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A SIMPLE PROOF OF IDENTITIES OF LEGENDRE AND RAMANUJAN

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Abstract. By using two simple theta function identities we prove both Ramanujan's celebrated identity and Legendre's identity.

1. Introduction

Different proofs have been given for the following identities [1, ch. 18, p. 407], [2, p. 30], [6,7,8].

$$\sum_{n=0}^{\infty}\left[\frac{q^{4n+1}}{(1-q^{4n+1})^2}-\frac{2q^{4n+2}}{(1-q^{4n+2})^2}+\frac{q^{4n+3}}{(1-q^{4n+3})^2}\right]=q\frac{(q^4;q^4)_\infty^4}{(q^2;q^4)_\infty^4}, \tag{1.1}$$

$$\sum_{n=0}^{\infty}\left[\frac{q^{5n+1}}{(1-q^{5n+1})^2}+\frac{q^{5n+4}}{(1-q^{5n+4})^2}-\frac{q^{5n+2}}{(1-q^{5n+2})^2}-\frac{q^{5n+3}}{(1-q^{5n+3})^2}\right]=q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty}, \tag{1.2}$$

and

$$\sum_{n=0}^{\infty}\left[q^{7n+1}\frac{1+q^{7n+1}}{(1-q^{7n+1})^3}+q^{7n+2}\frac{1+q^{7n+2}}{(1-q^{7n+2})^3}+q^{7n+4}\frac{1+q^{7n+4}}{(1-q^{7n+4})^3}-q^{7n+3}\frac{1+q^{7n+3}}{(1-q^{7n+3})^3}-\right.$$

$$\left.q^{7n+5}\frac{1+q^{7n+5}}{(1-q^{7n+5})^3}-q^{7n+6}\frac{1+q^{7n+6}}{(1-q^{7n+6})^3}\right]=q(q;q)_\infty^3(q^7;q^7)_\infty^3+8q^2\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty}. \tag{1.3}$$

Identities (1.2) and (1.3) are due to Ramanujan. They lead to Ramanujan's partition identities for modulo 5 and modulo 7. Identity (1.1) is due to Legendre. H.H. Chan [6] has given a new, though complicated, proof of (1.2) and (1.3).

My motivation in writing this paper is to give a unified simple proof of all these well-known identities by using the following two simple theta function identities, see [10, eq. (2.14)] and [11, eq.(7.1)], respectively :

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$$\left(\frac{\theta_1'}{\theta_1}\right)'(z|q) = 4\sum_{n=0}^{\infty}\frac{q^n e^{2iz}}{(1-q^n e^{2iz})^2} + 4\sum_{n=1}^{\infty}\frac{q^n e^{-2iz}}{(1-q^n e^{-2iz})^2}, \tag{1.4}$$

and

$$\left(\frac{\theta_1'}{\theta_1}\right)'(a|q) - \left(\frac{\theta_1'}{\theta_1}\right)'(b|q) = \theta_1'(q)^2\,\frac{\theta_1(a-b|q)\theta_1(a+b|q)}{\theta_1^2(a|q)\theta_1^2(b|q)}. \tag{1.5}$$

2. Basic Preliminaries

Throughout this paper we use q to denote $e^{2\pi i\tau}$, $Im(\tau) > 0$.
We will use the following standard q-notation, $|q| < 1$:

$$(a;q^k)_n = (1-a)(1-aq^k).......(1-aq^{k(n-1)}), n \geq 1 \tag{2.1}$$

$$(a;q^k)_\infty = \prod_{n=0}^{\infty}(1-aq^{nk}), \tag{2.2}$$

$$(a,b,c....;q)_\infty = (a;q)_\infty(b;q)_\infty(c;q)_\infty...... \tag{2.3}$$

Easily, for any integer $n > 0$

$$(a,aq,.....,aq^{n-1};q^n)_\infty = (a;q)_\infty. \tag{2.4}$$

Jacobi theta function $\theta_1(z|q)$ is defined as [14, p.469]

$$\theta_1(z|q) = -iq^{\frac{1}{8}}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(n+1)}{2}}e^{(2n+1)iz} \tag{2.5}$$

$$= 2q^{\frac{1}{8}}\sum_{n=0}^{\infty}(-1)^n q^{\frac{n(n+1)}{2}} sin(2n+1)z. \tag{2.6}$$

From (2.6), we have

$$\theta_1(-z|q) = -\theta_1(z|q). \tag{2.7}$$

The function $\theta_1(z|q)$ can also be expressed in terms of an infinite product

$$\theta_1(z|q) = 2q^{\frac{1}{8}} sinz(q;q)_\infty(qe^{2iz};q)_\infty(qe^{-2iz};q)_\infty \tag{2.8}$$

$$= iq^{\frac{1}{8}}e^{-iz}(q;q)_\infty(e^{2iz};q)_\infty(qe^{-2iz};q)_\infty. \tag{2.9}$$

