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THREE-STEP ITERATION PROCESS FOR A FINITE FAMILY OF ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN THE INTERMEDIATE SENSE IN CONVEX METRIC SPACES

(COMMUNICATED BY DANIEL PELLEGRINO)

G. S. SALUJA

Abstract. The purpose of this paper is to establish some strong convergence theorems of three-step iteration process with errors for approximating common fixed points for a finite family of asymptotically quasi-nonexpansive mappings in the intermediate sense in the setting of convex metric spaces. The results obtained in this paper generalize, improve and unify some main results of [1]-[7], [9]-[11], [13]-[17], [19] and [21].

1. Introduction

Throughout this paper, we assume that E is a metric space, $F(T_i) = \{x \in E : T_i x = x\}$ is the set of all fixed points of the mappings T_i $(i = 1, 2, \dots, N)$, $D(T)$ is the domain of T and $\mathbb{N}$ is the set of all positive integers. The set of common fixed points of T_i $(i = 1, 2, \dots, N)$ denoted by F, that is, $F = \cap_{i=1}^{N} F(T_i)$.

Definition 1. ([1]) Let $T \colon D(T) \subset E \to E$ be a mapping.

(i) The mapping T is said to be L-Lipschitzian if there exists a constant $L > 0$ such that

$$d(Tx, Ty) \quad \leq \quad L d(x, y), \qquad \forall x, y \in D(T). \tag{1.1}$$

(ii) The mapping T is said to be nonexpansive if

$$d(Tx, Ty) \quad \leq \quad d(x, y), \qquad \forall x, y \in D(T). \tag{1.2}$$

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(iii) The mapping T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$d(Tx, p) \quad \leq \quad d(x, p), \qquad \forall x \in D(T), \ \forall p \in F(T). \tag{1.3}$$

(iv) The mapping T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x, T^n y) \quad \leq \quad k_n d(x, y), \qquad \forall x, y \in D(T), \ \forall n \in \mathbb{N}. \tag{1.4}$$

(v) The mapping T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x, p) \quad \leq \quad k_n d(x, p), \qquad \forall x \in D(T), \ \forall p \in F(T), \ \forall n \in \mathbb{N}. \tag{1.5}$$

(vi) T is said to be asymptotically nonexpansive type, if

$$\limsup_{n\to\infty} \left\{ \sup_{x,y \in D(T)} \Big(d(T^n x, T^n y) - d(x, y) \Big) \right\} \quad \leq \quad 0. \tag{1.6}$$

(vii) T is said to be asymptotically quasi-nonexpansive type, if $F(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \left\{ \sup_{x \in D(T), \ p \in F(T)} \Big(d(T^n x, p) - d(x, p) \Big) \right\} \quad \leq \quad 0. \tag{1.7}$$

Remark 1. *It is easy to see that if $F(T)$ is nonempty, then nonexpansive mapping, quasi-nonexpansive mapping, asymptotically nonexpansive mapping, asymptotically quasi-nonexpansive mapping and asymptotically nonexpansive type mapping all are the special cases of asymptotically quasi-nonexpansive type mappings.*

Now, we define asymptotically quasi-nonexpansive mapping in the intermediate sense in convex metric space.

T is said be asymptotically quasi-nonexpansive mapping in the intermediate sense provided that T is uniformly continuous and

$$\limsup_{n\to\infty} \left\{ \sup_{x \in D(T), \ p \in F(T)} \Big(d(T^n x, p) - d(x, p) \Big) \right\} \quad \leq \quad 0. \tag{1.8}$$

In recent years, the problem concerning convergence of iterative sequences (and sequences with errors) for asymptotically nonexpansive mappings or asymptotically quasi-nonexpansive mappings converging to some fixed points in Hilbert spaces or Banach spaces have been considered by many authors.

In 1973, Petryshyn and Williamson [13] obtained a necessary and sufficient condition for Picard iterative sequences and Mann iterative sequences to converge to a fixed point for quasi-nonexpansive mappings. In 1994, Tan and Xu [16] also proved some convergence theorems of Ishikawa iterative sequences satisfying Opial's condition [12] or having Frečhet differential norm. In 1997, Ghosh and Debnath [4] extended the result of Petryshyn and Williamson [13] and gave a necessary and sufficient condition for Ishikawa iterative sequences to converge to a fixed point of quasi-nonexpansive mappings. Also in 2001 and 2002, Liu [9, 10, 11] obtained some necessary and sufficient conditions for Ishikawa iterative sequences or Ishikawa iterative sequences with errors to converge to a fixed point for asymptotically quasi-nonexpansive mappings.

