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PROPERTY (Bw) AND WEYL TYPE THEOREMS

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Abstract. The paper introduces the notion of property (Bw), a version of generalized Weyl's theorem for a bounded linear operator T on an infinite dimensional Banach space X. A characterization of property (Bw) is also given. Certain conditions are explored on Hilbert space operators T and S so that $T \oplus S$ obeys property (Bw).

1. Introduction

Let $B(X)$ denote the algebra of all bounded linear operators on an infinite-dimensional complex Banach space X. For an operator $T \in B(X)$, we denote by T^*, $\sigma(T)$, $\sigma_{\text{iso}}(T)$, $N(T)$ and $R(T)$ the adjoint, the spectrum, the isolated points of $\sigma(T)$, the null space and the range space of T, respectively. Let $\alpha(T)$ and $\beta(T)$ denote the dimension of the kernel $N(T)$ and the codimension of the range $R(T)$, respectively. If the range $R(T)$ of T is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then T is called an upper semi-Fredholm (resp., a lower semi-Fredholm) operator.

If T is either an upper or a lower semi-Fredholm then T is called a semi-Fredholm operator, while T is said to be a Fredholm operator if it is both upper and lower semi-Fredholm. If $T \in B(X)$ is semi-Fredholm, then the index of T is defined as

$$\text{ind}(T) = \alpha(T) - \beta(T).$$

The descent $q(T)$ and the ascent $p(T)$ of T are given by

$$q(T) = \inf\{n : R(T^n) = R(T^{n+1})\},$$
$$p(T) = \inf\{n : N(T^n) = N(T^{n+1})\}.$$

An operator $T \in B(X)$ is called Weyl (resp., Browder) if it is a Fredholm operator of index 0 (resp., a Fredholm operator of finite ascent and descent). The Weyl spectrum $\sigma_W(T)$ (resp., Browder spectrum $\sigma_b(T)$) of T is the set of $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not Weyl (resp., $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not Browder).
Let

$$E_0(T) = \{\lambda \in \sigma_{\text{iso}}(T) : 0 < \alpha(T - \lambda I) < \infty\},$$

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then we say that T satisfies Weyl's theorem if $\sigma(T)\setminus\sigma_W(T) = E_0(T)$ and T satisfies Browder's theorem if $\sigma(T) \setminus \sigma_W(T) = \pi_0(T)$, where $\pi_0(T)$ is the set of poles of T of finite rank.

For a bounded linear operator T and a nonnegative integer n, we define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into itself (in particular $T_0 = T$). If for some integer n, the range space $R(T^n)$ is closed and T_n is an upper (resp., a lower) semi-Fredholm operator, then T is called an upper (resp., a lower) semi-B-Fredholm operator. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. Moreover, if T_n is a Fredholm operator, then T is called a B-Fredholm operator. From [4, Proposition 2.1] if T_n is a semi-Fredholm operator then T_m is also a semi-Fredholm operator for each $m \geq n$ and $\text{ind}(T_m) = \text{ind}(T_n)$. Thus the index of a semi-B-Fredholm operator T is defined as the index of the semi-Fredholm operator T_n (see [3, 4]).

An operator $T \in B(X)$ is called a B-Weyl operator if it is a B-Fredholm operator of index 0. The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined as

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Weyl operator}\}.$$

We say that generalized Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_{BW}(T) = E(T),$$

where $E(T)$ is the set of isolated eigen values of T and that generalized Browder's theorem holds for T if

$$\sigma(T) \setminus \sigma_{BW}(T) = \pi(T),$$

where $\pi(T)$ is the set of poles of T [3, Definition 2.13].

Berkani and Koliha [3] proved that generalized Weyl's theorem $\Rightarrow$ Weyl's theorem. Berkani and Arroud [2] established generalized Weyl's theorem for hyponormal operators acting on a Hilbert space.

