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CERTAIN NEW CLASSES OF ANALYTIC FUNCTIONS DEFINED BY USING THE SALAGEAN OPERATOR

(DEDICATED IN OCCASION OF THE 70-YEARS OF PROFESSOR HARI M. SRIVASTAVA)

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Abstract. New classes of analytic functions defined by using the Salagean operator are introduced and studied. We provide coefficient inequalities, distortion theorems, extreme points and radius of close-to-convexity, starlikeness and convexity of these classes.

1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$.

Let $\mathcal{A}^+$ denote the class of functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \ (a_j \geq 0), \tag{1.2}$$

which are analytic in $\mathbb{U}$.

We denote by $\mathcal{S}^*(A, B)$ and $\mathcal{K}(A, B)$ $(-1 \leq B < A \leq 1)$ the subclasses of starlike functions and the subclasses of convex functions, respectively, that is (see, for details, [1] and [2])

$$\mathcal{S}^*(A, B) = \left\{ f(z) \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \ (z \in \mathbb{U}; -1 \leq B < A \leq 1) \right\}$$

and

$$\mathcal{K}(A, B) = \left\{ f(z) \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz} \ (z \in \mathbb{U}; -1 \leq B < A \leq 1) \right\}.$$

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Clearly, we have

$$f(z) \in \mathcal{K}(A,B) \Longleftrightarrow zf'(z) \in \mathcal{S}^*(A,B).$$

A function $f(z) \in \mathcal{A}$ is said to be in the class of uniformly convex functions, denoted by $\mathcal{UK}$ (see [3-5]) if

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right| \tag{1.3}$$

and is said to be in a corresponding class denoted by $\mathcal{US}$ if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)}-1\right|. \tag{1.4}$$

A function $f(z) \in \mathcal{A}$ is said to be in the class of α-uniformly convex functions of order β, denoted by $\mathcal{UK}(\alpha,\beta)$ (see [6]) if

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha\left|\frac{zf''(z)}{f'(z)}\right| + \beta \ (\alpha \geq 0; 0 \leq \beta < 1) \tag{1.5}$$

and is said to be in a corresponding class denoted by $\mathcal{US}(\alpha,\beta)$ if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha\left|\frac{zf'(z)}{f(z)}-1\right| + \beta \ (\alpha \geq 0; 0 \leq \beta < 1). \tag{1.6}$$

It is obvious that $f(z) \in \mathcal{UK}(\alpha,\beta)$ if and only if $zf'(z) \in \mathcal{US}(\alpha,\beta)$ (see [6]). The properties of various subclasses of functions $\mathcal{UK}(\alpha,\beta)$ and $\mathcal{US}(\alpha,\beta)$ were studied in [7].

For $f(z) \in \mathcal{A}$, Salagean [8] introduced the following operator which is called the Salagean operator:

$$D^0f(z) = f(z), D^1f(z) = zf'(z), \cdots, D^nf(z) = D(D^{n-1}f(z)) \ (n \in N = \{1,2,\cdots\}).$$

We note that

$$D^nf(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j \ (n \in N_0 = N \cup \{0\}). \tag{1.7}$$

Let $\mathcal{U}_{m,n}(\alpha,A,B)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the following inequality:

$$\frac{D^mf(z)}{D^nf(z)} - \alpha\left|\frac{D^mf(z)}{D^nf(z)}-1\right| \prec \frac{1+Az}{1+Bz} \ (\alpha \geq 0, -1 \leq B < A \leq 1, m \in N, n \in N_0). \tag{1.8}$$

Also let $\mathcal{V}^s_{m,n}(\alpha,A,B)$ $(s \in N_0)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the following condition:

$$f(z) \in \mathcal{V}^s_{m,n}(\alpha,A,B) \Longleftrightarrow D^sf(z) \in \mathcal{U}_{m,n}(\alpha,A,B). \tag{1.9}$$

For $s=0$, it is easy to see that

$$\mathcal{V}^0_{m,n}(\alpha,A,B) = \mathcal{U}_{m,n}(\alpha,A,B).$$

When $m=1, n=0$ and $m=2, n=1$ of inequality (1.8), respectively, we get two classes of functions

$$\mathcal{US}(\alpha,A,B) = \left\{f(z) \in \mathcal{A} : \frac{zf'(z)}{f(z)} - \alpha\left|\frac{zf'(z)}{f(z)}-1\right| \prec \frac{1+Az}{1+Bz},\ \alpha \geq 0, -1 \leq B < A \leq 1\right\}$$

and

$\mathcal{UK}(\alpha,A,B)=\left\{f(z)\in\mathcal{A}:1+\frac{zf''(z)}{f'(z)}-\alpha\left|\frac{zf''(z)}{f'(z)}\right|\prec\frac{1+Az}{1+Bz},\ \alpha\geq 0,-1\leq B<A\leq 1\right\}.$

It is clear from two of the above definitions that

$$f(z)\in\mathcal{UK}(\alpha,A,B)\Longleftrightarrow zf'(z)\in\mathcal{US}(\alpha,A,B),$$

$$\mathcal{US}(1,1,-1)=\mathcal{US},\ \mathcal{UK}(1,1,-1)=\mathcal{UK}.$$

By specializing the parameters α,A,B,m and n involved in the class $\mathcal{U}_{m,n}(\alpha,A,B)$, we also obtain the following subclasses which were studied in many earlier works:

(1) $\mathcal{U}_{1,0}(\alpha,1-2\beta,-1)=\mathcal{US}(\alpha,\beta)$ *and* $\mathcal{U}_{2,1}(\alpha,1-2\beta,-1)=\mathcal{UK}(\alpha,\beta)$ (*see*[6]).