In 2004, Chang et al. [1] extended and improved the result of Liu [11] in convex metric space. Further in the same year, Kim et al. [7] gave a necessary and sufficient conditions for asymptotically quasi-nonexpansive mappings in convex metric spaces which generalized and improved some previous known results.

Very recently, Tian and Yang [18] gave some necessary and sufficient conditions for a new Noor-type iterative sequences with errors to approximate a common fixed point for a finite family of uniformly quasi-Lipschitzian mappings in convex metric spaces.

The purpose of this paper is to establish some strong convergence theorems of three-step iteration process with errors to approximate a common fixed point for a finite family of asymptotically quasi-nonexpansive mappings in the intermediate sense in the setting of convex metric spaces. The results obtained in this paper generalize, improve and unify some main results of [1]-[7], [9]-[11], [13]-[17], [19] and [21].

Let T be a given self mapping of a nonempty convex subset C of an arbitrary normed space. The sequence $\{x_n\}_{n=0}^{\infty}$ defined by:

$$
\begin{aligned}
& x_0 \in C, \\
x_{n+1} &= \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \quad n \geq 0, \\
y_n &= a_n x_n + b_n T z_n + c_n v_n, \\
z_n &= d_n x_n + e_n T x_n + f_n w_n,
\end{aligned} \tag{1.9}
$$

is called the Noor-type iterative procedure with errors [2], where $\alpha_n, \beta_n, \gamma_n, a_n, b_n, c_n$, d_n, e_n and f_n are appropriate sequences in $[0,1]$ with $\alpha_n+\beta_n+\gamma_n = a_n+b_n+c_n = d_n + e_n + f_n = 1$, $n \geq 0$ and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C. If $d_n = 1(e_n = f_n = 0)$, $n \geq 0$, then (1.9) reduces to the Ishikawa iterative procedure with errors [20] defined as follows:

$$
\begin{aligned}
& x_0 \in C, \\
x_{n+1} &= \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \quad n \geq 0, \\
y_n &= a_n x_n + b_n T x_n + c_n v_n.
\end{aligned} \tag{1.10}
$$

If $a_n = 1(b_n = c_n = 0)$, then (1.10) reduces to the following Mann type iterative procedure with errors [20]:

$$
\begin{aligned}
& x_0 \in C, \\
x_{n+1} &= \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \quad n \geq 0.
\end{aligned} \tag{1.11}
$$

For the sake of convenience, we first recall some definitions and notation.

Definition 2. (see [1]): Let (E, d) be a metric space and $I = [0, 1]$. A mapping $W\colon E^3 \times I^3 \to E$ is said to be a convex structure on E if it satisfies the following condition:

$$d(u, W(x, y, z; \alpha, \beta, \gamma)) \le \alpha d(u, x) + \beta d(u, y) + \gamma d(u, z),$$

for any $u, x, y, z \in E$ and for any $\alpha, \beta, \gamma \in I$ with $\alpha + \beta + \gamma = 1$.

If (E, d) is a metric space with a convex structure W, then (E, d) is called a *convex metric space* and is denoted by (E, d, W). Let (E, d) be a convex metric space, a nonempty subset C of E is said to be convex if

$$W(x, y, z, \lambda_1, \lambda_2, \lambda_3) \in C, \quad \forall (x, y, z, \lambda_1, \lambda_2, \lambda_3) \in C^3 \times I^3.$$

Remark 2. *It is easy to prove that every linear normed space is a convex metric space with a convex structure $W(x, y, z; \alpha, \beta, \gamma) = \alpha x + \beta y + \gamma z$, for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in I$ with $\alpha + \beta + \gamma = 1$. But there exist some convex metric spaces which can not be embedded into any linear normed spaces (see, Takahashi* [15]*).*

Definition 3. Let (E, d, W) be a convex metric space and $T_i\colon E \to E$ be a finite family of asymptotically quasi-nonexpansive mappings in the intermediate sense with $i = 1, 2, \ldots, N$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$ and $\{f_n\}$ be nine sequences in $[0, 1]$ with

$$\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = d_n + e_n + f_n = 1, \quad n = 0, 1, 2, \ldots. \tag{1.12}$$

