

## ON SOME COMMON FIXED POINT THEOREMS WITH $\varphi$ -MAPS ON $G$ -CONE METRIC SPACES

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ABSTRACT. In this paper, we prove some fixed point theorems for mappings in  $G$ -cone metric space which is defined by I. Beg, M. Abbas and T. Nazir in [5] via some contractive conditions related to  $\varphi$ -maps. Our results are generalization and extension of several well-known results related to fixed point theory.

### 1. INTRODUCTION

Since fixed point theory plays a major role in mathematics and applied sciences, such as optimization, mathematical models, economy and medicine. The metric fixed point theory has been researched extensively in the past two decades. The concept of the metric spaces different generalizations have been improved by Gahler [9],[10] and Dhage [1]. Gahler studied 2-metric spaces and also Dhage's theory was related to D-metric spaces.

In 2005, Mustafa and Sims [13] introduced a new structure of generalized metric spaces which are called  $G$ - metric spaces as a generalization of metric spaces. Afterwards Mustafa et al. [14]-[16] obtained several fixed point theorems for mappings satisfying different contractive conditions in  $G$ - metric spaces.

Later on, Huang and Zhang [4] generalized the concept of metric spaces, replacing the set of real numbers by an ordered Banach space, hence they have defined the cone metric spaces. They also described the convergence of sequences and introduced the notion of completeness in cone metric spaces. They have proved some fixed point theorems of contractive mappings on complete cone metric space with the assumption of normality of a cone. Subsequently, various authors have generalized the results of Huang and Zhang and have studied fixed point theorems for normal and non-normal cones.

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Recently, I. Beg, M. Abbas and T. Nazir [5] introduced  $G$ - cone metric spaces which are generalization of  $G$ - metric spaces and cone metric spaces. They proved some topological properties of these spaces such as convergence properties of sequences and completeness. Some fixed point theorems satisfying certain contractive conditions have been also obtained.

Some theorems which given with  $\varphi$ -maps have been proved by Cristina Di Pari and Pasquale Vetro [2] in cone metric spaces and W. Shatanawi [12] also obtained some fixed point results in  $G$ - metric spaces.

The purpose of this paper is to obtain some fixed point results which satisfy generalized contractive conditions defined by generalized  $\varphi$ -maps. Our results are generalizations of some theorems in [2], [6], [7], [12].

## 2. BASIC FACTS AND DEFINITIONS

We give some facts and definitions which we need them in the sequel. First we give definition of generalized cone.

Let  $B$  be a real Banach space and  $K$  be a subset of  $B$ .  $K$  is called a *cone* if and only if

- i.  $K$  is closed, nonempty and  $K \neq \{0\}$ ,
- ii.  $a, b \in \mathbb{R}$ ,  $a, b \geq 0$ ,  $x, y \in K \Rightarrow ax + by \in K$ , more generally if  $a, b, c \in \mathbb{R}$ ,  
 $a, b, c \geq 0$ ,  $x, y, z \in K \Rightarrow ax + by + cz \in K$ ,
- iii.  $x \in K$  and  $-x \in K \Rightarrow x = 0$ .

Given a cone  $K \subset E$ , we define a partial ordering  $\leq$  with respect to  $K$  by  $x \leq y$  if and only if  $y - x \in K$ . We write  $x < y$  if  $x \leq y$  but  $x \neq y$ ;  $x \ll y$  if  $y - x \in \text{int}K$ , where  $\text{int}K$  is the interior of  $K$ .

There exists two kinds of cones which are normal and non normal cones. The cone  $K$  is a normal cone if

$$\inf \{ \|x + y\| : x, y \in K \text{ and } \|x\| = \|y\| = 1 \} > 0 \quad (2.1)$$

or equivalently, if there is a number  $M > 0$  such that for all  $x, y \in E$ ,

$$0 \leq x \leq y \Rightarrow \|x\| \leq M \|y\|. \quad (2.2)$$

The least positive number satisfying (2.2) is called normal constant of  $K$ . From (2.1) one can conclude that  $K$  is a non normal if and only if there exist sequences  $x_n, y_n \in K$  such that

$$0 \leq x_n \leq x_n + y_n, \lim_{n \rightarrow \infty} (x_n + y_n) = 0, \text{ but } \lim_{n \rightarrow \infty} x_n \neq 0$$

Rezapour and Hamlbarani [11] proved that there are no normal cones with constants  $M < 1$  and for each  $k > 1$  there are cones with normal constants  $M > k$ .

