continuous* on M.

Theorem 1.2. (Schauder) (see [14]) Let X be a Banach space, $M \subset X$ a nonempty closed bounded convex subset and $P : M \to M$ is compact. Then P has a fixed point.

Recently, the subordination and superordination containing the Dziok-Srivastava linear operator are studied by many authors [15].

2. Subordination and superordination.

In this section, we study some important properties of the fractional differential and integral operators D_z^α, I_z^α, given by the authors [16] which are useful in the next results of the subordination and superordination.

Theorem 2.1[16] For $\alpha, \in (0, 1]$ and f is a continuous function, then

$$1 - DI_z^\alpha f(z) = \frac{(z)^{\alpha-1}}{\Gamma(\alpha)}f(0) + I_z^\alpha Df(z);\ \ D = \frac{d}{dz}.$$

$$2 - I_z^\alpha D_z^\alpha f(z) = D_z^\alpha I_z^\alpha f(z) = f(z).$$

But, first we consider the subordination results involving Dziok-Srivastava linear operator and fractional integral operator as the following:

Theorem 2.2. Let f, g be analytic in U. $[\frac{I_z^\alpha H_m^l[\alpha_1]g(z)}{\rho_\alpha(z)}]^\mu$ be univalent in U such that $\frac{I_z^\alpha H_m^l[\alpha_1]g(z)}{\rho_\alpha(z)} \neq 0$ and $z([\frac{I_z^\alpha H_m^l[\alpha_1]g(z)}{\rho_\alpha(z)}]^\mu)'$ be starlike univalent in U. If the subordination

$$[\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu\{1 + \mu(\frac{zI_z^\alpha[H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)} - \frac{z\rho'(z)}{\rho(z)})\}$$

$$\prec [\frac{I_z^\alpha H_m^l[\alpha_1]g(z)}{\rho_\alpha(z)}]^\mu\{1+\mu(\frac{zI_z^\alpha[H_m^l[\alpha_1]g(z)]'}{I_z^\alpha H_m^l[\alpha_1]g(z)}-\frac{z\rho'(z)}{\rho(z)}\}$$

holds and

$$\Re\{\frac{zG'(z)}{G(z)}+(\mu-1)\frac{zG(z)\rho_\alpha(z)}{I_z^\alpha H_m^l[\alpha_1]g(z)}\}>0,\ \ z\in U,$$

where

$$G(z):=[\frac{I_z^\alpha H_m^l[\alpha_1]g(z)}{\rho_\alpha(z)}]'.$$

Then

$$[\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu\prec[\frac{I_z^\alpha H_m^l[\alpha_1]g(z)}{\rho_\alpha(z)}]^\mu$$

and $[\frac{I_z^\alpha H_m^l[\alpha_1]g}{\rho_\alpha}]^\mu$ is the best dominant.

Proof. Setting

$$p(z):=[\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu,\ \ q(z):=[\frac{I_z^\alpha H_m^l[\alpha_1]g(z)}{\rho_\alpha(z)}]^\mu.$$

Our aim is to apply Lemma 1.1. First we show that $\Re\{1+\frac{zq''(z)}{q'(z)}\}>0$.

$$\Re\{1+\frac{zq''(z)}{q'(z)}\}=\Re\{1+\frac{zG'(z)}{G(z)}+(\mu-1)\frac{G(z)\rho_\alpha(z)}{I_z^\alpha H_m^l[\alpha_1]g(z)}\}>0.$$

Assume that

$$\theta(\omega):=\omega\ \ \textit{and}\ \ \phi(\omega):=1,$$

it can easily be observed that θ,ϕ are analytic in $\mathbb{C}$. Also, we let

$$Q(z):=zq'(z)\phi(z)=zq'(z),$$

$$h(z):=\theta(q(z))+Q(z)=q(z)+zq'(z).$$

By the assumptions of the theorem we find that Q is starlike univalent in U and that

$$\Re\{\frac{zh'(z)}{Q(z)}\}=\Re\{2+\frac{zq''(z)}{q'(z)}\}>0.$$

By using Theorem 2.1, a computation shows

$$\begin{aligned}p(z)+zp'(z)&=[\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu\{1+\mu(\frac{zI_z^\alpha[H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)}-\frac{z\rho'(z)}{\rho(z)})\}\\&\prec[\frac{I_z^\alpha H_m^l[\alpha_1]g(z)}{\rho_\alpha(z)}]^\mu\{1+\mu(\frac{zI_z^\alpha[H_m^l[\alpha_1]g(z)]'}{I_z^\alpha H_m^l[\alpha_1]g(z)})-\frac{z\rho'(z)}{\rho(z)})\}\\&=q(z)+zq'(z).\end{aligned}$$

Thus in view of Lemma 1.1, $p(z)\prec q(z)$ and q is the best dominant.