(2) $\mathcal{U}_{n+1,n}(\alpha,1-2\beta,-1)=\mathcal{US}_n(\alpha,\beta)$ (*see*[9], [10]).

(3) $\mathcal{U}_{m,n}(\alpha,1-2\beta,-1)=\mathcal{U}_{m,n}(\alpha,\beta)$ *and* $\mathcal{V}^s_{m,n}(\alpha,1-2\beta,-1)=\mathcal{V}^s_{m,n}(\alpha,\beta)$ $(0\leq\alpha,0\leq\beta<1)$(*see*[11], [12]).

Let

$$\tilde{\mathcal{U}}\mathcal{S}(\alpha,A,B)=\mathcal{A}^+\cap\mathcal{US}(\alpha,A,B);\ \tilde{\mathcal{U}}\mathcal{K}(\alpha,A,B)=\mathcal{A}^+\cap\mathcal{UK}(\alpha,A,B);$$

$$\tilde{\mathcal{U}}_{m,n}(\alpha,A,B)=\mathcal{A}^+\cap\mathcal{U}_{m,n}(\alpha,A,B);\ \tilde{\mathcal{V}}^s_{m,n}(\alpha,A,B)=\mathcal{A}^+\cap\mathcal{V}^{s,+}_{m,n}(\alpha,A,B).$$

Then we obtain contain relations and the close properties of integral operators. This paper mainly studies the classes $\mathcal{U}_{m,n}(\alpha,A,B)$ and $\mathcal{V}^s_{m,n}(\alpha,A,B)$. We provide coefficient inequalities, distortion inequalities, extreme points and radius of close-to-convexity, starlikeness and convexity for the above classes.

2. COEFFICIENT INEQUALITIES FOR CLASSES $\mathcal{U}_{m,n}(\alpha,A,B)$ AND $\mathcal{V}^s_{m,n}(\alpha,A,B)$

Theorem 1. If $f(z)\in\mathcal{A}$ satisfies

$$\textstyle\sum_{j=2}^{\infty}\phi(m,n,j,\alpha,A,B)|a_j|\leq A-B \tag{2.1}$$

where

$$\phi(m,n,j,\alpha,A,B)=(1+2\alpha)|j^m-j^n|+|Bj^m-Aj^n| \tag{2.2}$$

for some $\alpha\geq 0,-1\leq B<A\leq 1,m\in N,n\in N_0=N\cup\{0\}$, then $f(z)\in\mathcal{U}_{m,n}(\alpha,A,B)$.

Proof. Suppose that (2.1) is true for $\alpha\geq 0,-1\leq B<A\leq 1,m\in N,n\in N_0$. For $f(z)\in\mathcal{A}$, let us define the function $p(z)$ by

$$p(z)=\frac{D^mf(z)}{D^nf(z)}-\alpha\left|\frac{D^mf(z)}{D^nf(z)}-1\right|.$$

It suffices to show that

$$\left|\frac{p(z)-1}{A-Bp(z)}\right|<1\ (z\in\mathbb{U}).$$

We note that

$$\left|\frac{p(z)-1}{A-Bp(z)}\right|=\left|\frac{D^mf(z)-\alpha e^{i\theta}|D^mf(z)-D^nf(z)|-D^nf(z)}{AD^nf(z)-B(D^mf(z)-\alpha e^{i\theta}|D^mf(z)-D^nf(z)|)}\right|$$

$$= \left| \frac{(D^m f(z) - D^n f(z)) - \alpha e^{i\theta}|D^m f(z) - D^n f(z)|}{(A-B)D^n f(z) - B((D^m f(z) - D^n f(z)) - \alpha e^{i\theta}|D^m f(z) - D^n f(z)|)} \right|$$

$$= \left| \frac{\sum_{j=2}^{\infty}(j^m - j^n)a_j z^{j-1} - \alpha e^{i\theta}|\sum_{j=2}^{\infty}(j^m - j^n)a_j z^{j-1}|}{(A-B) - \sum_{j=2}^{\infty}(Bj^m - Aj^n)a_j z^{j-1} - \alpha e^{i\theta}|\sum_{j=2}^{\infty}(j^m - j^n)a_j z^{j-1}|)} \right|$$