For a given $x_0 \in E$, define a sequence $\{x_n\}$ as follows:

$$\begin{aligned} x_{n+1} &= W(x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), \quad n \ge 0, \\ y_n &= W(g(x_n), T_n^n z_n, v_n; a_n, b_n, c_n), \\ z_n &= W(g(x_n), T_n^n x_n, w_n; d_n, e_n, f_n), \end{aligned} \tag{1.13}$$

where $T_n^n = T_{n(mod\ N)}^n$, $g\colon E \to E$ is a Lipschitz continuous mapping with a Lipschitz constant $\xi > 0$ and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are any given three sequences in E. Then $\{x_n\}$ is called the Noor-type iterative sequence with errors for a finite family of asymptotically quasi-nonexpansive type mappings $\{T_i\}_{i=1}^N$. If $g = I$ (the identity mapping on E) in (1.13), then the sequence $\{x_n\}$ defined by (1.13) can be written as follows:

$$\begin{aligned} x_{n+1} &= W(x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), \quad n \ge 0, \\ y_n &= W(x_n, T_n^n z_n, v_n; a_n, b_n, c_n), \\ z_n &= W(x_n, T_n^n x_n, w_n; d_n, e_n, f_n), \end{aligned} \tag{1.14}$$

If $d_n = 1(e_n = f_n = 0)$ for all $n \geq 0$ in (1.13), then $z_n = g(x_n)$ for all $n \geq 0$ and the sequence $\{x_n\}$ defined by (1.13) can be written as follows:

$$\begin{aligned} x_{n+1} &= W(x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), \quad n \geq 0, \\ y_n &= W(g(x_n), T_n^n g(x_n), v_n; a_n, b_n, c_n). \end{aligned} \tag{1.15}$$

If $g = I$ and $d_n = 1(e_n = f_n = 0)$ for all $n \geq 0$, then the sequence $\{x_n\}$ defined by (1.13) can be written as follows:

$$\begin{aligned} x_{n+1} &= W(x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), \quad n \geq 0, \\ y_n &= W(x_n, T_n^n x_n, v_n; a_n, b_n, c_n), \end{aligned} \tag{1.16}$$

which is the Ishikawa type iterative sequence with errors considered in [17]. Further,

if $g = I$ and $d_n = a_n = 1(e_n = f_n = b_n = c_n = 0)$ for all $n \geq 0$, then $z_n = y_n = x_n$ for all $n \geq 0$ and (1.13) reduces to the following Mann type iterative sequence with errors [17]:

$$x_{n+1} = W(x_n, T_n^n x_n, u_n; \alpha_n, \beta_n, \gamma_n), \quad n \geq 0. \tag{1.17}$$

Lemma 1.1. *(see [10]): Let $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ be three nonnegative sequences of real numbers satisfying the following conditions:*

$$p_{n+1} \leq (1 + q_n)p_n + r_n, \quad n \geq 0, \quad \sum_{n=0}^{\infty} q_n < \infty, \quad \sum_{n=0}^{\infty} r_n < \infty. \tag{1.18}$$

Then

(1) $\lim_{n\to\infty} p_n$ *exists.*

(2) In addition, if $\liminf_{n\to\infty} p_n = 0$, *then* $\lim_{n\to\infty} p_n = 0$.

2. Main Results

Now we state and prove our main results of this paper.

Lemma 2.1. *Let (E, d, W) be a complete convex metric space and C be a nonempty closed convex subset of E. Let $T_i \colon C \to C$ be a finite family of asymptotically quasi-nonexpansive mappings in the intermediate sense for $i = 1, 2, \dots, N$ such that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ and $g \colon C \to C$ a contractive mapping with a contractive constant $\xi \in (0, 1)$. Put*

$$\begin{aligned} G_n = \max\Big\{ &\sup_{p\in F,\ n\geq 0} \Big(d(T_n^n x_n, p) - d(x_n, p)\Big) \vee \sup_{p\in F,\ n\geq 0} \Big(d(T_n^n y_n, p) - d(y_n, p)\Big) \\ &\vee \sup_{p\in F,\ n\geq 0} \Big(d(T_n^n z_n, p) - d(z_n, p)\Big) \vee 0\Big\}, \end{aligned} \tag{2.1}$$

such that $\sum_{n=0}^{\infty} G_n < \infty$. Let $\{x_n\}$ be the iterative sequence with errors defined by (1.13) and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ be three bounded sequences in C. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$, $\{f_n\}$ be sequences in $[0, 1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = d_n + e_n + f_n = 1, \quad \forall n \geq 0$;

(ii) $\sum_{n=0}^{\infty}(\beta_n+\gamma_n)<\infty$.

