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ON MODULAR EQUATIONS AND LAMBERT SERIES FOR A CONTINUED FRACTION OF RAMANUJAN

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Abstract. Modular equations and generalized Lambert series are given for theta functions $G_1(q)$ and $H_1(q)$.

1. Introduction

In [6], we considered the continued fraction of Ramanujan defined by

$$C(q) = \frac{1}{1+}\frac{(1+q)}{1+}\frac{q^2}{1+}\frac{(q+q^3)}{1+}\frac{q^4}{1+...}, |q| < 1 \tag{1.1}$$

$$= \frac{\sum_{n=0}^{\infty} \frac{q^{(n^2+n)/2}(-q;q)_n}{(q;q)_n}}{\sum_{n=0}^{\infty} \frac{q^{(n^2-n)/2}(-q;q)_n}{(q;q)_n}} \tag{1.2}$$

$$= \frac{(q;q^4)_\infty (q^3;q^4)_\infty}{(q^2;q^4)_\infty^2}, \tag{1.3}$$

and we called it analogous to the famous celebrated Rogers-Ramanujan continued fraction $R(q)$ defined by [1, p 9]:

$$R(q) = \frac{q^{1/5}}{1+}\frac{q}{1+}\frac{q^2}{1+....}, |q| < 1 \tag{1.4}$$

$$= q^{1/5}\frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{1.5}$$

Considering the closed form of the continued fraction$C(q)$, we define theta functions$G_1(q)$ and $H_1(q)$. We prove two relations for these theta functions $G_1(q)$ and $H_1(q)$ and from these relations prove three modular equations. This is done in section 3.

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In section 4, we prove a generalized Lambert series and write number of generalized Lambert series for these$G_1(q)$ and $H_1(q)$ and for the continued fraction $C(q)$.. Recall the following two identities of Slater [5, eq. 8 and 13]

$$\sum_{n=0}^{\infty} \frac{q^{(n^2+n)/2}(-q;q)_n}{(q;q)_n} = \frac{(-q;q)_\infty}{(q;q)_\infty}(q;q^4)_\infty(q^3;q^4)_\infty(q^4;q^4)_\infty \tag{1.6}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{(n^2-n)/2}(-q;q)_n}{(q;q)_n} = \frac{(-q;q)_\infty}{(q;q)_\infty}\left[(q;q^4)_\infty(q^3;q^4)_\infty(q^4;q^4)_\infty + (q^2;q^4)^2_\infty(q^4;q^4)_\infty\right]. \tag{1.7}$$

Writing them as

$$\frac{1}{G_1(q)} = \frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n=0}^{\infty} \frac{q^{(n^2+n)/2}(-q;q)_n}{(q;q)_n} = (q;q^4)_\infty(q^3;q^4)_\infty(q^4;q^4)_\infty \tag{1.8}$$

and

$$\begin{aligned}\frac{1}{H_1(q)} &= \left[\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n=0}^{\infty} \frac{q^{(n^2-n)/2}(-q;q)_n}{(q;q)_n} - (q;q^4)_\infty(q^3;q^4)_\infty(q^4;q^4)_\infty\right] \\ &= (q^2;q^4)^2_\infty(q^4;q^4)_\infty,\end{aligned} \tag{1.9}$$

we have

$$C(q) = \frac{H_1(q)}{G_1(q)}. \tag{1.10}$$

2. Preliminaries

We will be using the following standard notations:
If $|q| < 1$ and $x \neq 0$, then

$$j(x,q) = (x;q)_\infty(q/x;q)_\infty(q;q)_\infty. \tag{2.1}$$

If m is a positive integer and a is an integer, then for $m \geqslant 1$

$$J_{a,m} = j(q^a, q^m), \tag{2.2}$$

$$J_{a,m} = j(-q^a, q^m), \tag{2.3}$$

and

$$J_m = j(q^m, q^{3m}) = (q^m, q^m)_\infty. \tag{2.4}$$

The following identities follow easily from the above definitions:

$$j(q/x,q) = j(x,q), \tag{2.5}$$

$$j(x^{-1},q) = -x^{-1}j(x,q), \tag{2.6}$$

$$j(x,q)j(-x,q)=J_{1,2}j(x^2,q^2), \tag{2.7}$$

x not integral power of q.

$$j(x,q)=\frac{J_1}{J_n^n}j(x,q^n)j(qx,q^n)....j(q^{n-1}x,q^n), n\geq 1. \tag{2.8}$$