Corollary 2.1. Let f,g be analytic in U. $[\frac{I_z^\alpha L(a,c)g}{\rho_\alpha}]^\mu$ be univalent in U and $z([\frac{I_z^\alpha L(a,c)g}{\rho_\alpha}]^\mu)'$ be starlike univalent in U. If the subordination

$$[\frac{I_z^\alpha L(a,c)f(z)}{\rho_\alpha(z)}]^\mu\{1+\mu(\frac{zI_z^\alpha[L(a,c)f(z)]'}{I_z^\alpha L(a,c)f(z)}-\frac{z\rho'(z)}{\rho(z)})\}$$

$$\prec[\frac{I_z^\alpha L(a,c)g(z)}{\rho_\alpha(z)}]^\mu\{1+\mu(\frac{zI_z^\alpha[L(a,c)g(z)]'}{I_z^\alpha L(a,c)g(z)}-\frac{z\rho'(z)}{\rho(z)})\}$$

holds and
$$\Re\{\frac{zG'(z)}{G(z)}+(\mu-1)\frac{zG(z)\rho_\alpha(z)}{I_z^\alpha L(a,c)g(z)}\}>0,\ \ z\in U,$$
where
$$G(z):=[\frac{I_z^\alpha L(a,c)g(z)}{\rho_\alpha(z)}]'.$$
Then
$$[\frac{I_z^\alpha L(a,c)f(z)}{\rho_\alpha(z)}]^\mu\prec[\frac{I_z^\alpha L(a,c)g(z)}{\rho_\alpha(z)}]^\mu$$
and $[\frac{I_z^\alpha L(a,c)g}{\rho_\alpha}]^\mu$ is the best dominant.

Proof. By putting $l=2, m=1, \alpha_1=a, \alpha_2=1$ and $\beta_1=c$ in Theorem 2.2.

Corollary 2.2. Let f,g be analytic in U, $[\frac{I_z^\alpha g}{\rho_\alpha}]^\mu$ be univalent in U such that $\frac{I_z^\alpha H_m^l[\alpha_1]g(z)}{\rho_\alpha(z)}\neq 0$ and $z([\frac{I_z^\alpha g}{\rho_\alpha}]^\mu)'$ be starlike univalent in U. If the subordination
$$[\frac{I_z^\alpha f(z)}{\rho_\alpha(z)}]^\mu\{1+\mu(\frac{zI_z^\alpha[f(z)]'}{I_z^\alpha f(z)}-\frac{z\rho'(z)}{\rho(z)})\}\prec[\frac{I_z^\alpha g(z)}{\rho_\alpha(z)}]^\mu\{1+\mu(\frac{zI_z^\alpha[g(z)]'}{I_z^\alpha g(z)}-\frac{z\rho'(z)}{\rho(z)})\}$$
holds and
$$\Re\{\frac{zG'(z)}{G(z)}+(\mu-1)\frac{zG(z)\rho_\alpha(z)}{I_z^\alpha g(z)}\}>0,\ \ z\in U,\ \ \textit{where}\ \ G(z):=[\frac{I_z^\alpha g(z)}{\rho_\alpha(z)}]'.$$
Then
$$[\frac{I_z^\alpha f(z)}{\rho_\alpha(z)}]^\mu\prec[\frac{I_z^\alpha g(z)}{\rho_\alpha(z)}]^\mu$$
and $[\frac{I_z^\alpha g}{\rho_\alpha}]^\mu$ is the best dominant.

Proof. By putting $l=1, m=0, \alpha_1=1$, in Theorem 2.2.

Theorem 2.3. Let f,g be analytic in U, q be convex univalent in U with $\Re\{1+\frac{zq''(z)}{q'(z)}+\frac{1}{\gamma}\}$, $\gamma\in\mathbb{C}$ and $[\frac{I_z^\alpha H_m^l[\alpha_1]f}{\rho_\alpha}]^\mu$ be analytic in U. If the subordination
$$[\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu\{1+\mu\gamma(\frac{zI_z^\alpha[H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)}-\frac{z\rho'(z)}{\rho(z)})\}\prec q(z)+\gamma zq'(z)$$
holds. Then
$$[\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu\prec q(z)$$
and q is the best dominant.