**Definition 2.1.** [5] *Let  $X$  be nonempty set,  $B$  be a real Banach space and  $K \subset B$  be a cone. Suppose the mapping  $G : X \times X \times X \rightarrow B$  satisfies*

- G1.  $G(x, y, z) = 0$  if  $x = y = z$ ,
- G2.  $0 < G(x, x, y)$ ; whenever  $x \neq y$ , for all  $x, y \in X$ ,
- G3.  $G(x, x, y) \leq G(x, y, z)$ ; whenever  $y \neq z$ , for all  $x, y, z \in X$ ,
- G4.  $G(x, y, z) = G(x, z, y) = G(y, x, z) = \dots$  (Symmetric in all three variables),
- G5.  $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$  for all  $x, y, z, a \in X$ .

Then  $G$  is called a generalized cone metric on  $X$  and  $X$  is called a generalized cone metric space or more specifically a  $G$ -cone metric space. It is obvious that the concept of a  $G$ -cone metric space is more general than a  $G$ -metric space and a cone metric space.

**Definition 2.2.** [5] *A  $G$ -cone metric space  $X$  is symmetric if*

$$G(x, y, y) = G(y, x, x)$$

for all  $x, y \in X$ .

The following examples are symmetric and non symmetric  $G$ -cone metric spaces, respectively.

**Example 1.** [5] *Let  $(X, d)$  be a cone metric space. Define  $G : X \times X \times X \rightarrow B$ , by*

$$G(x, y, z) = d(x, y) + d(y, z) + d(x, z).$$

**Example 2.** [5] *Let  $X = \{a, b\}$ ,  $B = \mathbb{R}^3$ ,  $K = \{(x, y, z) \in B : x, y, z \geq 0\}$ . Define  $G : X \times X \times X \rightarrow B$  by*

$$\begin{aligned} G(a, a, a) &= (0, 0, 0) = G(b, b, b), \\ G(a, b, b) &= (0, 1, 1) = G(b, a, b) = G(b, b, a), \\ G(b, a, a) &= (0, 1, 0) = G(a, b, a) = G(a, a, b), \end{aligned}$$

$X$  is non symmetric  $G$ -cone metric space as  $G(a, a, b) \neq G(a, b, b)$ .

**Proposition 2.3.** [5] *Let  $X$  be a  $G$ -cone metric space, define*

$$d_G : X \times X \rightarrow B$$

by

$$d_G(x, y) = G(x, y, y) + G(y, x, x).$$

Then  $(X, d_G)$  is a cone metric space. Also it can be written that

$$G(x, y, y) \leq \frac{2}{3}d_G(x, y).$$

If  $X$  is a symmetric  $G$ -cone metric space, then

$$d_G(x, y) = 2G(x, y, y),$$

for all  $x, y \in X$ .

Throughout the paper we assume that  $B$  is a real Banach space and  $K$  is a non normal cone in  $B$  with  $\text{int}K \neq \emptyset$ . By this way, we uniquely determine the limit of a sequence.

**Definition 2.4.** [5] *Let  $X$  be a  $G$ -cone metric space and  $\{x_n\}$  be a sequence in  $X$  and  $x \in X$ . We say that  $\{x_n\}$  is a*

- i. *Convergent sequence if for every  $c \in B$  with  $0 \ll c$ , there is  $N \in \mathbb{N}$  such that for all  $m, n > N$ ,  $G(x_m, x_n, x) \ll c$  for some fixed  $x$  in  $X$ .*
- ii. *Cauchy sequence if for every  $c \in B$  with  $0 \ll c$ , there is  $N \in \mathbb{N}$  such that for all  $m, n, l > N$ ,  $G(x_m, x_n, x_l) \ll c$ .*

A  $G$ -cone metric space  $X$  is said to be complete if every Cauchy sequence in  $X$  is convergent in  $X$ .

**Proposition 2.5.** [5] *Let  $X$  be a  $G$ -cone metric space then the followings are equivalent;*

- i.  $\{x_n\}$  converges to  $x$ .
- ii.  $G(x_n, x_n, x) \rightarrow 0$  as  $n \rightarrow \infty$ .
- iii.  $G(x_n, x, x) \rightarrow 0$  as  $n \rightarrow \infty$ .
- iv.  $G(x_m, x_n, x) \rightarrow 0$  as  $m, n \rightarrow \infty$ .

The following lemmas are about topological structure of  $G$ -cone metric space and these lemmas have been proved in [5], so we give them without the proofs.