The single valued extension property was introduced by Dunford ([8], [9]) and it plays an important role in local spectral theory and Fredholm theory ([1], [10]).

The operator $T \in B(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at $\lambda_0 \in \mathbb{C}$) if for every open disc U of λ_0 the only analytic function $f : U \to X$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in U$, is the function $f \equiv 0$.

An operator $T \in B(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$. An operator $T \in B(X)$ has SVEP at every point of the resolvent $\rho(T) = \mathbb{C} \setminus \sigma(T)$. Every operator T has SVEP at an isolated point of the spectrum.

Duggal [5] gave the following important results:

Theorem 1.1 ([5, Proposition 3.9])**.** (a) *The following statements are equivalent.*
(i) *T satisfies generalized Browder's theorem.*
(ii) *T has SVEP at points $\lambda \notin \sigma_{BW}(T)$*
(b) *T satisfies generalized Browder's theorem if and only if T satisfies Browder's theorem.*

Remark 1.2. Duggal [5] proved that T^* satisfies generalized Browder's theorem if and only if T satisfies Browder's theorem as $\sigma(T) = \sigma(T^*)$, $\sigma_{BW}(T) = \sigma_{BW}(T^*)$ and $\pi(T) = \pi(T^*)$.

In this paper, we introduce a new variant of generalized Weyl's theorem called the property (Bw) (see Definition 2.1). We prove that T satisfies property (Bw) if and only if generalized Browder's theorem holds for T and $\pi(T) = E_0(T)$.

2. Property (Bw)

Let us define property (Bw) as follows:

Definition 2.1. A bounded linear operator $T \in B(X)$ is said to satisfy property (Bw) if

$$\sigma(T) \setminus \sigma_{BW}(T) = E_0(T).$$

We give an example of an operator satisfying property (Bw):

Example 2.2. Let $Q \in l^2(\mathbb{N})$ be the quasinilpotent operator $Q(x_0, x_1, x_2, \dots) = \left(\frac{1}{2}x_1, \frac{1}{3}x_2 \dots\right)$ and $N \in l^2(\mathbb{N})$ be a nilpotent operator. Let $T = Q \oplus N$. Then $\sigma(T) = \sigma_W(T) = \sigma_{BW}(T) = \{0\}$, $E(T) = \{0\}$ and $E_0(T) = \phi$, which implies that T satisfies property (Bw).

Next is an example of an operator which fails to satisfy property (Bw):

Example 2.3. Let $T \in l^2(\mathbb{N})$ be defined as

$$T(x_0, x_1, \dots) = \left(\frac{1}{2}x_1, \frac{1}{3}x_2, \dots\right) \quad \text{for all} \quad (x_n) \in l^2(\mathbb{N})\,.$$

Theorem 2.4. *Let $T \in B(X)$ satisfy property* (Bw). *Then generalized Browder's theorem holds for T and $\sigma(T) = \sigma_{BW}(T) \cup \sigma_{\text{iso}}(T)$.*

Proof. By Proposition 3.9 of [5] it is sufficient to prove that T has SVEP at every $\lambda \notin \sigma_{BW}(T)$. Let us assume that $\lambda \notin \sigma_{BW}(T)$.

If $\lambda \notin \sigma(T)$, then T has SVEP at λ. If $\lambda \in \sigma(T)$ and suppose that T satisfies property (Bw) then $\lambda \in \sigma(T) \setminus \sigma_{BW}(T) = E_0(T)$. Thus $\lambda \in \sigma_{\text{iso}}(T)$ which implies T has SVEP at λ. To prove that $\sigma(T) = \sigma_{BW}(T) \cup \sigma_{\text{iso}}(T)$, we observe that $\lambda \in \sigma(T) \setminus \sigma_{BW}(T) = E_0(T)$. Thus $\lambda \in \sigma_{\text{iso}}(T)$. Hence $\sigma(T) \subseteq \sigma_{BW}(T) \cup \sigma_{\text{iso}}(T)$. But $\sigma_{BW}(T) \cup \sigma_{\text{iso}}(T) \subseteq \sigma(T)$ for every $T \in B(X)$. Thus $\sigma(T) = \sigma_{BW}(T) \cup \sigma_{\text{iso}}(T)$. □