We define

$$\theta_1(q) = \theta_1(0|q). \tag{2.10}$$

Differentiating (2.8) with respect to z and then putting$z = 0$, we have

$$\theta_1'(q) = \theta_1'(0|q) = 2q^{\frac{1}{8}}(q;q)_\infty^3. \tag{2.11}$$

From (2.9) and (2.7) respectively, we have

$$\theta_1\left(n\pi\tau|q^k\right) = iq^{\frac{k-4n}{8}}(q^k;q^k)_\infty(q^n;q^k)_\infty(q^{k-n};q^k)_\infty, \tag{2.12}$$

$$\theta_1\left(-n\pi\tau|q^k\right)=-\theta_1\left(n\pi\tau|q^k\right). \tag{2.13}$$

Taking $n=1$,$k=5$, and $n=2$,$k=5$ in (2.12) and then multiplying the two resulting identities, we get

$$\theta_1\left(\pi\tau|q^5\right)\theta_1\left(2\pi\tau|q^5\right)=-q^{-\frac{1}{4}}(q;q)_\infty(q^5;q^5)_\infty. \tag{2.14}$$

Ramanujan defined general theta function $f(a,b)$as

$$f(a,b)=\sum_{n=-\infty}^{\infty}a^{\frac{n(n+1)}{2}}b^{\frac{n(n-1)}{2}},|ab|<1.$$

We then have [1, p. 35, Entry 19]

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

3. Proof of Identities (1.1) and (1.2)

Making $q\to q^4$ and then setting $z=\pi\tau$ and $z=2\pi\tau$, respectively, in (1.4), to obtain

$$\left(\frac{\theta_1'}{\theta_1}\right)'(\pi\tau|q^4)=4\sum_{n=0}^{\infty}\frac{q^{4n+1}}{(1-q^{4n+1})^2}+4\sum_{n=1}^{\infty}\frac{q^{4n-1}}{(1-q^{4n-1})^2} \tag{3.1}$$

and

$$\left(\frac{\theta_1'}{\theta_1}\right)'(2\pi\tau|q^4)=4\sum_{n=0}^{\infty}\frac{q^{4n+2}}{(1-q^{4n+2})^2}+4\sum_{n=1}^{\infty}\frac{q^{4n-2}}{(1-q^{4n-2})^2}. \tag{3.2}$$

Writing $n+1$for n in the second summation on the right hand side of equation (3.1) and (3.2) and then subtracting (3.2) from (3.1), we have

$$\left(\frac{\theta_1'}{\theta_1}\right)'(\pi\tau|q^4)-\left(\frac{\theta_1'}{\theta_1}\right)'(2\pi\tau|q^4)=4\sum_{n=0}^{\infty}\left[\frac{q^{4n+1}}{(1-q^{4n+1})^2}-\frac{2q^{4n+2}}{(1-q^{4n+2})^2}+\frac{q^{4n+3}}{(1-q^{4n+3})^2}\right]. \tag{3.3}$$

This identity we proved in [13, eq. (7(ii))].

Making $q\to q^4$ and then taking $a=\pi\tau$, $b=2\pi\tau$ in (1.5), we have

$$\begin{aligned}\left(\frac{\theta_1'}{\theta_1}\right)'(\pi\tau|q^4)-\left(\frac{\theta_1'}{\theta_1}\right)'(2\pi\tau|q^4)&=\theta_1'\left(q^4\right)^2\frac{\theta_1(-\pi\tau|q^4)\theta_1(3\pi\tau|q^4)}{\theta_1^2(\pi\tau|q^4)\theta_1^2(2\pi\tau|q^4)}\\&=4q\frac{(q^4;q^4)_\infty^4}{(q^2;q^4)_\infty^4}.\end{aligned} \tag{3.4}$$

We have used (2.11) and (2.12) in simplifying the right hand side of the above identity.

From (3.3) and (3.4), we have

$$\sum_{n=0}^{\infty}\left[\frac{q^{4n+1}}{(1-q^{4n+1})^2}-\frac{2q^{4n+2}}{(1-q^{4n+2})^2}+\frac{q^{4n+3}}{(1-q^{4n+3})^2}\right]=q\frac{(q^4;q^4)_\infty^4}{(q^2;q^4)_\infty^4},$$

which is (1.1).

Now we prove (1.2) using the same identity (1.4).

