Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 3 Issue 1(2011), Pages 27-34.

FIXED POINT THEOREMS FOR MAPPINGS UNDER GENERAL CONTRACTIVE CONDITION OF INTEGRAL TYPE

(COMMUNICATED BY PETER SEMRL)

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Abstract. In the present paper, we establish a fixed point theorem for a mapping and a common fixed point theorem for a pair of mappings. The mapping involved here generalizes various type of contractive mappings in integral setting.

1. Introduction and Preliminaries

Impact of fixed point theory in different branches of mathematics and its applications is immense. The first important result on fixed points for contractive type mapping was the much celebrated Banach's contraction principle by S.Banach [1] in 1922. In the general setting of complete metric space, this theorem runs as follows (see Theorem 2.1, [4] or, Theorem 1.2.2, [10]).

Theorem 1.1. (Banach's contraction principle)
Let (X, d) be a complete metric space, $c \in (0, 1)$ and $f : X \to X$ be a mapping such that for each $x, y \in X$,

$$d(fx, fy) \leq cd(x, y) \tag{1.1}$$

then f has a unique fixed point $a \in X$, such that for each $x \in X$, $\lim_{n\to\infty} f^n x = a$.

After this classical result, Kannan [5] gave a substantially new contractive mapping to prove the fixed point theorem. Since then a number of mathematicians have been working on fixed point theory dealing with mappings satisfying various type of contractive conditions (see [3], [5] [7], [8], [9] and [11] for details).

In 2002, A.Branciari [2] analyzed the existence of fixed point for mapping f defined on a complete metric space (X, d) satisfying a general contractive condition of integral type.

2000 *Mathematics Subject Classification.* Primary 47H10; Secondary 54H25.
Key words and phrases. fixed point, general contractive condition, integral type.

Submitted October 2, 2010. Published December 16, 2010.

Theorem 1.2. (Branciari)
Let (X, d) be a complete metric space, $c \in (0, 1)$ and let $f : X \to X$ be a mapping such that for each $x, y \in X$,

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq c \int_0^{d(x,y)} \varphi(t)dt \tag{1.2}$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lesbesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0, +\infty)$, nonnegative , and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$, then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n\to\infty} f^n x = a$.
After the paper of Branciari, a lot of research works have been carried out on generalising contractive conditions of integral type for different contractive mappings satisfying various known properties. A fine work has been done by Rhoades [6] extending the result of Branciari by replacing the condition (1.2) by the following

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq c \int_0^{\max\{d(x,y),d(x,fx),d(y,fy),\frac{[d(x,fy)+d(y,fx)]}{2}\}} \varphi(t)dt \tag{1.3}$$

The aim of this paper is to generalise some mixed type of contractive conditions to the mapping and then a pair of mappings satisfying a general contractive condition of integral type , which includes several known contractive mappings such as Kannan type[5], Chatterjea type [3], Zamfirescu type [11], etc.

2. Main results

Theorem 2.1. *Let f be a self mapping of a complete metric space (X, d) satisfying the following condition:*

$$\begin{aligned}\int_0^{d(fx,fy)} \varphi(t)dt \quad \leq \quad & \alpha \int_0^{[d(x,fx)+d(y,fy)]} \varphi(t)dt + \beta \int_0^{d(x,y)} \varphi(t)dt \\ & +\gamma \int_0^{\max\{d(x,fy),d(y,fx)\}} \varphi(t)dt \end{aligned} \tag{2.1}$$

for each $x, y \in X$ with nonnegative reals α, β, γ such that $2\alpha + \beta + 2\gamma < 1$, where $\varphi : \Re^+ \to \Re^+$ is a Lesbesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $\Re^+$, nonnegative , and such that

$$\textit{for each } \epsilon > 0, \int_0^{\epsilon} \varphi(t)dt > 0 \tag{2.2}$$

Then f has a unique fixed point $z \in X$ and for each $x \in X$, $\lim_n f^n x = z$.

Proof. Let $x_0 \in X$ and, for brevity, define $x_n = fx_{n-1}$. For each integer $n \geq 1$, from (2.1) we get,

$$\begin{aligned}
\int_0^{d(x_n,x_{n+1})} \varphi(t)dt &= \int_0^{d(fx_{n-1},fx_n)} \varphi(t)dt \\
&\leq \alpha \int_0^{[d(x_{n-1},x_n)+d(x_n,x_{n+1})]} \varphi(t)dt + \beta \int_0^{d(x_{n-1},x_n)} \varphi(t)dt \\
&\quad +\gamma \int_0^{\max\{d(x_{n-1},x_{n+1}),d(x_n,x_n)\}} \varphi(t)dt \\
&= (\alpha+\beta) \int_0^{d(x_{n-1},x_n)} \varphi(t)dt + \alpha \int_0^{d(x_n,x_{n+1})} \varphi(t)dt \\
&\quad +\gamma \int_0^{d(x_{n-1},x_{n+1})} \varphi(t)dt \\
&\leq (\alpha+\beta) \int_0^{d(x_{n-1},x_n)} \varphi(t)dt + \alpha \int_0^{d(x_n,x_{n+1})} \varphi(t)dt \\
&\quad +\gamma \int_0^{[d(x_{n-1},x_n)+d(x_n,x_{n+1})]} \varphi(t)dt \\
&= (\alpha+\beta) \int_0^{d(x_{n-1},x_n)} \varphi(t)dt + \alpha \int_0^{d(x_n,x_{n+1})} \varphi(t)dt \\
&\quad +\gamma \int_0^{d(x_{n-1},x_n)} \varphi(t)dt + \gamma \int_0^{d(x_n,x_{n+1})} \varphi(t)dt
\end{aligned}$$