**Lemma 2.6.** *Let  $X$  be a  $G$ -cone metric space,  $\{x_m\}$ ,  $\{y_n\}$  and  $\{z_l\}$  be sequences in  $X$  such that  $x_m \rightarrow x$ ,  $y_n \rightarrow y$  and  $z_l \rightarrow z$ , then  $G(x_m, y_n, z_l) \rightarrow G(x, y, z)$  as  $m, n, l \rightarrow \infty$ .*

**Lemma 2.7.** *Let  $\{x_n\}$  be sequence in a  $G$ -cone metric space  $X$  and  $x \in X$ . If  $\{x_n\}$  converges to  $x$  and  $\{x_n\}$  converges to  $y$ , then  $x = y$ .*

**Lemma 2.8.** *Let  $\{x_n\}$  be sequence in a  $G$ -cone metric space  $X$  and if  $\{x_n\}$  converges to  $x \in X$ , then  $G(x_m, x_n, x) \rightarrow 0$  as  $m, n \rightarrow \infty$ .*

**Lemma 2.9.** *Let  $\{x_n\}$  be sequence in a  $G$ -cone metric space  $X$  and  $x \in X$ . If  $\{x_n\}$  converges to  $x$ , then  $\{x_n\}$  is a Cauchy sequence.*

**Lemma 2.10.** *Let  $\{x_n\}$  be sequence in a  $G$ -cone metric space  $X$  and if  $\{x_n\}$  is a Cauchy sequence in  $X$ , then  $G(x_m, x_n, x_l) \rightarrow 0$  as  $m, n, l \rightarrow \infty$ .*

**Remark.** [3] *If  $B$  is a real Banach space with cone  $K$  and if  $a \leq \lambda a$  where  $a \in K$  and  $0 < \lambda < 1$  then  $a = 0$ .*

**Definition 2.11.** Let  $T$  and  $S$  be self mappings of a set  $X$ . If  $w = Tx = Sx$  for some  $x$  in  $X$ , then  $x$  is called a coincidence point of  $T$  and  $S$  and  $w$  is called a point of coincidence of  $f$  and  $g$ .

**Definition 2.12.** The mappings  $T, S : X \rightarrow X$  are weakly compatible, if for every  $x \in X$ , the following holds:

$$TSx = STx \text{ whenever } Sx = Tx. \tag{2.3}$$

**Definition 2.13.** [2] Let  $K$  be a cone defined as above. A nondecreasing function  $\varphi : K \rightarrow K$  is called a  $\varphi$ -map if the following conditions hold,

- i.  $\varphi(\theta) = \theta$  and  $\theta < \varphi(z) < z$  for  $z \in K \setminus \{\theta\}$ ,
- ii.  $z \in \text{int}K$  implies  $z - \varphi(z) \in \text{int}K$ ,
- iii.  $\lim_{n \rightarrow \infty} \varphi^n(z) = \theta$  for every  $z \in K \setminus \{\theta\}$ .

### 3. MAIN RESULTS

In [2] some fixed point theorems related to  $\varphi$ -maps have been obtained. Also in a  $G$ -metric space, fixed point theorems for contractive mappings satisfying  $\varphi$ -maps have been proved by W.Shatanawi [12] Now, we get some fixed point results with  $\varphi$ - pairs in a  $G$ -cone metric space.

**Theorem 3.1.** Let  $X$  be a  $G$ -cone metric space and let the mappings  $T, S : X \rightarrow X$  satisfy the following;

$$G(Tx, Ty, Tz) \leq \varphi(G(Sx, Sy, Sz)) \tag{3.1}$$

for all  $x, y, z \in X$ . Suppose that  $T$  and  $S$  are weakly compatible with  $T(X) \subset S(X)$ . If  $T(X)$  or  $S(X)$  is a complete subspace of  $X$ , then the mappings  $T$  and  $S$  have a unique common fixed point in  $X$ .

*Proof.* Let  $x_0 \in X$  be arbitrary. Choose  $x_1 \in X$  such that  $Tx_0 = Sx_1$ . This is true since  $T(X) \subset S(X)$ . Continuing this process, having chosen  $x_n \in X$ , we choose  $x_{n+1} \in X$  such that  $Tx_n = Sx_{n+1}$  for all  $n \in \mathbb{N}$ . If  $Tx_n = Tx_{n-1}$  for some  $n \in \mathbb{N}$ , then  $Tx_m = Tx_n$  for all  $m \in \mathbb{N}$  with  $m > n$  and so  $\{Tx_n\}$  is a Cauchy sequence. We assume that  $Tx_n \neq Tx_{n-1}$  for all  $n \in \mathbb{N}$ . By (3.1), we have

$$\begin{aligned} G(Tx_{n+1}, Tx_{n+1}, Tx_n) &\leq \varphi(G(Sx_{n+1}, Sx_{n+1}, Sx_n)) \\ &= \varphi(G(Tx_n, Tx_n, Tx_{n-1})) \\ &\leq \varphi^2(G(Sx_n, Sx_n, Sx_{n-1})) \\ &= \varphi^2(G(Tx_{n-1}, Tx_{n-1}, Tx_{n-2})) \\ &\vdots \\ &\vdots \\ &\vdots \\ &\leq \varphi^n(G(Tx_1, Tx_1, Tx_0)). \end{aligned}$$