A characterization of property (Bw) is given as follows:

Theorem 2.5. *Let $T \in B(X)$. Then the following statements are equivalent:*

(i) *T satisfies property* (Bw)*,*
(ii) *generalized Browder's theorem holds for T and $\pi(T) = E_0(T)$.*

Proof. (i) ⇒ (ii). Assume that T satisfies property (Bw). By Theorem 2.4 it is sufficient to prove the equality $\pi(T) = E_0(T)$.
If $\lambda \in E_0(T)$ then as T satisfies property (Bw), it implies that $\lambda \in \sigma(T) \setminus \sigma_{BW}(T) = \pi(T)$, because generalized Browder's theorem holds for T.
If $\lambda \in \pi(T) = \sigma(T) \setminus \sigma_{BW}(T) = E_0(T)$, therefore the equality $\pi(T) = E_0(T)$.

(ii)⇒(i). If $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$, then generalized Browder's theorem implies that $\lambda \in \pi(T) = E_0(T)$. Conversely, if $\lambda \in E_0(T)$ then $\lambda \in \pi(T) = \sigma(T) \setminus \sigma_{BW}(T)$. Thus $\sigma(T) \setminus \sigma_{BW}(T) = E_0(T)$. □

Theorem 2.6. *Let $T \in B(X)$. If T or T^* has SVEP at points in $\sigma(T) \setminus \sigma_{BW}(T)$, then T satisfies property (Bw) if and only if $E_0(T) = \pi(T)$.*

Proof. The hypothesis T or T^* has SVEP at points in $\sigma(T) \setminus \sigma_{BW}(T) = \sigma(T^*) \setminus \sigma_{BW}(T^*)$ implies that T satisfies generalized Browder's theorem (see Theorem 1.1 and Remark 1.2). Hence, if $\pi(T) = E_0(T)$, then $\sigma(T) \setminus \sigma_{BW}(T) = \pi(T) = E_0(T)$. □

Definition 2.7. Operators $S, T \in B(X)$ are said to be injectively interwined, denoted, $S \prec_i T$, if there exists an injection $U \in B(X)$ such that $TU = US$.

If $S \prec_i T$, then T has SVEP at a point λ implies S has SVEP at λ. To see this, let T have SVEP at λ, let U be an open neighbourhood of λ and let $f : U \to X$ be an analytic function such that $(S - \mu)f(\mu) = 0$ for every $\mu \in U$. Then $U(S - \mu)f(\mu) = (T - \mu)Uf(\mu) = 0 \Rightarrow Uf(\mu) = 0$. Since U is injective, $f(\mu) = 0$, i.e., S has SVEP at λ.

Theorem 2.8. *Let $S, T \in B(X)$. If T has SVEP and $S \prec_i T$, then S satisfies property (Bw) if and only if $E_0(S) = \pi(S)$.*

Proof. Suppose that T has SVEP. Since $S \prec_i T$, therefore S has SVEP. Hence the result follows from Theorem 2.6. □

Definition 2.9. An operator $T \in B(X)$ is said to be finitely isoloid if all the isolated points of its spectrum are eigenvalues of finite multiplicity i.e. $\sigma_{iso}(T) \subseteq E_0(T)$. An operator $T \in B(X)$ is said to be finitely polaroid (resp., polaroid) if all the isolated points of its spectrum are poles of finite rank i.e. $\sigma_{iso}(T) \subseteq \pi_0(T)$, (resp., $\sigma_{iso}(T) \subseteq \pi(T)$).

Theorem 2.10. *Let $T \in B(X)$ be a polaroid operator and satisfy property (Bw). Then generalized Weyl's theorem holds for T.*

Proof. T is polaroid and satisfies property (Bw) $\Leftrightarrow$.