Proof. Setting
$$p(z):=[\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu.$$
Our aim is to applied Lemma 1.2. Let $\psi:=1$, since
$$\begin{aligned}p(z)+\gamma zp'(z)&=[\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu+\gamma z([\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu)'\\&=[\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu\{1+\mu\gamma(\frac{zI_z^\alpha[H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)}-\frac{z\rho'(z)}{\rho(z)})\}\\&\prec q(z)+\gamma zq'(z)\end{aligned}$$

then, in view of Lemma 1.2, $p(z) \prec q(z)$ and q is the best dominant.

Corollary 2.3. Let f, g be analytic in U, $-1 \leq B \leq A \leq 1$, $q(z) := [\frac{1+Az}{1+Bz}]^\mu$ with $\Re\{1 + \frac{zq''(z)}{q'(z)} + \frac{1}{\gamma}\}$, $\gamma \in \mathbb{C}$ and $[\frac{I_z^\alpha H_m^l[\alpha_1]f}{\rho_\alpha}]^\mu$ be analytic in U. If the subordination

$$[\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu\{1+\mu(\frac{zI_z^\alpha[H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)}-\frac{z\rho'(z)}{\rho(z)})\} \prec [\frac{1+Az}{1+Bz}]^\mu\{1+\frac{\mu\gamma z(A-B)}{(1+Az)(1+Bz)}\}$$

holds. Then

$$[\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu \prec [\frac{1+Az}{1+Bz}]^\mu, \ \ -1 \leq B < A \leq 1$$

and $[\frac{1+Az}{1+Bz}]^\mu$ is the best dominant.

Next, applying Lemma 1.3 and Lemma 1.4 respectively, to obtain the following theorems.

Theorem 2.4. Let f, g be analytic in U,$[\frac{I_z^\alpha H_m^l[\alpha_1]g}{\rho_\alpha}]^\mu$ be convex univalent in U such that $\frac{I_z^\alpha H_m^l[\alpha_1]g(z)}{\rho_\alpha(z)} \neq 0$, $z([\frac{I_z^\alpha H_m^l[\alpha_1]g}{\rho_\alpha}]^\mu)'$ be starlike univalent in U and $(z[\frac{I_z^\alpha H_m^l[\alpha_1]f}{\rho_\alpha}]^\mu)'$ be univalent in U. If the subordination

$$[\frac{I_z^\alpha H_m^l[\alpha_1]g(z)}{\rho_\alpha(z)}]^\mu\{1+\mu(\frac{zI_z^\alpha[H_m^l[\alpha_1]g(z)]'}{I_z^\alpha H_m^l[\alpha_1]g(z)}-\frac{z\rho'(z)}{\rho(z)})\}$$

$$\prec [\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu\{1+\mu(\frac{zI_z^\alpha[H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)}-\frac{z\rho'(z)}{\rho(z)})\}$$

holds and $[\frac{z^{\alpha-1}}{\Gamma(\alpha)}]^\mu[\frac{H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu \in \mathcal{H}[0,1] \cap \mathbf{Q}$. Then

$$[\frac{I_z^\alpha H_m^l[\alpha_1]g(z)}{\rho_\alpha(z)}]^\mu \prec [\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu$$

and $[\frac{I_z^\alpha H_m^l[\alpha_1]g}{\rho_\alpha}]^\mu$ is the best subordinant.

Proof. Setting

$$p(z) := [\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu, \ \ q(z) := [\frac{I_z^\alpha H_m^l[\alpha_1]g(z)}{\rho_\alpha(z)}]^\mu.$$

Our aim is to apply Lemma 1.3. By taking

$$\vartheta(\omega) := \omega \ \ and \ \ \varphi(\omega) := 1,$$

it can easily observed that ϑ, φ are analytic in $\mathbb{C}$. Thus

$$\Re\{\frac{\vartheta'(q(z))}{\varphi(q(z))}\} = 1 > 0.$$

Now we must show that

$$q(z) + zq'(z) \prec p(z) + zp'(z).$$

a computation shows that

$$\begin{aligned} q(z) + zq'(z) &= [\frac{I_z^\alpha H_m^l[\alpha_1]g(z)}{\rho_\alpha(z)}]^\mu\{1+\mu(\frac{zI_z^\alpha[H_m^l[\alpha_1]g(z)]'}{I_z^\alpha H_m^l[\alpha_1]g(z)} - \frac{z\rho'(z)}{\rho(z)})\} \\ &\prec [\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu\{1+\mu(\frac{zI_z^\alpha[H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)} - \frac{z\rho'(z)}{\rho(z)})\} \\ &= p(z) + zp'(z). \end{aligned}$$

Thus in view of Lemma 1.3, $q(z) \prec p(z)$ and p is the best subordinant.