Making $q \to q^5$ and then setting $z = \pi\tau$ and $z = 2\pi\tau$, respectively, in (1.4), and then substracting, we have

$$\left(\frac{\theta_1'}{\theta_1}\right)'(\pi\tau|q^5) - \left(\frac{\theta_1'}{\theta_1}\right)'(2\pi\tau|q^5) = 4\sum_{n=0}^{\infty}\left[\frac{q^{5n+1}}{(1-q^{5n+1})^2} + \frac{q^{5n+4}}{(1-q^{5n+4})^2} - \frac{q^{5n+2}}{(1-q^{5n+2})^2} - \frac{q^{5n+3}}{(1-q^{5n+3})^2}\right]$$

$$= 4\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2}, \tag{3.5}$$

where $\left(\frac{n}{5}\right)$ is Legendre symbol.

Making $q \to q^5$ and then taking $a = \pi\tau$ and $b = 2\pi\tau$ in (1.5), using (2.11) and (2.14), we have

$$\left(\frac{\theta_1'}{\theta_1}\right)'(\pi\tau|q^5) - \left(\frac{\theta_1'}{\theta_1}\right)'(2\pi\tau|q^5) = 4q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty}. \tag{3.6}$$

From (3.5) and (3.6), we have

$$\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2} = q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty},$$

which is (1.2).

4. Proof of Identity (1.3)

In proving the identity (1.3) we use the same theta function identity (1.4) only we differentiate partially (1.4) with respect to z.

Differentiate partially with respect to z both side of (1.4), to get

$$\left(\frac{\theta_1'}{\theta_1}\right)''(z|q) = 8i\sum_{n=0}^{\infty}\frac{q^n e^{2iz}(1+q^n e^{2iz})}{(1-q^n e^{2iz})^3} - 8i\sum_{n=1}^{\infty}\frac{q^n e^{-2iz}(1+q^n e^{-2iz})}{(1-q^n e^{-2iz})^3}. \tag{4.1}$$

Making $q \to q^7$ and then writing $n+1$ for n in the second summation on the right hand side of (4.1), we get

$$\left(\frac{\theta_1'}{\theta_1}\right)''(z|q^7) = 8i\sum_{n=0}^{\infty}\frac{q^{7n} e^{2iz}(1+q^{7n} e^{2iz})}{(1-q^{7n} e^{2iz})^3} - 8i\sum_{n=0}^{\infty}\frac{q^{7n+7} e^{-2iz}(1+q^{7n+7} e^{-2iz})}{(1-q^{7n+7} e^{-2iz})^3}. \tag{4.2}$$

Put $z = \pi\tau$ and $z = 2\pi\tau$, respectively, in (4.2) and add to get

$$\left(\frac{\theta_1'}{\theta_1}\right)''(\pi\tau|q^7) + \left(\frac{\theta_1'}{\theta_1}\right)''(2\pi\tau|q^7)$$

$$= 8i\sum_{n=0}^{\infty}\frac{q^{7n+1}(1+q^{7n+1})}{(1-q^{7n+1})^3} - 8i\sum_{n=0}^{\infty}\frac{q^{7n+6}(1+q^{7n+6})}{(1-q^{7n+6})^3}$$

$$+ 8i\sum_{n=0}^{\infty}\frac{q^{7n+2}(1+q^{7n+2})}{(1-q^{7n+2})^3} - 8i\sum_{n=0}^{\infty}\frac{q^{7n+5}(1+q^{7n+5})}{(1-q^{7n+5})^3}. \tag{4.3}$$

Now put $z = 3\pi\tau$ in (4.2) and subtract from (4.3) to obtain

$$\begin{aligned}
&\left(\frac{\theta_1'}{\theta_1}\right)''(\pi\tau|q^7) + \left(\frac{\theta_1'}{\theta_1}\right)''(2\pi\tau|q^7) - \left(\frac{\theta_1'}{\theta_1}\right)''(3\pi\tau|q^7)\\
&= 8i\sum_{n=0}^{\infty}\frac{q^{7n+1}(1+q^{7n+1})}{(1-q^{7n+1})^3} - 8i\sum_{n=0}^{\infty}\frac{q^{7n+6}(1+q^{7n+6})}{(1-q^{7n+6})^3}\\
&+8i\sum_{n=0}^{\infty}\frac{q^{7n+2}(1+q^{7n+2})}{(1-q^{7n+2})^3} - 8i\sum_{n=0}^{\infty}\frac{q^{7n+5}(1+q^{7n+5})}{(1-q^{7n+5})^3}\\
&- 8i\sum_{n=0}^{\infty}\frac{q^{7n+3}(1+q^{7n+3})}{(1-q^{7n+3})^3} + 8i\sum_{n=0}^{\infty}\frac{q^{7n+4}(1+q^{7n+4})}{(1-q^{7n+4})^3}.
\end{aligned} \tag{4.4}$$

For evaluating the left hand side of (4.4) we use the second identity (1.5).