$\sigma(T) \setminus \sigma_{BW}(T) = E_0(T) \subseteq E(T) = \pi(T) = \sigma(T) \setminus \sigma_{BW}(T)$. (Since T satisfies generalized Browder's theorem by Theorem 2.5). □

Definition 2.11. The analytic core of an operator $T \in B(X)$ is the subspace (not necessarily closed) $K(T)$ of all $x \in X$ such that there exists a sequence $\{x_n\}$ and a constant $c > 0$ such that (i) $Tx_{n+1} = x_n$, $x = x_0$ (ii) $\|x_n\| \leq c^n\|x\|$ for $n = 1, 2, \ldots$.

Apparently, $\sigma_{BW}(T) \subseteq \sigma_W(T)$ for every $T \in B(X)$. Hence, if T satisfies property (Bw), then $\sigma(T) \setminus \sigma_W(T) \subseteq \sigma(T) \setminus \sigma_{BW}(T) = E_0(T)$. Thus, if $\sigma_{\text{iso}}(T) = \phi$, then $\sigma(T) = \sigma_W(T) = \sigma_{BW}(T)$ (and T satisfies Weyl's theorem and generalized Weyl's theorem). For a non-quasinilpotent $T \in B(X)$, a condition guaranteeing $\sigma_{\text{iso}}(T) = \phi$ is that $K(T) = \{0\}$.

Theorem 2.12. *Let $T \in B(X)$ be not quasinilpotent and $K(T) = \{0\}$, then $\sigma(T) = \sigma_W(T) = \sigma_{BW}(T)$ and T satisfies both property (Bw) and generalized Weyl's theorem.*

Proof. Let $T \in B(X)$ be not quasinilpotent and $K(T) = \{0\}$, then T has SVEP, $\sigma(T) = \sigma_W(T)$ is a connected set containing 0 and $\sigma_{\text{iso}}(T) = \phi$ [1, Theorem 3.121]. SVEP implies T satisfies generalized Browder's theorem. Hence $\sigma(T) \setminus \sigma_{BW}(T) = \pi(T) = \phi = E_0(T) = E(T)$, i.e., T satisfies property (Bw) and generalized Weyl's theorem (so also Weyl's theorem). □

Remark 2.13. Let $T \in B(X)$ be quasinilpotent, then $\sigma(T) = \sigma_{BW}(T) = \{0\}$; hence T satisfies property (Bw) is equivalent to T satisfies generalized Weyl's theorem.

Theorem 2.14. *Let $T \in B(X)$ be a finitely isoloid operator and satisfy generalized Weyl's theorem. Then T satisfies property (Bw).*

Proof. If T satisfies generalized Weyl's theorem then $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$. To show that T satisfies property (Bw), we need to prove that $E(T) = E_0(T)$. Suppose $\lambda \in E(T)$. It implies that $\lambda \in \sigma_{\text{iso}}(T) \subseteq E_0(T)$, as T is finitely isoloid. Thus $E(T) \subseteq E_0(T)$. Other inclusion is always true. □

Theorem 2.15. *Let $T \in B(X)$ be a finitely polaroid operator. If T or T^* has SVEP, then property (Bw) holds for T.*

Proof. If T or T^* has SVEP, then T satisfies generalized Browder's theorem. Suppose $\lambda \in E_0(T)$. It implies that $\lambda \in \sigma_{\text{iso}}(T) \subseteq \pi_0(T) \subseteq \pi(T)$, as T is finitely polaroid. Therefore $E_0(T) \subseteq \pi(T)$. For the reverse inclusion suppose $\lambda \in \pi(T)$, then $\lambda \in \sigma_{\text{iso}}(T) \subseteq \pi_0(T) \subseteq E_0(T)$. Thus $\pi(T) \subseteq E_0(T)$. Using Theorem 2.5, we have that T satisfies property (Bw). □

3. Property (Bw) for class of operators satisfying norm condition

The bounded linear operator $T \in B(X)$ is normaloid if

$$\|T\| = r(T) = v(T),$$

where $\|T\|$ is usual operator norm of T, $r(T)$ is its spectral radius and $v(T)$ is its numerical radius.