Then the following conclusions hold:

(1) for all $p\in F$ *and* $n\geq 0$,

$$d(x_{n+1},p) \leq (1+3\beta_n)d(x_n,p)+3G_n+M\theta_n, \tag{2.2}$$

where $\theta_n=\beta_n+\gamma_n$ *for all* $n\geq 0$ *and*

$$M=\sup_{p\in F,\ n\geq 0}\Big\{d(u_n,p)+d(v_n,p)+d(w_n,p)+2d(g(p),p)\Big\}.$$

(2) there exists a constant $M_1>0$ *such that*

$$\begin{aligned} d(x_{n+m},p) &\leq M_1 d(x_n,p)+3M_1\sum_{k=n}^{n+m-1}G_k \\ &\quad +MM_1\sum_{k=n}^{n+m-1}\theta_k, \quad \forall p\in F, \end{aligned} \tag{2.3}$$

for all $n,m\geq 0$.

Proof. For any $p\in F$, using (1.13) and (2.1), we have

$$\begin{aligned} d(x_{n+1},p) &= d(W(x_n,T_n^n y_n,u_n;\alpha_n,\beta_n,\gamma_n),p) \\ &\leq \alpha_n d(x_n,p)+\beta_n d(T_n^n y_n,p)+\gamma_n d(u_n,p) \\ &\leq \alpha_n d(x_n,p)+\beta_n[d(y_n,p)+G_n]+\gamma_n d(u_n,p) \\ &\leq \alpha_n d(x_n,p)+\beta_n d(y_n,p)+\beta_n G_n+\gamma_n d(u_n,p) \\ &\leq \alpha_n d(x_n,p)+\beta_n d(y_n,p)+G_n+\gamma_n d(u_n,p), \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} d(y_n,p) &= d(W(g(x_n),T_n^n z_n,v_n;a_n,b_n,c_n),p) \\ &\leq a_n d(g(x_n),p)+b_n d(T_n^n z_n,p)+c_n d(v_n,p) \\ &\leq a_n d(g(x_n),g(p))+a_n d(g(p),p) \\ &\quad +b_n[d(z_n,p)+G_n]+c_n d(v_n,p) \\ &\leq a_n\xi d(x_n,p)+a_n d(g(p),p)+b_n d(z_n,p) \\ &\quad +b_n G_n+c_n d(v_n,p) \\ &\leq a_n\xi d(x_n,p)+a_n d(g(p),p)+b_n d(z_n,p) \\ &\quad +G_n+c_n d(v_n,p), \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} d(z_n,p) &= d(W(g(x_n),T_n^n x_n,w_n;d_n,e_n,f_n),p)\\ &\le d_n d(g(x_n),p)+e_n d(T_n^n x_n,p)+f_n d(w_n,p)\\ &\le d_n d(g(x_n),g(p))+d_n d(g(p),p)\\ &\quad +e_n[d(x_n,p)+G_n]+f_n d(w_n,p)\\ &\le d_n\xi d(x_n,p)+d_n d(g(p),p)+e_n d(x_n,p)\\ &\quad +e_n G_n+f_n d(w_n,p)\\ &\le (d_n\xi+e_n)d(x_n,p)+d_n d(g(p),p)\\ &\quad +e_n G_n+f_n d(w_n,p)\\ &\le (d_n\xi+e_n)d(x_n,p)+d_n d(g(p),p)\\ &\quad +G_n+f_n d(w_n,p). \end{aligned} \tag{2.6}$$