Theorem 2.5. Let f, g be analytic in U, q be convex univalent in U, $[\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu \in \mathcal{H}[0,1] \cap \mathbf{Q}$ and

$$[\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu\{1+\mu\gamma(\frac{zI_z^\alpha[H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)} - \frac{z\rho'(z)}{\rho(z)})\}, \ \Re\{\overline{\gamma}\} > 0,$$

be univalent in U. If the subordination

$$q(z) + \gamma zq'(z) \prec [\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu\{1+\mu\gamma(\frac{zI_z^\alpha[H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)} - \frac{z\rho'(z)}{\rho(z)})\}$$

holds. Then

$$q(z) \prec [\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu$$

and q is the best subordinant.

Proof. Setting

$$p(z) := [\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu.$$

Our aim is to apply Lemma 1.4. Since

$$\begin{aligned} q(z) + \gamma zq'(z) &= [\frac{I_z^\alpha H_m^l[\alpha_1]g(z)}{\rho_\alpha(z)}]^\mu\{1+\mu\gamma(\frac{zI_z^\alpha[H_m^l[\alpha_1]g(z)]'}{I_z^\alpha H_m^l[\alpha_1]g(z)} - \frac{z\rho'(z)}{\rho(z)})\} \\ &\prec [\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu\{1+\mu\gamma(\frac{zI_z^\alpha[H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)} - \frac{z\rho'(z)}{\rho(z)})\} \\ &= p(z) + \gamma zp'(z) \end{aligned}$$

then, in view of Lemma 1.4, $q(z) \prec p(z)$ and q is the best subordinant.

Combining the results of differential subordination and superordination, we state the following sandwich theorems.

Theorem 2.6. Let f, g_1, g_2 be analytic in U, $[\frac{I_z^\alpha H_m^l[\alpha_1]g_1}{\rho_\alpha}]^\mu$ be convex univalent in U such that $\frac{I_z^\alpha H_m^l[\alpha_1]g(z)}{\rho_\alpha(z)} \neq 0$, $z([\frac{I_z^\alpha H_m^l[\alpha_1]g_1}{\rho_\alpha}]^\mu)'$, $z([\frac{I_z^\alpha H_m^l[\alpha_1]g_2}{\rho_\alpha}]^\mu)'$ be starlike univalent in U and let $(z[\frac{I_z^\alpha H_m^l[\alpha_1]f}{\rho_\alpha}]^\mu)'$, $[\frac{I_z^\alpha H_m^l[\alpha_1]g_2}{\rho_\alpha}]^\mu$ be univalent in U. If the subordination

$$[\frac{I_z^\alpha H_m^l[\alpha_1]g_1(z)}{\rho_\alpha(z)}]^\mu\{1+\mu(\frac{zI_z^\alpha[H_m^l[\alpha_1]g_1(z)]'}{I_z^\alpha H_m^l[\alpha_1]g_1(z)} - \frac{z\rho'(z)}{\rho(z)})\}$$

$$\prec [\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^\mu\{1+\mu(\frac{zI_z^\alpha[H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)} - \frac{z\rho'(z)}{\rho(z)})\}$$

$$\prec [\frac{I_z^{\alpha}H_m^l[\alpha_1]g_2(z)}{\rho_\alpha(z)}]^{\mu}\{1+\mu(\frac{zI_z^{\alpha}[H_m^l[\alpha_1]g_2(z)]'}{I_z^{\alpha}H_m^l[\alpha_1]g_2(z)}-\frac{z\rho'(z)}{\rho(z)})\}$$

holds, $[\frac{z^{\alpha-1}}{\Gamma(\alpha)}]^{\mu}[\frac{H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^{\mu} \in \mathcal{H}[0,1]\cap \mathbf{Q}$ and

$$\Re\{\frac{zG_2'(z)}{G_2(z)}+(\mu-1)\frac{zG_2(z)\rho_\alpha(z)}{I_z^{\alpha}H_m^l[\alpha_1]g_2(z)}\}>0,\ \ z\in U,$$

where

$$G_2(z):=[\frac{I_z^{\alpha}H_m^l[\alpha_1]g_2(z)}{\rho_\alpha(z)}]'.$$

Then

$$[\frac{I_z^{\alpha}H_m^l[\alpha_1]g_1(z)}{\rho_\alpha(z)}]^{\mu}\prec[\frac{I_z^{\alpha}H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^{\mu}\prec[\frac{I_z^{\alpha}H_m^l[\alpha_1]g_2(z)}{\rho_\alpha(z)}]^{\mu}$$

and $[\frac{I_z^{\alpha}H_m^l[\alpha_1]g_1}{\rho_\alpha}]^{\mu}$ and $[\frac{I_z^{\alpha}H_m^l[\alpha_1]g_2}{\rho_\alpha}]^{\mu}$ are respectively the best subordinant and dominant.