$$\leq \frac{\sum_{j=2}^{\infty}|j^m - j^n||a_j||z|^{j-1} + \alpha|e|^{i\theta}\sum_{j=2}^{\infty}|j^m - j^n||a_j||z|^{j-1}}{(A-B) - \sum_{j=2}^{\infty}|Bj^m - Aj^n||a_j||z|^{j-1} - \alpha|e|^{i\theta}\sum_{j=2}^{\infty}|j^m - j^n||a_j||z|^{j-1}}$$

$$\leq \frac{\sum_{j=2}^{\infty}|j^m - j^n||a_j| + \alpha\sum_{j=2}^{\infty}|j^m - j^n||a_j|}{(A-B) - \sum_{j=2}^{\infty}|Bj^m - Aj^n||a_j| - \alpha\sum_{j=2}^{\infty}|j^m - j^n||a_j|}.$$

The last expression is bounded above by 1, if

$$\sum_{j=2}^{\infty}|j^m - j^n||a_j| + \alpha\sum_{j=2}^{\infty}|j^m - j^n||a_j| \leq (A-B) - \sum_{j=2}^{\infty}|Bj^m - Aj^n||a_j| - \alpha\sum_{j=2}^{\infty}|j^m - j^n||a_j|$$

which is equivalent to the condition (2.1). This completes the proof of Theorem 1.

Corollary 1. If $f(z) \in \mathcal{A}$ satisfies

$$\sum_{j=2}^{\infty}\phi(1,0,j,\alpha,A,B)|a_j| \leq A - B$$

where

$$\phi(1,0,j,\alpha,A,B) = (1+2\alpha)(j-1) + |Bj - A|$$

for some $\alpha \geq 0, -1 \leq B < A \leq 1$, then $f(z) \in \mathcal{US}(\alpha, A, B)$.

Corollary 2. If $f(z) \in \mathcal{A}$ satisfies

$$\sum_{j=2}^{\infty}\phi(2,1,j,\alpha,A,B)|a_j| \leq A - B$$

where

$$\phi(2,1,j,\alpha,A,B) = (1+2\alpha)j(j-1) + j|Bj - A|$$

for some $\alpha \geq 0, -1 \leq B < A \leq 1$, then $f(z) \in \mathcal{UK}(\alpha, A, B)$.

By using Theorem 1, we have

Theorem 2. If $f(z) \in \mathcal{A}$ satisfies

$$\sum_{j=2}^{\infty}j^s\phi(m,n,j,\alpha,A,B)|a_j| \leq A - B$$

where $\phi(m,n,j,\alpha,A,B)$ is defined by (2.2) for some $\alpha \geq 0, -1 \leq B < A \leq 1, m \in N, n \in N_0$, then $f(z) \in \mathcal{V}_{m,n}^s(\alpha, A, B)$.

Proof. From (1.7), Replacing a_j by $j^s a_j$ in Theorem 1, we have Theorem 2.

Example 1. The function $f(z)$ given by

$$f(z) = z + \sum_{j=2}^{\infty}\frac{(A-B)(2+\delta)\varepsilon_j}{(j+\delta)(j+1+\delta)\phi(m,n,j,\alpha,A,B)}z^j = z + \sum_{j=2}^{\infty}A_j z^j$$

with

$$A_j = \frac{(A-B)(2+\delta)\varepsilon_j}{(j+\delta)(j+1+\delta)\phi(m,n,j,\alpha,A,B)}$$

belongs to the class $\mathcal{U}_{m,n}(\alpha, A, B)$ for $\delta > -2, \alpha \geq 0, -1 \leq B < A \leq 1, \varepsilon_j \in \mathbb{C}$ and $|\varepsilon_j| = 1$. Because, we know that

$$\sum_{j=2}^{\infty} \phi(m,n,j,\alpha,A,B)|A_j| \leq \sum_{j=2}^{\infty} \frac{(A-B)(2+\delta)}{(j+\delta)(j+1+\delta)}$$
$$= \sum_{j=2}^{\infty}(A-B)(2+\delta)\sum_{j=2}^{\infty}\left(\frac{1}{j+\delta} - \frac{1}{j+1+\delta}\right) = A - B.$$

Example 2. The function $f(z)$ given by

$$f(z) = z + \sum_{j=2}^{\infty} \frac{(A-B)(2+\delta)\varepsilon_j}{j^s(j+\delta)(j+1+\delta)\phi(m,n,j,\alpha,A,B)} z^j = z + \sum_{j=2}^{\infty} B_j z^j$$

with

$$B_j = \frac{(A-B)(2+\delta)\varepsilon_j}{j^s(j+\delta)(j+1+\delta)\phi(m,n,j,\alpha,A,B)}$$

belongs to the class $\mathcal{V}^s_{m,n}(\alpha, A, B)$ for $\delta > -2, \alpha \geq 0, -1 \leq B < A \leq 1, \varepsilon_j \in \mathbb{C}$ and $|\varepsilon_j| = 1$. Because, we know that

$$\sum_{j=2}^{\infty} j^s \phi(m,n,j,\alpha,A,B)|B_j| \leq \sum_{j=2}^{\infty} \frac{(A-B)(2+\delta)}{(j+\delta)(j+1+\delta)} = A - B.$$