Differentiating partially both side of (1.5) with respect to a and then putting $b = a$, and making $q \to q^7$, we obtain

$$\left(\frac{\theta_1'}{\theta_1}\right)''(a|q^7) = \theta_1'\left(q^7\right)^3\frac{\theta_1(2a|q^7)}{\theta_1^4(a|q^7)}. \tag{4.5}$$

Taking $a = \pi\tau$, $2\pi\tau$ and $3\pi\tau$, respectively, in (4.5) and using (2.11) and (2.12), we have

$$\left(\frac{\theta_1'}{\theta_1}\right)''(\pi\tau|q^7) = 8iq\frac{(q^7;q^7)_\infty^9(q^2;q^7)_\infty(q^5;q^7)_\infty(q^7;q^7)_\infty}{(q;q^7)_\infty^4(q^6;q^7)_\infty^4(q^7;q^7)_\infty^4}, \tag{4.6}$$

$$\left(\frac{\theta_1'}{\theta_1}\right)''(2\pi\tau|q^7) = 8iq^2\frac{(q^7;q^7)_\infty^9(q^3;q^7)_\infty(q^4;q^7)_\infty(q^7;q^7)_\infty}{(q^2;q^7)_\infty^4(q^5;q^7)_\infty^4(q^7;q^7)_\infty^4}, \tag{4.7}$$

and

$$\left(\frac{\theta_1'}{\theta_1}\right)''(3\pi\tau|q^7) = 8iq^3\frac{(q^7;q^7)_\infty^9(q;q^7)_\infty(q^6;q^7)_\infty(q^7;q^7)_\infty}{(q^3;q^7)_\infty^4(q^4;q^7)_\infty^4(q^7;q^7)_\infty^4}. \tag{4.8}$$

Using (4.6), (4.7) and (4.8) the left hand side of (4.4) equals

$$\begin{aligned}
&8iq(q^7;q^7)_\infty^9\left[q\frac{(q^2;q^7)_\infty(q^5;q^7)_\infty(q^7;q^7)_\infty}{(q;q^7)_\infty^4(q^6;q^7)_\infty^4(q^7;q^7)_\infty^4} + q^2\frac{(q^3;q^7)_\infty(q^4;q^7)_\infty(q^7;q^7)_\infty}{(q^2;q^7)_\infty^4(q^5;q^7)_\infty^4(q^7;q^7)_\infty^4} - \right.\\
&\qquad\left. q^3\frac{(q;q^7)_\infty(q^6;q^7)_\infty(q^7;q^7)_\infty}{(q^3;q^7)_\infty^4(q^4;q^7)_\infty^4(q^7;q^7)_\infty^4}\right]\\
&= 8iq^2(q^7;q^7)_\infty^9\left[q^{-1}\frac{f(-q^2,-q^5)}{f^4(-q,-q^6)} + \frac{f(-q^3,-q^4)}{f^4(-q^2,-q^5)} - q\frac{f(-q,-q^6)}{f^4(-q^3,-q^4)}\right].
\end{aligned} \tag{4.9}$$

Using the following identity [4, eq.(4.22)] to evaluate the right hand side of (4.9)

$$(q;q)_\infty (q^7;q^7)_\infty^2 \left[q^{-1} \frac{f(-q^2,-q^5)}{f^4(-q,-q^6)} + \frac{f(-q^3,-q^4)}{f^4(-q^2,-q^5)} - q \frac{f(-q,-q^6)}{f^4(-q^3,-q^4)} \right]$$
$$= \frac{f^4(-q)}{qf^4(-q^7)} + 8$$

we have

$$\left(\frac{\theta_1'}{\theta_1}\right)''(\pi\tau|q^7) + \left(\frac{\theta_1'}{\theta_1}\right)''(2\pi\tau|q^7) - \left(\frac{\theta_1'}{\theta_1}\right)''(3\pi\tau|q^7)$$
$$= 8i \left[q(q;q)_\infty^3 (q^7;q^7)_\infty^3 + 8q^2 \frac{(q^7;q^7)_\infty^7}{(q;q)_\infty} \right].$$

Now by (4.4)

$$\sum_{n=0}^{\infty} \left[q^{7n+1} \frac{1+q^{7n+1}}{(1-q^{7n+1})^3} + q^{7n+2} \frac{1+q^{7n+2}}{(1-q^{7n+2})^3} + q^{7n+4} \frac{1+q^{7n+4}}{(1-q^{7n+4})^3} - q^{7n+3} \frac{1+q^{7n+3}}{(1-q^{7n+3})^3} - \right.$$
$$\left. q^{7n+5} \frac{1+q^{7n+5}}{(1-q^{7n+5})^3} - q^{7n+6} \frac{1+q^{7n+6}}{(1-q^{7n+6})^3} \right] = q(q;q)_\infty^3 (q^7;q^7)_\infty^3 + 8q^2 \frac{(q^7;q^7)_\infty^7}{(q;q)_\infty},$$

which is (1.3).

This identity has also been proved by Liu [9], using these two identities.

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