Theorem 2.7. Let $f, g_1, g_2 \in \mathcal{A}$, q_1, q_2 be convex univalent in U, with $\Re\{1+\frac{zq_2''(z)}{q_2'(z)}+\frac{1}{\gamma}\}$, $\gamma\in\mathbb{C}$, $[\frac{I_z^{\alpha}H_m^l[\alpha_1]f}{\rho_\alpha}]^{\mu}\in\mathcal{H}[0,1]\cap\mathbf{Q}$, and analytic in U and

$$[\frac{I_z^{\alpha}H_m^l[\alpha_1]f}{\rho_\alpha}]^{\mu}\{1+\mu(\frac{zI_z^{\alpha}[H_m^l[\alpha_1]f]'}{I_z^{\alpha}H_m^l[\alpha_1]f}-\frac{z\rho'}{\rho})\},\ \ \Re\{\overline{\gamma}\}>0,$$

be univalent in U. If the subordination

$$q_1(z)+\gamma zq_1'(z)\prec[\frac{I_z^{\alpha}H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^{\mu}\{1+\mu\gamma(\frac{zI_z^{\alpha}[H_m^l[\alpha_1]f(z)]'}{I_z^{\alpha}H_m^l[\alpha_1]f(z)}-\frac{z\rho'(z)}{\rho(z)})\}\prec q_2(z)+\gamma zq_2'(z)$$

holds. Then

$$q_1(z)\prec[\frac{I_z^{\alpha}H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)}]^{\mu}\prec q_2(z)$$

and q_1, q_2 are respectively the best subordinant and the best dominant.

3. Existence of univalent solution.

Let $\mathcal{B}:=\mathcal{C}[U,\mathbb{C}]$ be a Banach space of all continuous functions on U endowed with the sup. norm

$$\|u\|:=sup_{z\in U}|u(z)|.$$

By using the properties in Theorem 2.1, we can easily obtain the following result:

Lemma 3.1. If the function $f\in\mathcal{A}$, then the initial value problem (1.1) is equivalent to the nonlinear integral equation

$$u(z)=\frac{1}{\rho_\alpha(z)}\int_0^z\frac{(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)}H_m^l[\alpha_1]f(\zeta)d\zeta. \tag{3.1}$$

In other words, every solution of the equation (3.1) is also a solution of the initial value problem (1.1) and vice versa.

Theorem 3.1.(Existence) Assume that $\frac{1}{|\rho_\alpha(z)|}\le M$; $M>0$. Then there exists a univalent function $u:U\to\mathbb{C}$ solving the problem (1.1).

Proof. Define an operator $P:\mathbb{C}\to\mathbb{C}$

$$(Pu)(z):=\frac{1}{\rho_\alpha(z)}\int_0^z\frac{(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)}H_m^l[\alpha_1]f(\zeta)d\zeta. \tag{3.2}$$

Denotes $B_n := \frac{(\alpha_1)_{n-1}...(\alpha_l)_{n-1}}{(\beta_1)_{n-1}...(\beta_m)_{n-1}} \frac{1}{(n-1)!}$. Our aim is to apply Theorem 2.1. First we show that P is bounded operator:

$$\begin{aligned}|(Pu)(z)| &= |\frac{1}{\rho_\alpha(z)} \int_0^z \frac{(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)} H_m^l[\alpha_1]f(\zeta)d\zeta| \\ &\le |\frac{1}{\rho_\alpha(z)}|| \int_0^z \frac{(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)} H_m^l[\alpha_1]f(\zeta)d\zeta| \\ &< M(1+\sum_{n=2}^{\infty} B_n|a_n|)| \int_0^z \frac{(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)} d\zeta| \\ &= M(1+\sum_{n=2}^{\infty} B_n|a_n|) \frac{|z^\alpha|}{\Gamma(\alpha+1)} \\ &< \frac{M(1+\sum_{n=2}^{\infty} B_n|a_n|)}{\Gamma(\alpha+1)}\end{aligned}$$