We shall use the following standard q-hypergeometric notation:

For $|q^k|<1$,

$$(a;q^k)_n=\prod_{m=1}^{n}(1-aq^{(m-1)k}),$$

$$(a;q^k)_\infty=\prod_{m=1}^{\infty}(1-aq^{(m-1)k}),$$

$$(a;q^k)_0=1.$$

Lastly, define

$$\chi(-q)=(q;q^2)_\infty.$$

Using the notation given in (2.1),

$$H_1(q)=\frac{1}{j(q^2,q^4)} \tag{2.9}$$

and

$$G_1(q)=\frac{1}{j(q,q^4)}. \tag{2.10}$$

3. Two Identities for $G_1(q)$ and $H_1(q)$

We shall prove the following two identities:

$$G_1^2(q)H_1(q^2)-H_1^2(q)G_1(q^2)=2qj^2(q,q^8)G_1(q)H_1(q)G_1(q^2)H_1^2(q^2) \tag{3.1}$$

and

$$G_1^2(q)H_1(q^2)+H_1^2(q)G_1(q^2)=\frac{2(-q^4;q^4)_\infty^4j^2(q^3,q^8)}{(-q;q)_\infty}G_1^2(q)G_1^3(q^2). \tag{3.2}$$

In proving the identities we shall use the following theorem of Hickerson [4, eq.(1.19), p. 644].

For $0<|q|<1$, $x\neq 0$, $y\neq 0$,

$$j(-x,q)j(y,q)-j(x,q)j(-y,q)=2xj(y/x,q^2)j(xyq,q^2). \tag{3.3}$$

Proof of (3.1) and (3.2)

First we prove (3.1).

Replacing q by q^4, x by q and y by q^2 in (3.3), we have

$$j(-q,q^4)j(q^2,q^4)-j(q,q^4)j(-q^2,q^4)=2qj^2(q,q^8). \tag{3.4}$$

Multiplying both sides of (3.4) by $\frac{1}{j(q,q^4)j(q^2,q^4)j(q^2,q^8)j(q^4,q^8)}$, we obtain

$$\frac{j(-q,q^4)}{j(q,q^4)j(q^2,q^8)j(q^4,q^8)}-\frac{j(-q^2,q^4)}{j(q^2,q^4)j(q^2,q^8)j(q^4,q^8)}=\frac{2qj^2(q,q^8)}{j(q,q^4)j(q^2,q^4)j(q^2,q^8)j(q^4,q^8)}. \tag{3.5}$$

By using (2.9) and (2.10), (3.5) simplifies to

$$G_1^2(q)H_1(q^2)-H_1^2(q)G_1(q^2)=2qj^2(q,q^8)G_1(q)H_1(q)G_1(q^2)H_1^2(q^2), \tag{3.6}$$

which proves (3.1).

Now we prove (3.2).

Replacing q by q^4, x by $\frac{1}{q}$ and y by q^2 in (3.3), we have

$$j(-q,q^4)j(q^2,q^4)+j(q,q^4)j(-q^2,q^4)=2qj^2(q^3,q^8). \tag{3.7}$$

Multiplying both sides of (3.7) by $\frac{1}{j(q,q^4)j(q^2,q^4)j(q^2,q^8)j(q^4,q^8)}$ and using (2.9) and (2.10), we have

$$G_1^2(q)H_1(q^2)+H_1^2(q)G_1(q^2)=\frac{2(-q^4;q^4)_\infty^4 j^2(q^3,q^8)}{(-q;q)_\infty}G_1^2(q)G_1^3(q^2), \tag{3.8}$$

which proves (3.2). Dividing (3.6) by (3.8)

$$B(q)\frac{G_1^2(q)H_1(q^2)-G_1(q^2)H_1^2(q)}{G_1^2(q)H_1(q^2)+G_1(q^2)H_1^2(q)}=C(q)C^2(q^2), \tag{3.9}$$

where

$$B(q)=\frac{(-q^4;q^4)_\infty^4 j^2(q^3,q^8)}{q(-q;q)_\infty j^2(q,q^8)}. \tag{3.10}$$

4. Applications

We now prove the following modular equations.

Theorem 1

Let

$$u=C(q^2), v=C(q)$$

then

$$(i)B(q)=\frac{u-v^2}{u+v^2}=u^2v, \tag{4.1}$$

$$(ii)k\left(\frac{1-k/B(q)}{1+k/B(q)}\right)^2=C^5(q), \tag{4.2}$$

where $k=C(q)C^2(q^2)$

$$(iii)B(q)\frac{1}{u^2v}-\frac{1}{B(q)}uv^2 \quad = \quad \frac{\chi(-q^2)\chi^6(-q^4)}{q\chi^2(-q)}. \tag{4.3}$$

$B(q)$ is as given in (3.10).