Substituting (2.5) into (2.4) and simplifying it, we have

$$\begin{aligned} d(x_{n+1},p) &\le \alpha_n d(x_n,p)+\beta_n\Big[a_n\xi d(x_n,p)+a_n d(g(p),p)\\ &\quad +b_n d(z_n,p)+G_n+c_n d(v_n,p)\Big]+G_n+\gamma_n d(u_n,p)\\ &\le (\alpha_n+a_n\beta_n\xi)d(x_n,p)+a_n\beta_n d(g(p),p)+\beta_n G_n\\ &\quad +b_n\beta_n d(z_n,p)+c_n\beta_n d(v_n,p)+G_n+\gamma_n d(u_n,p)\\ &= (\alpha_n+a_n\beta_n\xi)d(x_n,p)+a_n\beta_n d(g(p),p)+(1+\beta_n)G_n\\ &\quad +b_n\beta_n d(z_n,p)+c_n\beta_n d(v_n,p)+\gamma_n d(u_n,p)\\ &\le (\alpha_n+a_n\beta_n\xi)d(x_n,p)+a_n\beta_n d(g(p),p)+2G_n\\ &\quad +b_n\beta_n d(z_n,p)+c_n\beta_n d(v_n,p)+\gamma_n d(u_n,p). \end{aligned} \tag{2.7}$$

Substituting (2.6) into (2.7) and simplifying it, we have

$$\begin{aligned} d(x_{n+1},p) &\leq (\alpha_n + a_n\beta_n\xi)d(x_n,p) + a_n\beta_n d(g(p),p) + 2G_n \\ &\quad + b_n\beta_n\Big[(d_n\xi + e_n)d(x_n,p) + d_n d(g(p),p) + G_n \\ &\quad + f_n d(w_n,p)\Big] + c_n\beta_n d(v_n,p) + \gamma_n d(u_n,p) \\ &\leq \Big[\alpha_n + a_n\beta_n\xi + b_n\beta_n(d_n\xi + e_n)\Big]d(x_n,p) \\ &\quad + \beta_n(a_n + b_n d_n)d(g(p),p) + G_n(2 + b_n\beta_n) \\ &\quad + b_n\beta_n f_n d(w_n,p) + c_n\beta_n d(v_n,p) + \gamma_n d(u_n,p) \\ &\leq \Big[\alpha_n + \beta_n(a_n\xi + b_n d_n\xi + b_n e_n)\Big]d(x_n,p) \\ &\quad + 2\beta_n d(g(p),p) + 3G_n + \beta_n d(w_n,p) \\ &\quad + \beta_n d(v_n,p) + \gamma_n d(u_n,p) \\ &\leq (1 + 3\beta_n)d(x_n,p) + 2\beta_n d(g(p),p) \\ &\quad + 2\gamma_n d(g(p),p) + 3G_n + \beta_n d(w_n,p) \\ &\quad + \gamma_n d(w_n,p) + \beta_n d(v_n,p) + \gamma_n d(v_n,p) \\ &\quad + \beta_n d(u_n,p) + \gamma_n d(u_n,p) \\ &= (1 + 3\beta_n)d(x_n,p) + 3G_n + 2(\beta_n + \gamma_n)d(g(p),p) \\ &\quad + (\beta_n + \gamma_n)\Big[d(u_n,p) + d(v_n,p) + d(w_n,p)\Big] \\ &= (1 + 3\beta_n)d(x_n,p) + 3G_n + (\beta_n + \gamma_n)\Big[d(u_n,p) \\ &\quad + d(v_n,p) + d(w_n,p) + 2d(g(p),p)\Big] \\ &\leq (1 + 3\beta_n)d(x_n,p) + 3G_n + M\theta_n, \quad \forall n \geq 0, \;\; p \in F, \qquad (2.8) \end{aligned}$$

where

$$M = \sup_{p\in F \;\; n\geq 0}\Big\{d(u_n,p) + d(v_n,p) + d(w_n,p) + 2d(g(p),p)\Big\}, \quad \theta_n = \beta_n + \gamma_n.$$

This completes the proof of part (1).