Theorem 3. If $f(z) \in \mathcal{U}_{m,n}(\alpha, A, B)$, then for $|z| = r < 1$

$$\frac{1-(A-B)r-ABr^2}{1-B^2r^2} \leq \operatorname{Re}\left\{\frac{D^m f(z)}{D^n f(z)} - \alpha\left|\frac{D^m f(z)}{D^n f(z)} - 1\right|\right\}$$
$$\leq \frac{1+(A-B)r-ABr^2}{1-B^2r^2}, B \neq 0, \ (2.4)$$

$$1 - Ar \leq \operatorname{Re}\left\{\frac{D^m f(z)}{D^n f(z)} - \alpha\left|\frac{D^m f(z)}{D^n f(z)} - 1\right|\right\} \leq 1 + Ar, \ B = 0. \ (2.5)$$

Proof. Janowski [13] proved that if

$$p(z) \prec \frac{1+Az}{1+Bz}, \ |z| = r < 1,$$

then

$$\left|p(z) - \frac{1-ABr^2}{1-B^2r^2}\right| < \frac{(A-B)r}{1-B^2r^2}, \ B \neq 0, \ (2.6)$$

$$|p(z) - 1| < Ar, \ B = 0. \ (2.7)$$

Using the definition of the class $\mathcal{U}_{m,n}(\alpha, A, B)$, the inequality (2.6) and (2.7) can be rewritten in the form

$$\left|\frac{D^m f(z)}{D^n f(z)} - \alpha\left|\frac{D^m f(z)}{D^n f(z)} - 1\right| - \frac{1-ABr^2}{1-B^2r^2}\right| < \frac{(A-B)r}{1-B^2r^2},\ B \neq 0,\ (2.8)$$

$$\left|\frac{D^m f(z)}{D^n f(z)} - \alpha\left|\frac{D^m f(z)}{D^n f(z)} - 1\right| - 1\right| < Ar,\ B = 0.\ (2.9)$$

From (2.8) and (2.9), we get (2.4) and (2.5) of Theorem 3.

Theorem 4 below follows easily from Theorem 3.

Theorem 4. If $f(z) \in \mathcal{V}^s_{m,n}(\alpha, A, B)$, then for $|z| = r < 1$

$$\frac{1-(A-B)r-ABr^2}{1-B^2r^2} \leq \mathrm{Re}\left\{\frac{D^mD^s f(z)}{D^nD^s f(z)} - \alpha\left|\frac{D^mD^s f(z)}{D^nD^s f(z)} - 1\right|\right\} \leq \frac{1+(A-B)r-ABr^2}{1-B^2r^2}, B \neq 0,\ (2.10)$$

$$1 - Ar \leq \mathrm{Re}\left\{\frac{D^mD^s f(z)}{D^nD^s f(z)} - \alpha\left|\frac{D^mD^s f(z)}{D^nD^s f(z)} - 1\right|\right\} \leq 1 + Ar,\ B = 0.\ (2.11)$$

Corollary 3. If $f(z) \in \mathcal{US}(\alpha, A, B)$, then for $|z| = r < 1$

$$\frac{1-(A-B)r-ABr^2}{1-B^2r^2} \leq \mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\left|\frac{zf'(z)}{f(z)} - 1\right|\right\} \leq \frac{1+(A-B)r-ABr^2}{1-B^2r^2}, B \neq 0,$$

$$1 - Ar \leq \mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\left|\frac{zf'(z)}{f(z)} - 1\right|\right\} \leq 1 + Ar,\ B = 0.$$

Corollary 4. If $f(z) \in \mathcal{UK}(\alpha, A, B)$, then for $|z| = r < 1$

$$\frac{1-(A-B)r-ABr^2}{1-B^2r^2} \leq \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\left|\frac{zf''(z)}{f'(z)}\right|\right\} \leq \frac{1+(A-B)r-ABr^2}{1-B^2r^2}, B \neq 0,$$

$$1 - Ar \leq \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\left|\frac{zf''(z)}{f'(z)}\right|\right\} \leq 1 + Ar,\ B = 0.$$

3. DISTORTION INEQUALITIES

Lemma 1. If $f(z) \in \tilde{\mathcal{U}}_{m,n}(\alpha, A, B)$, then we have

$$\sum_{j=p+1}^{\infty} a_j \leq \frac{(A-B) - \sum_{j=2}^{p} \phi(m,n,j,\alpha,A,B)a_j}{\phi(m,n,p+1,\alpha,A,B)},\ (3.1)$$

where $\phi(m, n, j, \alpha, A, B)$ is defined by (2.2).