Proof of (i)

Dividing the numerator and denominator of the left side of (3.9) by $G_1^2(q)G_1(q^2)$ and applying (1.10), we have

$$B(q)\frac{C(q^2)-C^2(q)}{C(q^2)+C^2(q)} = C(q)C^2(q^2).$$

By the definition of u and v, we have

$$B(q)\frac{u-v^2}{u+v^2} = u^2v,$$

which proves (4.1).

Proof of (ii)

Writing k for $C(q)C^2(q^2)$, we obtain from (3.9)

$$k(q)\left(\frac{1-k/B(q)}{1+k/B(q)}\right)^2 = C(q)C^2(q^2)\frac{\left[1-\frac{G_1^2(q)H_1(q^2)-G_1(q^2)H_1^2(q)}{G_1^2(q)H_1(q^2)+G_1(q^2)H_1^2(q)}\right]^2}{\left[1+\frac{G_1^2(q)H_1(q^2)-G_1(q^2)H_1^2(q)}{G_1^2(q)H_1(q^2)+G_1(q^2)H_1^2(q)}\right]^2}$$

$$= C(q)C^2(q^2)\left[\frac{2G_1(q^2)H_1^2(q)}{2G_1^2(q)H_1(q^2)}\right]^2$$

$$= C(q)C^2(q^2)\left[\frac{C^2(q)}{C(q^2)}\right]^2$$

$$= C^5(q),$$

which proves (4.2).

Proof of (iii)

$$B(q)\frac{1}{u^2v} - \frac{1}{B(q)}u^2v$$

$$= \frac{G_1^2(q)H_1(q^2)+H_1^2(q)G_1(q^2)}{G_1^2(q)H_1(q^2)-H_1^2(q)G_1(q^2)} - \frac{G_1^2(q)H_1(q^2)-H_1^2(q)G_1(q^2)}{G_1^2(q)H_1(q^2)+H_1^2(q)G_1(q^2)}$$

$$= \frac{\left[G_1^2(q)H_1(q^2)+H_1^2(q)G_1(q^2)\right]^2-\left[G_1^2(q)H_1(q^2)-H_1^2(q)G_1(q^2)\right]^2}{\left[G_1^2(q)H_1(q^2)-H_1^2(q)G_1(q^2)\right]\left[G_1^2(q)H_1(q^2)+H_1^2(q)G_1(q^2)\right]}$$

$$= \frac{4G_1^2(q)H_1^2(q)G_1(q^2)H_1(q^2)}{\left[2qj^2(q,q^8)G_1(q)H_1(q)G_1(q^2)H_1^2(q^2)\right]\left[\frac{2(-q^4;q^4)_\infty^4 j^2(q^3,q^8)}{(-q;q)_\infty}G_1^2(q)G_1^3(q^2)\right]}$$

$$= \frac{(-q;q)_\infty}{q(-q^4;q^4)_\infty^4 j^2(q,q^8)j^2(q^3,q^8)}\frac{G_1(q)H_1(q)}{G_1^2(q)G_1^2(q^2)G_1(q^2)H_1(q^2)}. \tag{4.4}$$

Now

$$G_1(q)H_1(q) = \frac{1}{(q;q)_\infty(q^2;q^2)_\infty}, \tag{4.5}$$

$$G_1(q)G_1(q^2) = \frac{1}{(q;q)_\infty(q^8;q^8)_\infty} \tag{4.6}$$

and

$$j(q,q^8)j(q^3,q^8)=(q;q^2)_\infty(q^8;q^8)^2_\infty. \tag{4.7}$$

Putting the values from (4.5), (4.6) and (4.7) in (4.4), we get

$$B(q)\frac{1}{u^2v}-\frac{1}{B(q)}u^2v=\frac{\chi(-q^2)\chi^6(-q^4)}{q\chi^2(-q)},$$

which proves (4.3).

5. Generalized Lambert Series

Series of the form

$$\sum_{-\infty}^{\infty}(-1)^{\epsilon n}q^{\lambda n^2}R(q^n),$$

where $\epsilon=0$ or 1, $\lambda>0$ and $R(x)$is a rational function of x,is called a generalized Lambert series.