(2) Since $1+x \leq e^x$ for all $x \geq 0$, it follows from (2.8) that, for $n, m \geq 0$ and $p \in F$, we have

$$
\begin{aligned}
d(x_{n+m}, p) &\leq (1+3\beta_{n+m-1})d(x_{n+m-1}, p) + 3G_{n+m-1} + M\theta_{n+m-1} \\
&\leq e^{3\beta_{n+m-1}} d(x_{n+m-1}, p) + 3G_{n+m-1} + M\theta_{n+m-1} \\
&\leq e^{3\beta_{n+m-1}} \Big[e^{3\beta_{n+m-2}} d(x_{n+m-2}, p) + 3G_{n+m-2} + M\theta_{n+m-2}\Big] \\
&\quad +3G_{n+m-1} + M\theta_{n+m-1} \\
&\leq e^{3(\beta_{n+m-1}+\beta_{n+m-2})} d(x_{n+m-2}, p) + 3\Big[e^{3\beta_{n+m-1}} G_{n+m-2} \\
&\quad +G_{n+m-1}\Big] + M\Big[e^{3\beta_{n+m-1}}\theta_{n+m-2} + \theta_{n+m-1}\Big] \\
&\leq e^{3(\beta_{n+m-1}+\beta_{n+m-2})} d(x_{n+m-2}, p) + 3e^{3\beta_{n+m-1}}\Big(G_{n+m-2} \\
&\quad +G_{n+m-1}\Big) + Me^{3\beta_{n+m-1}}\Big(\theta_{n+m-2} + \theta_{n+m-1}\Big) \\
&\leq \dots \\
&\leq \dots \\
&\leq M_1 d(x_n, p) + 3M_1 \sum_{k=n}^{n+m-1} G_k + MM_1 \sum_{k=n}^{n+m-1} \theta_k, \qquad (2.9)
\end{aligned}
$$

where

$$M_1 = e^{3\sum_{k=0}^{\infty}\beta_k}.$$

This completes the proof of part (2). □

Theorem 2.2. *Let (E, d, W) be a complete convex metric space and C be a nonempty closed convex subset of E. Let $T_i \colon C \to C$ be a finite family of uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings in the intermediate sense for $i = 1, 2, \dots, N$ such that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$ and $g \colon C \to C$ a contractive mapping with a contractive constant $\xi \in (0, 1)$. Put*

$$
\begin{aligned}
G_n = \max\Big\{ &\sup_{p\in F,\ n\geq 0} \Big(d(T_n^n x_n, p) - d(x_n, p)\Big) \vee \sup_{p\in F,\ n\geq 0} \Big(d(T_n^n y_n, p) - d(y_n, p)\Big) \\
&\vee \sup_{p\in F,\ n\geq 0} \Big(d(T_n^n z_n, p) - d(z_n, p)\Big) \vee 0\Big\},
\end{aligned}
$$

such that $\sum_{n=0}^{\infty} G_n < \infty$. Let $\{x_n\}$ be the iterative sequence with errors defined by (1.13) and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ be three bounded sequences in C, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$ and $\{f_n\}$ be nine sequences in $[0, 1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = d_n + e_n + f_n = 1, \quad \forall n \geq 0$;

(ii) $\sum_{n=0}^{\infty}(\beta_n + \gamma_n) < \infty$.

Then the sequence $\{x_n\}$ converges strongly to a common fixed point p of the mappings $\{T_i\}_{i=1}^N$ if and only if

$$\liminf_{n\to\infty} d(x_n, F) = 0,$$

where $d(x, F) = \inf_{p \in F} d(x, p)$.

Proof. The necessity is obvious. Now, we prove the sufficiency. In fact, from Lemma 2.1, we have

$$d(x_{n+1}, F) \quad \leq \quad (1 + 3\beta_n) d(x_n, F) + 3G_n + M\theta_n, \quad \forall n \geq 0, \tag{2.10}$$

where $\theta_n = \beta_n + \gamma_n$. By assumption and conditions (i) and (ii), we know that

$$\sum_{n=0}^{\infty} \theta_n < \infty, \quad \sum_{n=0}^{\infty} \beta_n < \infty, \quad \sum_{n=0}^{\infty} G_n < \infty. \tag{2.11}$$

It follows from Lemma 1.1 that $\lim_{n\to\infty} d(x_n, F)$ exists. Since $\liminf_{n\to\infty} d(x_n, F) = 0$, we have

$$\lim_{n\to\infty} d(x_n, F) = 0. \tag{2.12}$$

Next, we prove that $\{x_n\}$ is a Cauchy sequence in C. In fact, for any given $\varepsilon > 0$, there exists a positive integer n_1 such that for any $n \geq n_1$, we have

$$d(x_n, F) < \frac{\varepsilon}{12M_1}, \quad \sum_{n=n_1}^{\infty} G_n < \frac{\varepsilon}{18M_1}, \quad \forall n \geq 0. \tag{2.13}$$

and

$$\sum_{n=n_1}^{\infty} \theta_n < \frac{\varepsilon}{6MM_1}, \quad \forall n \geq 0. \tag{2.14}$$