A part of an operator is its restriction to a closed invariant subspace. We say that an operator $T \in B(X)$ is totally hereditarily normaloid, $T \in THN$, if every part of T, and the inverse of every part of T (whenever it exists), is normaloid. Hereditarily normaloid operators are simply polaroid (i.e., isolated points of the spectrum are simple poles of the resolvent) [6, Exampe 2.2] and have SVEP [6, Theorem 2.8]. We say that T is polynomially THN if there exists a non-constant polynomial $p(\cdot)$ such that $p(T) \in THN$.

Theorem 3.1. *Let $T \in B(X)$ be a polynomially THN operator. Then T and T^* satisfy property* (Bw) *if and only if $E(T) = E_0(T)$.*

Proof. If $p(T) \in THN$ for some non-constant polynomial $p(\cdot)$, then $p(T)$ has SVEP and $p(T)$ is simply polaroid. But then T has SVEP [1, Theorem 2.40] and T is polaroid [6, Example 2.5]. Hence $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$. This implies that T satisfies property (Bw) if and only if $E(T) = E_0(T)$. Observe that T has SVEP implies that T^* satisfies generalized Browder's theorem, i.e., $\sigma(T^*) \setminus \sigma_{BW}(T^*) = \pi(T^*)$. Since T polaroid implies T^* polaroid, we also have that $E(T) = \sigma(T) \setminus \sigma_{BW}(T) = \sigma(T^*) \setminus \sigma_{BW}(T^*) = \pi(T^*) = E(T^*)$. Clearly, if $\alpha(T - \lambda) \prec \infty$ and $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$, then $\alpha(T^* - \lambda I^*) = \beta(T - \lambda I) \prec \infty$. Hence T^* satisfies property (Bw) if and only if $E(T) = E_0(T)$.

4. Property (Bw) for direct sums

Let H and K be infinite-dimensional Hilbert spaces. In this section we show that if T and S are two operators on H and K respectively and at least one of them satisfies property (Bw) then their direct sum $T \oplus S$ obeys property (Bw). We have also explored various conditions on T and S so that $T \oplus S$ satisfies property (Bw).

Theorem 4.1. *Suppose that property* (Bw) *holds for $T \in B(H)$ and $S \in B(K)$. If T and S are isoloid and $\sigma_{BW}(T \oplus S) = \sigma_{BW}(T) \cup \sigma_{BW}(S)$, then property (Bw) holds for $T \oplus S$.*

Proof. We know $\sigma(T \oplus S) = \sigma(T) \cup \sigma(S)$ for any pair of operators.

If T and S are isoloid, then

$$E_0(T \oplus S) = [E_0(T) \cap \rho(S)] \cup [\rho(T) \cap E_0(S)] \cup [E_0(T) \cap E_0(S)]$$

where $\rho(.) = \mathbb{C} \setminus \sigma(.)$.

If property (Bw) holds for T and S, then

$$\begin{aligned} &[\sigma(T) \cup \sigma(S)] \setminus [\sigma_{BW}(T) \cup \sigma_{BW}(S)] \\ &\quad = \quad [E_0(T) \cap \rho(S)] \cup [\rho(T) \cap E_0(S)] \cup [E_0(T) \cap E_0(S)]. \end{aligned}$$

Thus $\sigma(T \oplus S) \setminus \sigma_{BW}(T \oplus S) = E_0(T \oplus S)$.

Hence property (Bw) holds for $T \oplus S$.