In [6] we proved two identities

$$\sum_{n=0}^{\infty}\frac{q^{in}}{1-q^{4n+i}}=\sum_{n=0}^{\infty}q^{4n^2+2in}\frac{1+q^{4n+i}}{1-q^{4n+i}}, \tag{5.1}$$

where $0<i\le 3$
and

$$\sum_{n=-\infty}^{\infty}\frac{q^{in}}{1-q^{4n+i}}=\frac{(q^4;q^4)^2_\infty(q^{2i};q^4)_\infty(q^{4-2i};q^4)_\infty}{(q^{4-i};q^4)^2_\infty(q^i;q^4)^2_\infty}, \tag{5.2}$$

where $0<i\le 3$,$i\ne 2$.

This (5.2) can be generalized to

$$\sum_{n=-\infty}^{\infty}\frac{q^{in}}{1-q^{4n+j}}=\frac{(q^4;q^4)^2_\infty(q^{i+j};q^4)_\infty(q^{4-i-i};q^4)_\infty}{(q^j;q^4)_\infty(q^{4-j};q^4)_\infty(q^i;q^4)_\infty(q^{4-i};q^4)_\infty}, \tag{5.3}$$

where $0<i\le 3$, $0<j\le 3$ and $i+j\ne 4$.

The proof (5.2) and (5.3) depends on the summation formula of Ramanujan:

$${}_1\psi_1(a;b;q,z)=\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}z^n=\frac{(b/a;q)_\infty(az;q)_\infty(q/az;q)_\infty(q;q)_\infty}{(q/a;q)_\infty(b/az;q)_\infty(b;q)_\infty(z;q)_\infty}.| \tag{5.4}$$

We now prove Lambert series for $G_1(q)$, $H_1(q)$ and $C(q)$.

We recall the definition of $G_1(q)$ and $H_1(q)$ given in (1.8) and (1.9):

$$G_1(q)=\frac{1}{(q;q)_\infty(q^3;q^4)_\infty(q^4;q^4)_\infty}$$

and

$$H_1(q)=\frac{1}{(q^2;q^4)^2_\infty(q^4;q^4)_\infty}.$$

We list identities for $G_1(q)$and $H_1(q)$ below and after each identity list the specialization of (5.2) and (5.3) in brackets:

$$(q^4;q^4)_\infty^3 H_1(q) = \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{4n+2}} \tag{5.5}$$

$(i=1, j=2$ in (5.3)).

$$(q^4;q^4)_\infty^3 \frac{G_1^2(q)}{H_1(q)} = \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{4n+1}} \tag{5.6}$$

$(i=1$ in (5.2)).

$$(q^4;q^4)_\infty^3 H_1(q) = \sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{4n+1}} \tag{5.7}$$

$(i=2, j=1$ in (5.3)).

$$(q^2;q^2)_\infty^2 (q^4;q^4)_\infty^2 G_1^2(q) = \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{4n+1}} \tag{5.8}$$

$(i=1$ in (5.2)).

$$(q^4;q^4)_\infty^3 \frac{G_1^2(q)}{H_1(q)} = \sum_{n=-\infty}^{\infty} q^{4n^2+2n} \frac{1+q^{4n+1}}{1-q^{4n+1}} \tag{5.9}$$

$(i=1$ in (5.1) and using (5.6)).

$$(q^2;q^2)_\infty^2 (q^4;q^4)_\infty^2 G_1^2(q) = \sum_{n=-\infty}^{\infty} q^{4n^2+2n} \frac{1+q^{4n+1}}{1-q^{4n+1}} \tag{5.10}$$

$(i=1$ in (5.1) and using (5.8)).

$$C^2(q) = \frac{\sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{4n+2}}}{\sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{4n+1}}} \tag{5.11}$$

(divide (5.5) by (5.6)).

$$C^2(q) = \frac{\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{4n+1}}}{\sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{4n+1}}} \tag{5.12}$$

(divide (5.7) by (5.6)).

$$(q^4;q^4)_\infty^3 G_1(q) = \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{8n+3}} \tag{5.13}$$

$(q \to q^2, i=\frac{1}{2}, j=\frac{3}{2}$ in (5.3)).

$$(q^4;q^4)_\infty^3 H_1(q) = \sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{8n+2}} \tag{5.14}$$

$(q \to q^2,\ i=1$ in (5.2)).

$$C(q) = \frac{\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{8n+2}}}{\sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{8n+3}}} \tag{5.15}$$

(divide(5.14) by (5.13)).

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