Given  $\theta \ll c$  and we choose a positive real number  $\delta$  such that  $c - \varphi(c) + N(\theta + \delta) \subset \text{int}K$ , where  $N(\theta + \delta) = \{y \in B : \|y\| < \delta\}$ . Also choose a natural number  $N$  such that  $\varphi^m(G(Tx_1, Tx_1, Tx_0)) \ll c - \varphi(c)$  for all  $m \geq N$ . Consequently

$$G(Tx_{m+1}, Tx_{m+1}, Tx_m) \ll c - \varphi(c)$$

for all  $m \geq N$ . Fix  $m \geq N$  and we prove

$$G(Tx_m, Tx_{n+1}, Tx_{n+1}) \ll c \quad (3.2)$$

for all  $n \geq m$ . We write (3.2) holds when  $n = m$ . We suppose that (3.2) holds for some  $n \geq m$ . Then we have by using G5,

$$\begin{aligned} G(Tx_m, Tx_{n+2}, Tx_{n+2}) &\leq G(Tx_m, Tx_{m+1}, Tx_{m+1}) + G(Tx_{m+1}, Tx_{n+2}, Tx_{n+2}) \\ &\leq G(Tx_m, Tx_{m+1}, Tx_{m+1}) + G(Tx_{m+1}, Tx_{n+2}, Tx_{n+2}) \\ &\ll c - \varphi(c) + \varphi(G(Sx_{m+1}, Sx_{n+2}, Sx_{n+2})) \\ &\ll c - \varphi(c) + \varphi(G(Tx_m, Tx_{n+1}, Tx_{n+1})) \\ &\ll c - \varphi(c) + \varphi(c) = c. \end{aligned}$$

Therefore, (3.2) holds when  $m = n + 1$ . By induction, we deduce (3.2) holds for all  $m, n \geq N$ . Hence  $\{Tx_n\}$  is a Cauchy sequence. Suppose  $T(X)$  is a complete subspace of  $X$ , then there exists  $w \in T(X) \subset S(X)$  such that  $Tx_n \rightarrow w$  and also  $Sx_n \rightarrow w$ . Let  $v \in X$  be such that  $Sv = w$ . We prove that  $Sv = Tv$ .

Fix  $\theta \ll c$  and we choose a natural number  $N$  such that  $G(w, Tx_n, Tx_n) \ll \frac{c}{2}$  and  $G(Sx_n, Sv, Sv) \ll \frac{c}{2}$ . Then by using G5,

$$\begin{aligned} G(w, Tv, Tv) &\leq G(w, Tx_n, Tx_n) + G(Tx_n, Tv, Tv) \\ &\leq G(w, Tx_n, Tx_n) + \varphi(G(Sx_n, Sv, Sv)) \end{aligned}$$

by using property of  $\varphi$  we get

$$\begin{aligned} F(G(w, Tv, Tv)) &< G(w, Tx_n, Tx_n) + G(Sx_n, Sv, Sv) \\ &\ll \frac{c}{2} + \frac{c}{2} = c. \end{aligned}$$

Thus,  $G(w, Tv, Tv) \ll \frac{c}{i}$  for all  $i \geq 1$ . Since  $\frac{c}{i} - G(w, Tv, Tv) \in K$ , for all  $i$ , as  $i \rightarrow \infty$  we get  $-G(w, Tv, Tv) \in K$ . But  $G(w, Tv, Tv) \in K$ . Therefore  $G(w, Tv, Tv) = \theta$  which implies that  $Sv = Tv = w$ , that is  $w$ , is a coincidence point of  $T$  and  $S$ . To show that  $w$  is a common fixed point of  $T$  and  $S$ , we need to use the hypothesis of weak compatibility of the mappings. As  $Tv = Sv$ , by weak compatibility of  $T$  and  $S$ , it follows that

$$Tw = TSv = STv = Sw.$$

We show that  $Tw = Sw = w$ . If  $Sw \neq w$ , by condition (3.1), we get

$$\begin{aligned} G(Tw, Tw, Tw) &\leq \varphi(G(Sw, Sw, Sv)) \\ &< G(Sw, Sw, Sv) \\ &= G(Tw, Tw, Tw) \end{aligned}$$

which gives us that  $Tw = w = Sw$ . Then  $w$  is a common fixed point for the mappings  $T$  and  $S$ .