Proof. In view of Theorem 1, we can write

$$\sum_{j=p+1}^{\infty}\phi(m,n,j,\alpha,A,B)a_j \leq (A-B)-\sum_{j=2}^{p}\phi(m,n,j,\alpha,A,B)a_j. \quad (3.2)$$

Clearly, $\phi(m,n,j,\alpha,A,B)$ is an increasing function for j. Then from (2.2) and (3.2), we have

$$\phi(m,n,p+1,\alpha,A,B)\sum_{j=p+1}^{\infty}a_j \leq (A-B)-\sum_{j=2}^{p}\phi(m,n,j,\alpha,A,B)a_j.$$

Thus, we obtain

$$\sum_{j=p+1}^{\infty}a_j \leq \frac{(A-B)-\sum_{j=2}^{p}\phi(m,n,j,\alpha,A,B)a_j}{\phi(m,n,p+1,\alpha,A,B)} = A_j.$$

Lemma 2. If $f(z)\in\tilde{\mathcal{U}}_{m,n}(\alpha,A,B)$, then

$$\sum_{j=p+1}^{\infty}ja_j \leq \frac{(A-B)-\sum_{j=2}^{p}\phi(m,n,j,\alpha,A,B)a_j}{\phi(m-1,n-1,p+1,\alpha,A,B)} = B_j, \quad (3.3)$$

where $\phi(m,n,j,\alpha,A,B)$ is defined by (2.2).

Corollary 5. If $f(z)\in\tilde{\mathcal{V}}^{s}_{m,n}(\alpha,A,B)$, then

$$\sum_{j=p+1}^{\infty}a_j \leq \frac{(A-B)-\sum_{j=2}^{p}j^s\phi(m,n,j,\alpha,A,B)a_j}{(p+1)^s\phi(m,n,p+1,\alpha,A,B)} = C_j \quad (3.4)$$

and

$$\sum_{j=p+1}^{\infty}ja_j \leq \frac{(A-B)-\sum_{j=2}^{p}j^s\phi(m,n,j,\alpha,A,B)a_j}{(p+1)^s\phi(m-1,n-1,p+1,\alpha,A,B)} = D_j. \quad (3.5)$$

Theorem 5. Let $f(z)\in\tilde{\mathcal{U}}_{m,n}(\alpha,A,B)$. Then for $|z|=r<1$

$$r-\sum_{j=2}^{p}a_j|z|^j-A_jr^{p+1} \leq |f(z)| \leq r+\sum_{j=2}^{p}a_j|z|^j+A_jr^{p+1} \quad (3.6)$$

and

$$1-\sum_{j=2}^{p}ja_j|z|^{j-1}-B_jr^{p} \leq |f'(z)| \leq 1+\sum_{j=2}^{p}ja_j|z|^{j-1}+B_jr^{p} \quad (3.7)$$

where A_j and B_j are given by Lemma 1 and Lemma 2.

Proof. Let $f(z)$ given by (1.2). For $|z|=r<1$, using Lemma 1, we have

$$|f(z)| \leq |z|+\sum_{j=2}^{p}a_j|z|^j+\sum_{j=p+1}^{\infty}a_j|z|^j \leq |z|+\sum_{j=2}^{p}a_j|z|^j+|z|^{p+1}\sum_{j=p+1}^{\infty}a_j$$

$$\le r + \sum_{j=2}^{p} a_j |z|^j + A_j r^{p+1}$$

and

$$|f(z)| \ge |z| - \sum_{j=2}^{p} a_j |z|^j - \sum_{j=p+1}^{\infty} a_j |z|^j \ge |z| - \sum_{j=2}^{p} a_j |z|^j - |z|^{p+1} \sum_{j=p+1}^{\infty} a_j$$

$$\ge r - \sum_{j=2}^{p} a_j |z|^j - A_j r^{p+1}.$$

Furthermore, for $|z| = r < 1$, using Lemma 2, we also obtain

$$|f'(z)| \le 1 + \sum_{j=2}^{p} j a_j |z|^{j-1} + \sum_{j=p+1}^{\infty} j a_j |z|^{j-1} \le 1 + \sum_{j=2}^{p} j a_j |z|^{j-1} + |z|^p \sum_{j=p+1}^{\infty} j a_j$$

$$\le 1 + \sum_{j=2}^{p} j a_j |z|^{j-1} + B_j r^p$$

and

$$|f'(z)| \ge 1 - \sum_{j=2}^{p} j a_j |z|^{j-1} - \sum_{j=p+1}^{\infty} j a_j |z|^{j-1} \ge 1 - \sum_{j=2}^{p} j a_j |z|^{j-1} - |z|^p \sum_{j=p+1}^{\infty} j a_j$$

$$\ge 1 - \sum_{j=2}^{p} j a_j |z|^{j-1} - B_j r^p.$$

This completes the assertion of Theorem 5.