which implies that

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \leq \left(\frac{\alpha+\beta+\gamma}{1-\alpha-\gamma}\right) \int_0^{d(x_{n-1},x_n)} \varphi(t)dt$$

and so,

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \leq h \int_0^{d(x_{n-1},x_n)} \varphi(t)dt \tag{2.3}$$

where $\frac{\alpha+\beta+\gamma}{1-\alpha-\gamma} = h(\text{ say}) < 1$.
Thus by routine calculation,

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \leq h^n \int_0^{d(x_0,x_1)} \varphi(t)dt \tag{2.4}$$

Taking limit of (2.4) as $n \to \infty$, we get

$$\lim_n \int_0^{d(x_n,x_{n+1})} \varphi(t)dt = 0$$

which, from (2.2) implies that

$$\lim_n d(x_n, x_{n+1}) = 0 \tag{2.5}$$

We now show that $\{x_n\}$ is a Cauchy sequence. Suppose that it is not. Then there exists an $\epsilon > 0$ and subsequences $\{m(p)\}$ and $\{n(p)\}$ such that $m(p) < n(p) < m(p+1)$ with

$$d(x_{m(p)}, x_{n(p)}) \geq \epsilon,\ d(x_{m(p)}, x_{n(p)-1}) < \epsilon \tag{2.6}$$

Now

$$\begin{aligned} d(x_{m(p)-1}, x_{n(p)-1}) &\leq d(x_{m(p)-1}, x_{m(p)}) + d(x_{m(p)}, x_{n(p)-1}) \\ &< d(x_{m(p)-1}, x_{m(p)}) + \epsilon \end{aligned} \tag{2.7}$$

Hence

$$\lim_{p} \int_0^{d(x_{m(p)-1}, x_{n(p)-1})} \varphi(t)dt \leq \int_0^{\epsilon} \varphi(t)dt \tag{2.8}$$

Using (2.3), (2.6) and (2.8) we get

$$\int_0^{\epsilon} \varphi(t)dt \leq \int_0^{d(x_{m(p)}, x_{n(p)})} \varphi(t)dt \leq h \int_0^{d(x_{m(p)-1}, x_{n(p)-1})} \varphi(t)dt \leq h \int_0^{\epsilon} \varphi(t)dt$$

which is a contradiction, since $h \in (0,1)$. Therefore, $\{x_n\}$ is Cauchy, hence convergent. Call the limit z.
From (2.1) we get

$$\begin{aligned} \int_0^{d(fz, x_{n+1})} \varphi(t)dt &\leq \alpha \int_0^{[d(z,fz)+d(x_n,x_{n+1})]} \varphi(t)dt + \beta \int_0^{d(z,x_n)} \varphi(t)dt \\ &\quad + \gamma \int_0^{\max\{d(z,x_{n+1}), d(x_n, fz)\}} \varphi(t)dt \end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\int_0^{d(fz,z)} \varphi(t)dt \leq (\alpha + \gamma) \int_0^{d(z,fz)} \varphi(t)dt$$

As $2\alpha + \beta + 2\gamma < 1$,

$$\int_0^{d(fz,z)} \varphi(t)dt = 0$$

which, from (2.2), implies that $d(fz, z) = 0$ or, $fz = z$.
Next suppose that $w(\neq z)$ be another fixed point of f. Then from (2.1) we have

$$\begin{aligned} \int_0^{d(z,w)} \varphi(t)dt &= \int_0^{d(fz,fw)} \varphi(t)dt \\ &\leq \alpha \int_0^{[d(z,fz)+d(w,fw)]} \varphi(t)dt + \beta \int_0^{d(z,w)} \varphi(t)dt \\ &\quad + \gamma \int_0^{\max\{d(z,fw), d(w,fz)\}} \varphi(t)dt \\ &\leq (\beta + \gamma) \int_0^{d(z,w)} \varphi(t)dt \end{aligned}$$

Since, $\beta + \gamma < 1$, this implies that

$$\int_0^{d(z,w)} \varphi(t)dt = 0$$

which, from (2.2), implies that $d(z, w) = 0$ or, $z = w$ and so the fixed point is unique. □

Remark. *On setting $\varphi(t) = 1$ over $\Re^+$, the contractive condition of integral type transforms into a general contractive condition not involving integrals.*