Finally, let us suppose that  $u$  is another common fixed point of  $T$  and  $S$ . For the proof we use (3.1).

$$\begin{aligned} G(w, w, u) &= G(Tw, Tw, Tu) \\ &\leq \varphi(G(Sw, Sw, Su)) \\ &< G(Sw, Sw, Su) = G(w, w, u) \end{aligned}$$

which is a contradiction, so uniqueness is obtained.  $\square$

**Theorem 3.2.** *Let  $X$  be a complete symmetric  $G$ -cone metric space. Suppose that the mappings  $T, S : X \rightarrow X$  satisfy the following*

$$G(Tx, Ty, Tz) \leq kG(Sx, Sy, Sz) \quad (3.3)$$

for all  $x, y, z \in X$ , where  $k \in [0, 1)$  is a constant. If  $T(X) \subset S(X)$  and  $S(X)$  is a complete subspace of  $X$ , then  $T$  and  $S$  have a unique point of coincidence in  $X$ . Moreover if  $T$  and  $S$  are weakly compatible,  $T$  and  $S$  have a unique common fixed point.

*Proof.* Assume that  $T$  satisfies the inequality (3.3), then for all  $x, y \in X$

$$G(Tx, Ty, Ty) \leq kG(Sx, Sy, Sy) \quad (3.4)$$

and

$$G(Ty, Tx, Tx) \leq kG(Sy, Sx, Sx). \quad (3.5)$$

Since  $X$  is a symmetric  $G$ -cone metric space, by adding (3.4) and (3.5) we have

$$d_G(Tx, Ty) \leq kd_G(Sx, Sy) \quad (3.6)$$

for all  $x, y \in X$ .

Let  $x_0 \in X$  be an arbitrary point in  $X$ . Choose  $x_1 \in X$  such that  $Tx_0 = Sx_1$ . This is true since  $T(X) \subset S(X)$ . Continuing this process, having chosen  $x_n \in X$ , we choose  $x_{n+1} \in X$  such that  $Tx_n = Sx_{n+1}$  for all  $n \in \mathbb{N}$ . Then we have

$$\begin{aligned} d_G(Sx_{n+1}, Sx_n) &= d_G(Tx_n, Tx_{n-1}) \leq kd_G(Sx_n, Sx_{n-1}) \\ &\leq k^2 d_G(Sx_{n-1}, Sx_{n-2}) \\ &\leq \dots \leq k^n d_G(Sx_1, Sx_0). \end{aligned}$$

Then, for  $n > m$ , we have

$$\begin{aligned} d_G(Sx_n, Sx_m) &\leq d_G(Sx_n, Sx_{n-1}) + d_G(Sx_{n-1}, Sx_{n-2}) + \dots + d_G(Sx_{m+1}, Sx_m) \\ &\leq (k^{n-1} + k^{n-2} + \dots + k^m) d_G(Sx_1, Sx_0) \\ &\leq \frac{k^m}{1-k} d_G(Sx_1, Sx_0). \end{aligned}$$

Let  $0 \ll c$  be given. Following similar arguments to those given in [11], we conclude that  $\frac{k^m}{1-k} d_G(Sx_1, Sx_0) \ll c$ . So we have  $d_G(Sx_n, Sx_m) \ll c$ , for all  $n > m$ . Therefore  $\{Sx_n\}$  is a Cauchy sequence. Since  $S(X)$  is a complete subspace of  $X$ , then there exists a  $w \in S(X)$  such that  $Sx_n \rightarrow w$  as  $n \rightarrow \infty$ . Hence we can find  $v$  in  $X$  such that  $Sv = w$ . We show that  $Sv = Tv$ . Given  $\theta \ll c$  and we choose a natural number  $N$  such that

$$d_G(Sx_n, w) \ll \frac{c}{2} \text{ and } d_G(Tx_n, w) \ll \frac{c}{2}.$$

Then,

$$\begin{aligned} d_G(w, Tv) &\leq d_G(w, Tx_n) + d_G(Tx_n, Tv) \\ &\leq d_G(w, Tx_n) + kd_G(Sx_n, Sv) \\ &\ll \frac{c}{2} + \frac{c}{2} = c. \end{aligned}$$

Thus,  $d_G(w, Tv) \ll \frac{c}{i}$  for all  $i \geq 1$ . Since  $\frac{c}{i} - d_G(w, Tv) \in K$  for all  $i$ , as  $i \rightarrow \infty$  we obtain  $-d_G(w, Tv) \in K$ . But  $d_G(w, Tv) \in K$ . Hence,  $d_G(w, Tv) = \theta$  which implies that  $Tv = Sv = w$ . So that  $v$  is a coincidence of  $T$  and  $S$ .