Theorem 4.2. *Suppose $T \in B(H)$ has no isolated point in its spectrum and $S \in B(K)$ satisfies property* (Bw). *If $\sigma_{BW}(T \oplus S) = \sigma(T) \cup \sigma_{BW}(S)$, then property* (Bw) *holds for $T \oplus S$.*

Proof. As $\sigma(T \oplus S) = \sigma(T) \cup \sigma(S)$ for any pair of operators, we have

$$\begin{aligned} \sigma(T \oplus S) \setminus \sigma_{BW}(T \oplus S) &= [\sigma(T) \cup \sigma(S)] \setminus [\sigma(T) \cup \sigma_{BW}(S)] \\ &= \sigma(S) \setminus [\sigma(T) \cup \sigma_{BW}(S)] \\ &= [\sigma(S) \setminus \sigma_{BW}(S)] \setminus \sigma(T) \\ &= E_0(S) \cap \rho(T) \end{aligned}$$

where $\rho(T) = \mathbb{C} \setminus \sigma(T)$.

Now $\sigma_{\text{iso}}(T)$ is the set of isolated points of $\sigma(T)$ and $\sigma_{\text{iso}}(T \oplus S)$ is the set of isolated points of $\sigma(T \oplus S) = \sigma(T) \cup \sigma(S)$. If $\sigma_{\text{iso}}(T) = \phi$, it implies that $\sigma(T) = \sigma_{\text{acc}}(T)$, where $\sigma_{\text{acc}}(T) = \sigma(T) \setminus \sigma_{\text{iso}}(T)$ is the set of all accumulation points of $\sigma(T)$. Thus we have

$$\begin{aligned} \sigma_{\text{iso}}(T \oplus S) &= [\sigma_{\text{iso}}(T) \cup \sigma_{\text{iso}}(S)] \setminus [(\sigma_{\text{iso}}(T) \cap \sigma_{\text{acc}}(S)) \cup (\sigma_{\text{acc}}(T) \cap \sigma_{\text{iso}}(S))] \\ &= (\sigma_{\text{iso}}(T) \setminus \sigma_{\text{acc}}(S)) \cup (\sigma_{\text{iso}}(S) \setminus \sigma_{\text{acc}}(T)) \\ &= \sigma_{\text{iso}}(S) \setminus \sigma(T) \\ &= \sigma_{\text{iso}}(S) \cap \rho(T). \end{aligned}$$

Let $\sigma_p(T)$ denote the point spectrum of T and $\sigma_{PF}(T)$ denote the set of all eigenvalues of T of finite multiplicity.

We have that $\sigma_p(T \oplus S) = \sigma_p(T) \cup \sigma_p(S)$ and $\dim N(T \oplus S) = \dim N(T) + \dim N(S)$ for every pair of operators, so that

$$\sigma_{PF}(T \oplus S) = \{\lambda \in \sigma_{PF}(T) \cup \sigma_{PF}(S) : \dim N(\lambda I - T) + \dim N(\lambda I - S) < \infty\}.$$

Therefore

$$\begin{aligned} E_0(T \oplus S) &= \sigma_{\text{iso}}(T \oplus S) \cap \sigma_{PF}(T \oplus S) \\ &= \sigma_{\text{iso}}(S) \cap \rho(T) \cap \sigma_{PF}(S) \\ &= E_0(S) \cap \rho(T). \end{aligned}$$

Thus $\sigma(T \oplus S) \setminus \sigma_{BW}(T \oplus S) = E_0(T \oplus S)$. Hence $T \oplus S$ satisfies property (Bw).

Let $\sigma_1(T)$ denote the complement of $\sigma_{BW}(T)$ in $\sigma(T)$ i.e. $\sigma_1(T) = \sigma(T) \setminus \sigma_{BW}(T)$. A straight forward application of Theorem 4.2 leads to the following corollaries.