From (2.13), there exists $p_1 \in F$ and positive integer $n_2 \geq n_1$ such that

$$d(x_{n_2}, p_1) < \frac{\varepsilon}{6M_1}. \tag{2.15}$$

Thus Lemma 2.1(2) implies that, for any positive integer n, m with $n \geq n_2$, we have

$$\begin{aligned}
d(x_{n+m}, x_n) \quad &\leq \quad d(x_{n+m}, p_1) + d(x_n, p_1) \\
&\leq \quad M_1 d(x_{n_2}, p_1) + 3M_1 \sum_{k=n_2}^{n+m-1} G_k + MM_1 \sum_{k=n_2}^{n+m-1} \theta_k \\
&\quad\quad + M_1 d(x_{n_2}, p_1) + 3M_1 \sum_{k=n_2}^{n+m-1} G_k + MM_1 \sum_{k=n_2}^{n+m-1} \theta_k \\
&\leq \quad 2M_1 d(x_{n_2}, p_1) + 6M_1 \sum_{k=n_2}^{n+m-1} G_k + 2MM_1 \sum_{k=n_2}^{n+m-1} \theta_k \\
&< \quad 2M_1 . \frac{\varepsilon}{6M_1} + 6M_1 . \frac{\varepsilon}{18M_1} + 2MM_1 . \frac{\varepsilon}{6MM_1} \\
&< \quad \varepsilon.
\end{aligned} \tag{2.16}$$

This shows that $\{x_n\}$ is a Cauchy sequence in a nonempty closed convex subset C of a complete convex metric space E. Without loss of generality, we can assume that $\lim_{n\to\infty} x_n = q \in E$. Now we will prove that $q \in F$. Since $x_n \to q$ and

$d(x_n, F) \to 0$ as $n \to \infty$, for any given $\varepsilon_1 > 0$, there exists a positive integer $n_3 \geq n_2$ such that for $n \geq n_3$, we have

$$d(x_n, q) < \varepsilon_1, \quad d(x_n, F) < \varepsilon_1. \tag{2.17}$$

Again from the second inequality of (2.17), there exists $q_1 \in F$ such that

$$d(x_{n_3}, q_1) < 2\varepsilon_1. \tag{2.18}$$

Moreover, it follows from (2.1) that for any $n \geq n_3$, we have

$$d(T_n^n q, q_1) - d(q, q_1) < G_n. \tag{2.19}$$

Thus for any $i = 1, 2, \dots, N$, from (2.17) - (2.19) and for any $n \geq n_3$, we have

$$\begin{aligned} d(T_i^n q, q) &\leq d(T_i^n q, q_1) + d(q_1, q) \\ &\leq d(q, q_1) + G_n + d(q_1, q) \\ &= G_n + 2d(q, q_1) \\ &\leq G_n + 2[d(q, x_{n_3}) + d(x_{n_3}, q_1)] \\ &< G_n + 2(\varepsilon_1 + 2\varepsilon_1) = G_n + 6\varepsilon_1 = \varepsilon', \end{aligned} \tag{2.20}$$

where $\varepsilon' = G_n + 6\varepsilon_1$, since $G_n \to 0$ as $n \to \infty$ and $\varepsilon_1 > 0$, it follows that $\varepsilon' > 0$. By the arbitrariness of $\varepsilon' > 0$, we know that $T_i^n q = q$ for all $i = 1, 2, \dots, N$.

Again since for any $n \geq n_3$, we have

$$\begin{aligned} d(T_i^n q, T_i q) &\leq d(T_i^n q, q_1) + d(T_i q, q_1) \\ &\leq d(q, q_1) + G_n + d(T_i q, q_1) \\ &\leq d(q, q_1) + G_n + Ld(q, q_1) \\ &= (1 + L)d(q, q_1) + G_n \\ &\leq (1 + L)[d(q, x_{n_3}) + d(x_{n_3}, q_1)] + G_n \\ &< (1 + L)[\varepsilon_1 + 2\varepsilon_1] + G_n \\ &= 3(1 + L)\varepsilon_1 + G_n = \varepsilon'', \end{aligned} \tag{2.21}$$