Now we use the hypothesis that  $T$  and  $S$  are weakly compatible to deduce that  $w$  is a common fixed point. Since  $Tv = Sv$ , by weak compatibility of  $T$  and  $S$ , this gives that

$$Tw = TSv = STv = Sw.$$

We show that  $Tw = Sw = w$ . If  $Sw \neq w$ , by condition (3.6) we get

$$d_G(Tw, Tv) \leq kd_G(Sw, Sv) = kd_G(Tw, Tv)$$

which gives that  $Tw = Sw = w$ . Then  $w$  is a common fixed point for  $T$  and  $S$ .

The uniqueness can be obtained easily, so we omit it.  $\square$

**Corollary 3.3.** *Let  $X$  be a complete symmetric  $G$ -cone metric space. Suppose that the mapping  $T : X \rightarrow X$  satisfies the following*

$$G(Tx, Ty, Tz) \leq kG(x, y, z) \quad (3.7)$$

for all  $x, y, z \in X$ , where  $k \in [0, 1)$  is a constant. Then  $T$  has unique common fixed point.

*Proof.* The proof can be obtained from Theorem 3.2 by taking  $S = I$  where  $I$  is identity map.  $\square$

**Theorem 3.4.** *Let  $X$  be a complete  $G$ -cone metric space. Suppose that the map  $T : X \rightarrow X$  satisfies*

$$G(Tx, Ty, Tz) \leq \varphi(M(x, y, z)) \quad (3.8)$$

where

$$M(x, y, z) \in \{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(Tx, y, z)\} \quad (3.9)$$

for all  $x, y, z \in X$ . Then  $T$  has a unique fixed point in  $X$ .

*Proof.* Choose  $x_0 \in X$ . Let  $x_n = Tx_{n-1}$ , for  $n \in \mathbb{N}$ . Suppose that  $x_n \neq x_{n-1}$ , for each  $n \in \mathbb{N}$ . Thus we have

$$G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n) \leq \varphi(M(x_{n-1}, x_n, x_n))$$

where

$$\begin{aligned}
M(x_{n-1}, x_n, x_n) &\in \{G(x_{n-1}, x_n, x_n), G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), \\
&\quad G(x_n, Tx_n, Tx_n), G(Tx_{n-1}, x_n, x_n)\} \\
&= \{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), \\
&\quad G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_n, x_n)\} \\
&= \{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), \theta\}.
\end{aligned}$$

If  $M(x_{n-1}, x_n, x_n) = G(x_n, x_{n+1}, x_{n+1})$ , then

$$G(x_n, x_{n+1}, x_{n+1}) \leq \varphi(G(x_n, x_{n+1}, x_{n+1}))$$

by the property of  $\varphi$  we have

$$G(x_n, x_{n+1}, x_{n+1}) < G(x_n, x_{n+1}, x_{n+1})$$

which is impossible. If  $M(x_{n-1}, x_n, x_n) = \theta$ , then

$$G(x_n, x_{n+1}, x_{n+1}) \leq \varphi(\theta) < \theta$$

which is a contradiction. And at last, if  $M(x_{n-1}, x_n, x_n) = G(x_{n-1}, x_n, x_n)$ , then

$$G(x_n, x_{n+1}, x_{n+1}) \leq \varphi(G(x_{n-1}, x_n, x_n))$$

and by using the same technique as in Theorem 3.1, we conclude that  $\{x_n\}$  is a Cauchy sequence. Since  $X$  is complete,  $x_n$  is convergent to  $u \in X$ . Now we show that  $u = Tu$ . For  $n \in \mathbb{N}$ , we have by using G5

$$\begin{aligned}
G(u, u, Tu) &\leq G(u, u, x_n) + G(x_n, x_n, Tu) \\
&= G(u, u, x_n) + G(Tx_{n-1}, Tx_{n-1}, Tu) \\
&\leq G(u, u, x_n) + \varphi(M(x_{n-1}, x_{n-1}, u))
\end{aligned}$$

and

$$\begin{aligned}
M(x_{n-1}, x_{n-1}, u) &\in \{G(x_{n-1}, x_{n-1}, u), G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), \\
&\quad G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), G(Tx_{n-1}, x_{n-1}, u)\} \\
&= \{G(x_{n-1}, x_{n-1}, u), G(x_{n-1}, x_n, x_n), \\
&\quad G(x_n, x_{n-1}, u)\}.
\end{aligned}$$

Choose a natural number  $N_1$  such that  $G(u, u, x_n) \ll \frac{c}{2}$ , for all  $n \geq N_1$ . We investigate these situations as follows;