Corollary 4.3. *Suppose $T \in B(H)$ is such that $\sigma_{\text{iso}}(T) = \phi$ and $S \in B(K)$ satisfies property* (Bw) *with $\sigma_{\text{iso}}(S) \cap \sigma_{PF}(S) = \phi$ and $\sigma_1(T \oplus S) = \phi$, then $T \oplus S$ satisfies property* (Bw)*.*

Proof. Since S satisfies property (Bw), therefore given condition $\sigma_{\text{iso}}(S) \cap \sigma_{PF}(S) = \phi$ implies that $\sigma(S) = \sigma_{BW}(S)$. Now $\sigma_1(T \oplus S) = \phi$ gives that $\sigma(T \oplus S) = \sigma_{BW}(T \oplus S) = \sigma(T) \cup \sigma_{BW}(S)$. Thus from Theorem 4.2 we have that $T \oplus S$ satisfies property (Bw).

Corollary 4.4. *Suppose $T \in B(H)$ is such that $\sigma_1(T) \cup \sigma_{\text{iso}}(T) = \phi$ and $S \in B(K)$ satisfies property* (Bw)*. If $\sigma_{BW}(T \oplus S) = \sigma_{BW}(T) \cup \sigma_{BW}(S)$, then property* (Bw) *holds for $T \oplus S$.*

Theorem 4.5. *Suppose $T \in B(H)$ is an isoloid operator that satisfies property* (Bw)*, then $T \oplus S$ satisfies property* (Bw) *whenever $S \in B(K)$ is a normal operator and satisfies property* (Bw)*.*

Proof. If $S \in B(K)$ is normal, then S (also, S^*) has SVEP, and $\text{ind}(S - \lambda) = 0$ for every λ such that $S - \lambda$ is B-Fredholm. Observe that $\lambda \notin \sigma_{BW}(T \oplus S) \Leftrightarrow T - \lambda$ and $S - \lambda$ are B-Fredholm and $\text{ind}(T - \lambda) + \text{ind}(S - \lambda) = \text{ind}(T - \lambda) = 0$.

$\Leftrightarrow \lambda \notin \{\sigma(T) \setminus \sigma_{BW}(T)\} \cap \{\sigma(S) \setminus \sigma_{BW}(S)\}$. Hence $\sigma_{BW}(T \oplus S) = \sigma_{BW}(T) \cup \sigma_{BW}(S)$. It is well known that the isolated points of the spectrum of a normal operator are simple poles of the resolvent of the operator (implies S is isoloid). Hence the result follows from Theorem 4.1.

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References

1. P. Aiena, *Fredholm and local spectral theory with applications to multipliers*, Kluwer Acad. Publishers, 2004.
2. M. Berkani and A. Arroud, *Generalized Weyl's theorem and hyponormal operators*, J. Aust. Math. Soc. **76**(2004) 291–302.
3. M. Berkani and J.J. Koliha, *Weyl type theorems for bounded linear operators*, Acta Sci. Math. (Szeged) **69**(2003) 359–376.
4. M. Berkani and M. Sarih, *On semi-B-Fredholm operators*, Glasgow Math. J. **43**(3)(2001) 457–465.
5. B.P. Duggal, *Polaroid Operators and generalized Browder, Weyl theorems*, Math Proc. Royal Irish Acad. **108** A(2008) 149–163.
6. B.P. Duggal, *Hereditarily Polaroid operators,SVEP and Weyl's theorem*, J. Math. Anal. Appl. **340** (2008) 366–373.
7. B.P. Duggal and C.S. Kubrusly, *Weyl's theorem for direct sums*, Studia Scientiarum Mathematicarum Hungarica **44**(2) (2007) 275–290.
8. N. Dunford, *Spectral theory I*, Resolution of the Identity. Pacific J. Math. **2** (1952) 559–614.
9. N. Dunford, *Spectral operators*, Pacific J. Math. **4** (1954) 321–354.
10. K.B. Laursen and M.M. Neumann, *An introduction to local spectral theory*, Clarendon Press, Oxford, 2000.

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