Theorem 6. Let $f(z) \in \tilde{\mathcal{V}}^s_{m,n}(\alpha, A, B)$. Then for $|z| = r < 1$

$$r - \sum_{j=2}^{p} a_j |z|^j - C_j r^{p+1} \le |f(z)| \le r + \sum_{j=2}^{p} a_j |z|^j + C_j r^{p+1} \quad (3.8)$$

and

$$1 - \sum_{j=2}^{p} j a_j |z|^{j-1} - D_j r^p \le |f'(z)| \le 1 + \sum_{j=2}^{p} j a_j |z|^{j-1} + D_j r^p \quad (3.9)$$

where C_j and D_j are given by Corollary 5.

Proof. Using a similar method to that in the proof of Theorem 5 and making use Corollary 5, we get our result.

Taking $p = 1$ in Theorem 5 and Theorem6, we have

Corollary 6. Let $f(z) \in \tilde{\mathcal{U}}_{m,n}(\alpha, A, B)$. Then for $|z| = r < 1$

$$r - \frac{A - B}{\phi(m, n, 2, \alpha, A, B)} r^2 \le |f(z)| \le r + \frac{A - B}{\phi(m, n, 2, \alpha, A, B)} r^2$$

and

$$1-\frac{2(A-B)}{\phi(m,n,2,\alpha,A,B)}r \le |f'(z)| \le 1+\frac{2(A-B)}{\phi(m,n,2,\alpha,A,B)}r.$$

Corollary 7. Let $f(z) \in \tilde{\mathcal{V}}^s_{m,n}(\alpha,A,B)$. Then for $|z|=r<1$

$$r-\frac{A-B}{2^s\phi(m,n,2,\alpha,A,B)}r^2 \le |f(z)| \le r+\frac{A-B}{2^s\phi(m,n,2,\alpha,A,B)}r^2$$

and

$$1-\frac{A-B}{2^{s-1}\phi(m,n,2,\alpha,A,B)}r \le |f'(z)| \le 1+\frac{A-B}{2^{s-1}\phi(m,n,2,\alpha,A,B)}r.$$

Taking $p=1, m=1$ and $n=0$ in Theorem 5 and Theorem 6, we also have
Corollary 8. Let $f(z) \in \tilde{\mathcal{U}}\mathcal{S}(\alpha,A,B)$. Then for $|z|=r<1$

$$r-\frac{A-B}{\phi(1,0,2,\alpha,A,B)}r^2 \le |f(z)| \le r+\frac{A-B}{\phi(1,0,2,\alpha,A,B)}r^2$$

and

$$1-\frac{2(A-B)}{\phi(1,0,2,\alpha,A,B)}r \le |f'(z)| \le 1+\frac{2(A-B)}{\phi(1,0,2,\alpha,A,B)}r.$$

Corollary 9. Let $f(z) \in \tilde{\mathcal{U}}\mathcal{K}(\alpha,A,B)$. Then for $|z|=r<1$

$$r-\frac{A-B}{\phi(2,1,2,\alpha,A,B)}r^2 \le |f(z)| \le r+\frac{A-B}{\phi(2,1,2,\alpha,A,B)}r^2$$

and

$$1-\frac{2(A-B)}{\phi(2,1,2,\alpha,A,B)}r \le |f'(z)| \le 1+\frac{2(A-B)}{\phi(2,1,2,\alpha,A,B)}r.$$

4. EXTREME POINTS

The determination of the extreme points of a family $f(z)$ of univalent functions enables us to solve many extreme problems for $f(z)$. Now, let us determine extreme points of the classes $\tilde{\mathcal{U}}_{m,n}(\alpha,A,B)$ and $\tilde{\mathcal{V}}^s_{m,n}(\alpha,A,B)$.

Theorem 7. Let $f_1(z)=z$ and

$$f_j(z)=z+\frac{A-B}{\phi(m,n,j,\alpha,A,B)}z^j \ (j=2,3,\cdots),$$

where $\phi(m,n,j,\alpha,A,B)$ is defined by (2.2). Then $f(z) \in \tilde{\mathcal{U}}_{m,n}(\alpha,A,B)$ if and only if it can be expressed in the form

$$f(z)=\sum_{j=1}^{\infty}\lambda_j f_j(z), \quad (4.1)$$

where $\lambda_j>0$ and $\sum_{j=1}^{\infty}\lambda_j=1$.

Proof. Suppose that

$$f(z)=\sum_{j=1}^{\infty}\lambda_j f_j(z)=z+\sum_{j=1}^{\infty}\lambda_j\frac{A-B}{\phi(m,n,j,\alpha,A,B)}z^j.$$

Then

$$\sum_{j=2}^{\infty}\phi(m,n,j,\alpha,A,B)\frac{A-B}{\phi(m,n,j,\alpha,A,B)}\lambda_j=\sum_{j=2}^{\infty}(A-B)\lambda_j=(A-B)(1-\lambda_1)<A-B.$$

Thus, $f(z)\in\tilde{\mathcal{U}}_{m,n}(\alpha,A,B)$ from the definition of the class of $f(z)\in\tilde{\mathcal{U}}_{m,n}(\alpha,A,B)$.