where $\varepsilon'' = 3(1 + L)\varepsilon_1 + G_n$, since $G_n \to 0$ as $n \to \infty$ and $\varepsilon_1 > 0$, it follows that $\varepsilon'' > 0$. By the arbitrariness of $\varepsilon'' > 0$, we know that $T_i^n q = T_i q$ for all $i = 1, 2, \dots, N$. From the uniqueness of limit, we have $q = T_i q$ for all $i = 1, 2, \dots, N$, that is, $q \in F$. This shows that q is a common fixed point of the mappings $\{T_i\}_{i=1}^N$. This completes the proof. □

Taking $g = I$ in Theorem 2.2, then we have the following theorem.

Theorem 2.3. *Let (E, d, W) be a complete convex metric space and C be a nonempty closed convex subset of E. Let $T_i\colon C \to C$ be a finite family of uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings in the intermediate sense for $i = 1, 2, \dots, N$ such that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. Put*

$$\begin{aligned} G_n \;=\; \max\Big\{ &\sup_{p \in F,\ n \geq 0} \Big(d(T_n^n x_n, p) - d(x_n, p)\Big) \vee \sup_{p \in F,\ n \geq 0} \Big(d(T_n^n y_n, p) - d(y_n, p)\Big) \\ &\vee \sup_{p \in F,\ n \geq 0} \Big(d(T_n^n z_n, p) - d(z_n, p)\Big) \vee 0\Big\}, \end{aligned}$$

such that $\sum_{n=0}^{\infty} G_n < \infty$. Let $\{x_n\}$ be the iterative sequence with errors defined by (1.14) and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ be three bounded sequences in C and $\{\alpha_n\}$, $\{\beta_n\}$,

$\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$ and $\{f_n\}$ be nine sequences in $[0,1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = d_n + e_n + f_n = 1, \quad \forall n \geq 0$;

(ii) $\sum_{n=0}^{\infty}(\beta_n + \gamma_n) < \infty$.

Then the sequence $\{x_n\}$ converges strongly to a common fixed point p of the mappings $\{T_i\}_{i=1}^N$ if and only if

$$\liminf_{n\to\infty} d(x_n, F) = 0,$$

where $d(x,F) = \inf_{p\in F} d(x,p)$.

Taking $d_n = 1 (e_n = f_n = 0)$ for all $n \geq 0$ in Theorem 2.2, then we have the following theorem.

Theorem 2.4. *Let (E,d,W) be a complete convex metric space and C be a nonempty closed convex subset of E. Let $T_i\colon C \to C$ be a finite family of uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings in the intermediate sense for $i = 1,2,\dots,N$ such that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$ and $g\colon C \to C$ a contractive mapping with a contractive constant $\xi \in (0,1)$. Put*

$$G_n = \max\Big\{ \sup_{p\in F,\ n\geq 0} \Big(d(T_n^n x_n, p) - d(x_n,p)\Big) \vee \sup_{p\in F,\ n\geq 0} \Big(d(T_n^n y_n, p) - d(y_n,p)\Big) \vee 0\Big\},$$

such that $\sum_{n=0}^{\infty} G_n < \infty$. Let $\{x_n\}$ be the iterative sequence with errors defined by (1.15) and $\{u_n\}$, $\{v_n\}$ be two bounded sequences in C and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be six sequences in $[0,1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = 1$, *for all* $n \geq 0$;

(ii) $\sum_{n=0}^{\infty}(\beta_n + \gamma_n) < \infty$.

Then the sequence $\{x_n\}$ converges to a common fixed point p in F if and only if

$$\liminf_{n\to\infty} d(x_n, F) = 0,$$

where $d(x,F) = \inf_{p\in F} d(x,p)$.

Remark 3. *Theorems 2.2 - 2.4 generalize, improve and unify some corresponding result in* [1]-[7], [9]-[11], [13]-[17], [19] *and* [21].

Remark 4. *Our results also extend the corresponding results of* [18] *to the case of more general class of uniformly quasi-Lipschitzian mappings considered in this paper.*

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G. S. Saluja

Department of Mathematics & Information Technology, Govt. Nagarjuna P.G. College of Science, Raipur - 492010 (C.G.), India.

E-mail address: saluja_1963@rediffmail.com, saluja1963@gmail.com