Thus we obtain that

$$\|P\| < \frac{M(1+\sum_{n=2}^{\infty} B_n|a_n|)}{\Gamma(\alpha+1)} := r$$

that is $P : B_r \to B_r$. Then P maps B_r into itself. Now we proceed to prove that P is equicontinuous. For $z_1, z_2 \in U$ such that $z_1 \neq z_2$, $|z_2 - z_1| < \delta,\ \ \delta > 0$ Then for all $u \in S$, where

$$S := \{u \in \mathbb{C}, : |u| \le \frac{M(1+\sum_{n=2}^{\infty} B_n|a_n|)}{\Gamma(\alpha+1)} := r,\, r > 0\},$$

we obtain

$$\begin{aligned}&|(Pu)(z_1) - (Pu)(z_2)| \\ &\le M(1+\textstyle\sum_{n=2}^{\infty} B_n|a_n|)| \int_0^{z_1} \frac{(z_1-\zeta)^{\alpha-1}}{\Gamma(\alpha)} d\zeta - \int_0^{z_2} \frac{(z_2-\zeta)^{\alpha-1}}{\Gamma(\alpha)} d\zeta| \\ &\le M(1+\textstyle\sum_{n=2}^{\infty} B_n|a_n|)| \int_0^{z_1} \frac{[(z_1-\zeta)^{\alpha-1}-(z_2-\zeta)^{\alpha-1}]}{\Gamma(\alpha)} d\zeta + \int_{z_1}^{z_2} \frac{(z_2-\zeta)^{\alpha-1}}{\Gamma(\alpha)} d\zeta| \\ &= \frac{M(1+\sum_{n=2}^{\infty} B_n|a_n|)}{\Gamma(\alpha+1)} |[2(z_2-z_1)^\alpha + z_2^\alpha - z_1^\alpha]| \\ &< \frac{2M(1+\sum_{n=2}^{\infty} B_n|a_n|)}{\Gamma(\alpha+1)} |z_2 - z_1|^\alpha \\ &< \frac{2M(1+\sum_{n=2}^{\infty} B_n|a_n|)}{\Gamma(\alpha+1)} \delta^\alpha,\end{aligned}$$

which is independent on u. Hence P is an equicontinuous mapping on S. By the assumption of the theorem we can show that P is a univalent function (see [17]). The Arzela-Ascoli theorem yields that every sequence of functions from $P(S)$ has got a uniformly convergent subsequence, and therefore $P(S)$ is relatively compact. Schauder's fixed point theorem asserts that P has a fixed point. By construction, a fixed point of P is a univalent solution of the initial value problem (1.1).

The next theorems show the relation between univalent solutions and the subordination for a class of fractional differential problem.

Theorem 3.2. Let the assumptions of Theorem 2.6 be satisfied. Then univalent solutions u_1, u, u_2, of the problem

$$D_z^\alpha u(z) = F(z, u(z)), \tag{3.3}$$

subject to the initial condition $u(0) = 0$, where $u : U \to \mathbb{C}$ is an analytic function for all $z \in U$ and $F : U \times \mathbb{C} \to \mathbb{C}$, is an analytic functions in $z \in U$, are satisfying the subordination $u_1 \prec u \prec u_2$.

Proof. Setting $\mu = 1$ and let $F(z, u_1(z)) := \frac{H^l_m[\alpha_1]g_1(z)}{\rho_\alpha(z)}$, $F(z, u(z)) := \frac{H^l_m[\alpha_1]f(z)}{\rho_\alpha(z)}$, and $F(z, u_2(z)) := \frac{H^l_m[\alpha_1]g_2(z)}{\rho_\alpha(z)}$ where $\rho_\alpha(z) \neq 0, \ \forall\, z \in U$.

Theorem 3.3. Let the assumptions of Theorem 2.7 be satisfied. Then every univalent solution $u(z)$ of the problem (3.3) satisfies the subordination $q_1(z) \prec u(z) \prec q_2(z)$, where $q_1(z)$ and $q_2(z)$ are univalent function in U.

Proof. Setting $\mu = 1, \ F(z, u(z)) := \frac{H^l_m[\alpha_1]f(z)}{\rho_\alpha(z)}$.

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Rabha W. Ibrahim and Maslina Darus
School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600 Selangor D.E., Malaysia
E-mail address: saibahmath@yahoo.com
E-mail address: maslina@ukm.my (corresponding author)