**Case 1:** If  $M(x_{n-1}, x_{n-1}, u) = G(x_{n-1}, x_{n-1}, u)$ , then

$$\begin{aligned}
G(u, u, Tu) &\leq G(u, u, x_n) + \varphi(G(x_{n-1}, x_{n-1}, u)) \\
&< G(u, u, x_n) + G(x_{n-1}, x_{n-1}, u) \\
&\ll \frac{c}{2} + \frac{c}{2} = c.
\end{aligned}$$

**Case 2:** If  $M(x_{n-1}, x_{n-1}, u) = G(x_{n-1}, x_n, x_n)$ , then

$$\begin{aligned}
G(u, u, Tu) &\leq G(u, u, x_n) + \varphi(G(x_{n-1}, x_n, x_n)) \\
&< G(u, u, x_n) + G(x_{n-1}, x_n, x_n) \\
&\ll c.
\end{aligned}$$

**Case 3:** If  $M(x_{n-1}, x_{n-1}, u) = G(x_n, x_{n-1}, u)$ , then

$$\begin{aligned} G(u, u, Tu) &\leq G(u, u, x_n) + \varphi(G(x_n, x_{n-1}, u)) \\ &< G(u, u, x_n) + G(x_n, x_{n-1}, u) \\ &\leq G(u, u, x_n) + G(x_n, x_{n-1}, x_{n-1}) + G(x_{n-1}, x_{n-1}, u) \\ &\ll c. \end{aligned}$$

whenever  $n \in \mathbb{N}$ . Thus in all cases  $G(u, u, Tu) \ll \frac{c}{i}$ , for all  $i \geq 1$ . So  $\frac{c}{i} - G(u, u, Tu) \in K$ , for all  $i \geq 1$ . Since  $\frac{c}{i} \rightarrow 0$  as  $i \rightarrow \infty$  and  $K$  is closed, hence  $-G(u, u, Tu) \in K$  and  $G(u, u, Tu) = \theta$  therefore  $u = Tu$ .

And this gives us the desired result.  $\square$

The following theorem is an extension of Theorem 2.1 and 2.2 in [8] to  $G$ -cone metric spaces.

**Theorem 3.5.** *Let  $X$  be a complete  $G$ -cone metric space,  $T$  a self map of  $X$  satisfying for all  $x, y, z \in X$*

$$G(Tx, Ty, Tz) \leq kM(x, y, z) \quad (3.10)$$

where

$$\begin{aligned} M(x, y, z) \in \{ &G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz) \\ &[G(x, Ty, Ty) + G(z, Tx, Tx)]/2, [G(x, Ty, Ty) + G(y, Tx, Tx)]/2, \\ &[G(y, Tz, Tz) + G(z, Ty, Ty)]/2, [G(x, Tz, Tz) + G(z, Tx, Tx)]/2 \} \end{aligned}$$

and  $k$  is a constant satisfying  $0 \leq k < 1$ . Then  $T$  has a unique fixed point.

*Proof.* Applying the similar method as in Theorem 3.4 with taking  $\varphi(x) = kx$ , where  $k \in [0, 1)$ .  $\square$

**Theorem 3.6.** *Let  $X$  be a complete symmetric  $G$ -cone metric space,  $T$  a self map of  $X$  satisfying for all  $x, y, z \in X$*

$$G(Tx, Ty, Tz) \leq km(x, y, z) \quad (3.11)$$

where

$$\begin{aligned} m(x, y, z) \in \{ &G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty) \\ &G(x, Ty, Ty), G(y, Tx, Tx), G(z, Tz, Tz) \} \end{aligned} \quad (3.12)$$

or

$$\begin{aligned} m^*(x, y, z) \in \{ &G(x, y, z), G(x, x, Tx), G(y, y, Ty) \\ &G(x, x, Ty), G(y, y, Tx), G(z, z, Tz) \} \end{aligned} \quad (3.13)$$

here  $k$  is a constant satisfying  $k \in [0, 1)$ . Then  $T$  has a unique fixed point.

*Proof.* Assume that  $T$  satisfies (3.11). Using (3.11) with  $z = y$  we have

$$G(Tx, Ty, Ty) \leq km'(x, y, y)$$

we have

$$\begin{aligned} m'(x, y, y) \in \{ &G(x, y, y), G(x, Tx, Tx), G(y, Ty, Ty), G(x, Ty, Ty), \\ &G(y, Tx, Tx) \} \end{aligned}$$

and

$$m''(x, y, y) \in \{G(x, y, y), G(x, x, Tx), G(y, y, Ty), G(x, x, Ty), G(y, y, Tx)\}.$$