Conversely, suppose that $f(z)\in\tilde{\mathcal{U}}_{m,n}(\alpha,A,B)$. Since

$$a_j\le\frac{A-B}{\phi(m,n,j,\alpha,A,B)}\ \ (j=2,3,\cdots),$$

we may set

$$\lambda_j=\frac{\phi(m,n,j,\alpha,A,B)}{A-B}a_j\ \ (j=2,3,\cdots)$$

and

$$\lambda_1=1-\sum_{j=2}^{\infty}\lambda_j.$$

Then

$$f(z)=\sum_{j=1}^{\infty}\lambda_j f_j(z).$$

This completes the proof of Theorem 7.

Corollary 10. Let $g_1(z)=z$ and

$$g_j(z)=z+\frac{A-B}{j^s\phi(m,n,j,\alpha,A,B)}z^j\ \ (j=2,3,\cdots).$$

Then $g(z)\in\tilde{\mathcal{V}}^s_{m,n}(\alpha,A,B)$ if and only if it can be expressed in the form

$$g(z)=\sum_{j=1}^{\infty}\lambda_j g_j(z),$$

where $\lambda_j>0$ and $\sum_{j=1}^{\infty}\lambda_j=1$.

Corollary 11. The extreme points of $\tilde{\mathcal{U}}_{m,n}(\alpha,A,B)$ are the functions $f_1(z)=z$ and

$$f_j(z)=z+\frac{A-B}{\phi(m,n,j,\alpha,A,B)}z^j\ \ (j=2,3,\cdots).$$

Corollary 12. The extreme points of $\tilde{\mathcal{V}}^s_{m,n}(\alpha, A, B)$ are the functions $g_1(z) = z$ and

$$g_j(z) = z + \frac{A-B}{j^s \phi(m,n,j,\alpha,A,B)} z^j \ (j = 2, 3, \cdots).$$

Corollary 13. The extreme points of $\tilde{\mathcal{U}}\mathcal{S}(\alpha, A, B)$ are the functions $f_1(z) = z$ and

$$f_j(z) = z + \frac{A-B}{\phi(1,0,j,\alpha,A,B)} z^j \ (j = 2, 3, \cdots).$$

Corollary 14. The extreme points of $\tilde{\mathcal{U}}\mathcal{K}(\alpha, A, B)$ are the functions $f_1(z) = z$ and

$$f_j(z) = z + \frac{A-B}{\phi(2,1,j,\alpha,A,B)} z^j \ (j = 2, 3, \cdots).$$

5. RADIUS OF CLOSE-TO-CONVEXITY, STARLIKENESS AND CONVEXITY

We concentrate upon getting the radius of close-to-convexity, starlikeness and convexity.

Theorem 8. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{U}_{m,n}(\alpha, A, B)$. Then $f(z)$ is close-to-convex of $\mu (0 \le \mu < 1)$ in $|z| < r_\mu(m,n,j,\alpha,A,B)$ where

$$r_\mu(m,n,j,\alpha,A,B) = \inf_j \{ \frac{(1-\mu)\phi(m,n,j,\alpha,A,B)}{j(A-B)} \}^{\frac{1}{j-1}} \ (j \ge 2) \ (5.1)$$

and $\phi(m,n,j,\alpha,A,B)$ is defined by (2.2).

Proof. We must show that $|f'(z) - 1| < 1 - \mu$ for $|z| < r_\mu(m,n,j,\alpha,A,B)$. We have

$$|f'(z) - 1| \le \sum_{j=2}^{\infty} j|a_j||z|^{j-1}.$$

Thus $|f'(z) - 1| < 1 - \mu$ if

$$\sum_{j=2}^{\infty} (\frac{j}{1-\mu}) |a_j||z|^{j-1} \le 1. \ (5.2)$$

By Theorem 1, we have

$$\sum_{j=2}^{\infty} \frac{(1-\mu)\phi(m,n,j,\alpha,A,B)}{A-B} |a_j| \le 1. \ (5.3)$$

Hence (5.2) will be true if

$$\frac{j|z|^{j-1}}{1-\mu} \leq \frac{(1-\mu)\phi(m,n,j,\alpha,A,B)}{j(A-B)}$$

Equivalently if

$$|z| \leq \{\frac{(1-\mu)\phi(m,n,j,\alpha,A,B)}{j(A-B)}\}^{\frac{1}{j-1}} \ (j \geq 2). \ (5.4)$$

The theorem follows from (5.4).

Theorem 9. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{U}_{m,n}(\alpha, A, B)$. Then $f(z)$ is starlike of $\eta(0 \leq \eta < 1)$ in $|z| < r_\eta(m,n,j,\alpha,A,B)$ where

$$r_\eta(m,n,j,\alpha,A,B) = \inf_j\{\frac{(1-\eta)\phi(m,n,j,\alpha,A,B)}{(j-\eta)(A-B)}\}^{\frac{1}{j-1}} \ (j \geq 2) \ (5.5)$$

and $\phi(m,n,j,\alpha,A,B)$ is defined by (2.2).
Proof. It suffices to show that $|\frac{zf'(z)}{f(z)} - 1| < 1-\eta$ for $|z| < r_\eta(m,n,j,\alpha,A,B)$. We have

$$|\frac{zf'(z)}{f(z)} - 1| \leq \frac{\sum_{j=2}^{\infty}(j-1)|a_j||z|^{j-1}}{1-\sum_{j=2}^{\infty}|a_j||z|^{j-1}}.$$

Thus $|\frac{zf'(z)}{f(z)} - 1| < 1-\eta$ if

$$\sum_{j=2}^{\infty}\frac{(j-1)|a_j||z|^{j-1}}{(1-\eta)} \leq 1 \ (5.6)$$

By using (5.3), (5.6), we have

$$\frac{(j-\eta)|z|^{j-1}}{(1-\eta)} \leq \frac{\phi(m,n,j,\alpha,A,B)}{(A-B)}$$

or Equivalently

$$|z| \leq \{\frac{(1-\eta)\phi(m,n,j,\alpha,A,B)}{(j-\eta)(A-B)}\}^{\frac{1}{j-1}} \ (j \geq 2). \ (5.7)$$

Theorem 10. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{U}_{m,n}(\alpha, A, B)$. Then $f(z)$ is convex of $\xi(0 \leq \xi < 1)$ in $|z| < r_\xi(m,n,j,\alpha,A,B)$ where

$$r_\xi(m,n,j,\alpha,A,B) = \inf_j\{\frac{(1-\xi)\phi(m,n,j,\alpha,A,B)}{j(j-\xi)(A-B)}\}^{\frac{1}{j-1}} \ (j \geq 2) \ (5.8)$$

and $\phi(m,n,j,\alpha,A,B)$ is defined by (2.2).
Proof. It suffices to show that

$$|\frac{zf''(z)}{f'(z)}| < 1-\xi \ for \ |z| < r_\eta(m,n,j,\alpha,A,B). \ (5.9)$$

We have

$$|\frac{zf'(z)}{f(z)}-1|=|\frac{\sum_{j=2}^{\infty}j(j-1)a_jz^{j-1}}{1+\sum_{j=2}^{\infty}ja_jz^{j-1}}|\leq\frac{\sum_{j=2}^{\infty}j(j-1)|a_j||z|^{j-1}}{1-\sum_{j=2}^{\infty}j|a_j||z|^{j-1}}.$$

The last expression above is bounded by $(1-\xi)$ if

$$\sum_{j=2}^{\infty}\frac{j(j-\xi)|a_j||z|^{j-1}}{(1-\xi)}\leq 1. \ (5.10)$$

In view of (5.9), it follows that (5.10) is true if

$$\frac{j(j-\xi)|z|^{j-1}}{(1-\xi)}\leq\frac{\phi(m,n,j,\alpha,A,B)}{(A-B)}$$

or Equivalently

$$|z|\leq\{\frac{(1-\xi)\phi(m,n,j,\alpha,A,B)}{j(j-\xi)(A-B)}\}^{\frac{1}{j-1}}\ (j\geq 2).$$

And this completes the proof.

Corollary 15. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{V}_{m,n}^{s}(\alpha,A,B)$. Then $f(z)$ is close-to-convex of $\mu(0\leq\mu<1)$ in $|z|<r_{\mu,s}(m,n,j,\alpha,A,B)$ where

$$r_{\mu,s}(m,n,j,\alpha,A,B)=\inf_j\{\frac{(1-\mu)j^s\phi(m,n,j,\alpha,A,B)}{j(A-B)}\}^{\frac{1}{j-1}}\ (j\geq 2).$$

Corollary 16. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{V}_{m,n}^{s}(\alpha,A,B)$. Then $f(z)$ is starlike of $\eta(0\leq\eta<1)$ in $|z|<r_{\eta,s}(m,n,j,\alpha,A,B)$ where

$$r_{\eta,s}(m,n,j,\alpha,A,B)=\inf_j\{\frac{(1-\eta)j^s\phi(m,n,j,\alpha,A,B)}{(j-\eta)(A-B)}\}^{\frac{1}{j-1}}\ (j\geq 2).$$

Corollary 17. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{V}_{m,n}^{s}(\alpha,A,B)$. Then $f(z)$ is convex of $\xi(0\leq\xi<1)$ in $|z|<r_{\xi,s}(m,n,j,\alpha,A,B)$ where

$$r_{\xi,s}(m,n,j,\alpha,A,B)=\inf_j\{\frac{(1-\xi)j^s\phi(m,n,j,\alpha,A,B)}{j(j-\xi)(A-B)}\}^{\frac{1}{j-1}}\ (j\geq 2).$$

We remark in conclusion that, by suitably specializing the parameters involved in the results presented in this paper, we can deduce numerous further corollaries and consequences of each of these results.

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