By Proposition 2.3 we know that  $d_G(x, y) = 2G(x, y, y)$  makes  $X$  to cone metric space

$$m'(x, y) \in \{d_G(x, y), d_G(x, Tx), d_G(y, Ty), d_G(x, Ty), d_G(y, Tx)\}.$$

Let  $x_0 \in X$  and  $x_n = Tx_{n-1}$ . Suppose that  $x_n \neq x_{n+1}$ , then

$$d_G(x_n, x_{n+1}) = d_G(Tx_{n-1}, Tx_n) \leq km^{**}(x_{n-1}, x_n)$$

where

$$\begin{aligned} m^{**}(x_{n-1}, x_n) &\in \{d_G(x_{n-1}, x_n), d_G(x_{n-1}, Tx_{n-1}), d_G(x_n, Tx_n), \\ &\quad d_G(x_{n-1}, Tx_n), d_G(x_n, Tx_{n-1})\} \\ &\in \{d_G(x_{n-1}, x_n), d_G(x_n, x_{n+1}), d_G(x_{n-1}, x_{n+1}), \theta\}. \end{aligned}$$

We investigate these possibilities with four cases:

**Case 1:** If  $m^{**}(x_{n-1}, x_n) = d_G(x_{n-1}, x_{n+1})$ , then

$$\begin{aligned} d_G(x_n, x_{n+1}) &\leq kd_G(x_{n-1}, x_{n+1}) \\ &\leq k[d_G(x_{n-1}, x_n) + kd_G(x_n, x_{n+1})] \\ &\leq \frac{k}{1-k}d_G(x_{n-1}, x_n) \leq kd_G(x_{n-1}, x_n). \end{aligned}$$

**Case 2:** If  $m^{**}(x_{n-1}, x_n) = d_G(x_{n-1}, x_{n+1})$ , then

$$d_G(x_n, x_{n+1}) \leq kd_G(x_n, x_{n+1})$$

we have  $d_G(x_n, x_{n+1})(1-k) \leq \theta$ , since  $k \in [0, 1)$  this a contradiction.

**Case 3:** If  $m^{**}(x_{n-1}, x_n) = \theta$ , then

$$d_G(x_n, x_{n+1}) \leq k\theta$$

which contradict with the assumption of  $x_n \neq x_{n+1}$ .

And the last case we have

**Case 4:** If  $m^{**}(x_{n-1}, x_n) = d_G(x_{n-1}, x_n)$ , then

$$\begin{aligned} d_G(x_n, x_{n+1}) &\leq kd_G(x_{n-1}, x_n) \leq k^2d_G(x_{n-2}, x_{n-1}) \\ &\leq \dots \leq k^n d_G(x_0, x_1). \end{aligned}$$

So we get the desired result. And the continuation of proof is same with the technique as in [11] Theorem 2.3. By this way we obtain that  $T$  has a unique fixed point.  $\square$

And last we give an example for Theorem 3.1.

**Example 3.** Let  $E = \mathbb{R}$  and  $K = \{x \in \mathbb{R} : x \geq 0\}$  be a cone. Let  $X = [1, \infty)$  with the following metric

$$G(x, y, z) = d(x, y) + d(y, z) + d(x, z)$$

and the usual metric  $d(x, y) = |x - y|$ . Define the two maps  $T, S : X \rightarrow X$  by

$$\begin{aligned} Tx &= x, \\ Sx &= 2x - 1, \end{aligned}$$

for all  $x \in X$ . And let define the function  $\varphi : K \rightarrow K$  by  $\varphi t = \frac{2}{3}t$ , for all  $t \in K$ . Then we have

- i.  $TX \subset SX$ ,
- ii.  $T$  and  $S$  are weakly compatible maps,
- iii. the condition (3.1) holds as,

$$\begin{aligned} G(Tx, Ty, Tz) &= d(Tx, Ty) + d(Ty, Tz) + d(Tx, Tz) \\ &= |Tx - Ty| + |Ty - Tz| + |Tx - Tz| \\ &= |x - y| + |y - z| + |x - z| \\ &\leq \frac{4}{3}(|x - y| + |y - z| + |x - z|) \\ &= \frac{2}{3}(|2x - 2y| + |2y - 2z| + |2x - 2z|) \\ &= \frac{2}{3}(|2x - 1 - 2y + 1| + |2y - 1 - 2z + 1| + |2x - 1 - 2z + 1|) \\ &= \frac{2}{3}(|Sx - Sy| + |Sy - Sz| + |Sx - Sz|) \\ &= \frac{2}{3}G(Sx, Sy, Sz) \\ G(Tx, Ty, Tz) &\leq \varphi(G(Sx, Sy, Sz)). \end{aligned}$$

- iv.  $T1 = S1 = 1$ .

Hence we have the conditions of Theorem 3.1 and we see that  $x = 1$  is unique common fixed point for  $T$  and  $S$ .

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