Remark. *From condition (2.1) of integral type, several contractive mappings of integral type can be obtained.*
I. $\beta = \gamma = 0$ *and* $\alpha \in (0, \frac{1}{2})$ *gives Kannan mappings of integral type.*
II. $\alpha = \beta = 0$ *and* $\gamma \in (0, \frac{1}{2})$ *gives Chatterjea* [3] *map of integral type.*
III. $\beta \in (0, 1)$ *and* $\alpha, \gamma \in (0, \frac{1}{2})$, *atleast one of the following conditions hold:*
$(z_1) : \int_0^{d(fx,fy)} \varphi(t)dt \leq \beta \int_0^{d(x,y)} \varphi(t)dt$
$(z_2) : \int_0^{d(fx,fy)} \varphi(t)dt \leq \alpha \int_0^{[d(x,fx)+d(y,fy)]} \varphi(t)dt$
$(z_3) : \int_0^{d(fx,fy)} \varphi(t)dt \leq \gamma \int_0^{[d(x,fy)+d(y,fx)]} \varphi(t)dt$
gives Zamfirescu [11] *mapping of integral type.*

Now we set an example verifying the Theorem 2.1

Example 2.2. *Let* $X = [0, 1]$ *and* d *be usual metric with* $d(x, y) = |x - y|$. *Clearly* (X, d) *is a complete metric space. Let* $f : X \to X$ *be given by* $fx = \frac{x}{2}$ *for all* $x \in [0, 1]$. *Again let* $\varphi : \Re^+ \to \Re^+$ *be given by* $\varphi(t) = \frac{t^2}{2}$ *for all* $t \in \Re^+$.
Then for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt = \int_0^{\epsilon} \frac{t^2}{2} dt = \frac{\epsilon^3}{6} > 0.$$

Now taking $\alpha = \gamma = \frac{1}{16}$ *and* $\beta = \frac{1}{8}$, *one can easily verify that the condition (2.1) of Theorem 2.1 is satisfied with* $2\alpha + \beta + 2\gamma < 1$ *and* 0 *is, of course, the unique fixed point of* f.

Next we extend the result for a pair of mappings.

Theorem 2.3. *Let* f *and* g *be self mappings of a complete metric space* (X, d) *satisfying the following condition:*

$$\begin{aligned}\int_0^{d(fx,gy)} \varphi(t)dt \quad &\leq \quad \alpha \int_0^{[d(x,fx)+d(y,gy)]} \varphi(t)dt + \beta \int_0^{d(x,y)} \varphi(t)dt \\ &\quad + \gamma \int_0^{\max\{d(x,gy),d(y,fx)\}} \varphi(t)dt \qquad (2.9)\end{aligned}$$

for each $x, y \in X$ *with nonnegative reals* α, β, γ *such that* $2\alpha + \beta + 2\gamma < 1$, *where* $\varphi : \Re^+ \to \Re^+$ *is a Lesbesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of* $\Re^+$, *nonnegative , and such that*

$$\text{for each } \epsilon > 0, \int_0^{\epsilon} \varphi(t)dt > 0 \qquad (2.10)$$

Then f *and* g *have a unique common fixed point* $z \in X$.

Proof. Let $x_0 \in X$ and, for brevity, define $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$. For each integer $n \geq 0$, from (2.9) we get,

$$\begin{aligned}
\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt &= \int_0^{d(fx_{2n},gx_{2n+1})} \varphi(t)dt \\
&\leq \alpha \int_0^{[d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})]} \varphi(t)dt + \beta \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt \\
&\quad +\gamma \int_0^{\max\{d(x_{2n},x_{2n+2}),d(x_{2n+1},x_{2n+1})\}} \varphi(t)dt \\
&= (\alpha+\beta) \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt + \alpha \int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt \\
&\quad +\gamma \int_0^{d(x_{2n},x_{2n+2})} \varphi(t)dt \\
&\leq (\alpha+\beta) \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt + \alpha \int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt \\
&\quad +\gamma \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt + \gamma \int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt
\end{aligned}$$

which implies that

$$\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt \leq \left(\frac{\alpha+\beta+\gamma}{1-\alpha-\gamma}\right) \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt$$

and so,

$$\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt \leq h \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt \tag{2.11}$$

where $\frac{\alpha+\beta+\gamma}{1-\alpha-\gamma} = h(\text{ say}) < 1$.
Similarly

$$\int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt \leq h \int_0^{d(x_{2n-1},x_{2n})} \varphi(t)dt \tag{2.12}$$

Thus in general, for all $n = 1, 2, ...$

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \leq h \int_0^{d(x_{n-1},x_n)} \varphi(t)dt \tag{2.13}$$

Then by routine calculation, we have

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \leq h^n \int_0^{d(x_0,x_1)} \varphi(t)dt$$

Taking limit as $n \to \infty$, we get

$$\lim_n \int_0^{d(x_n,x_n+1)} \varphi(t)dt = 0$$

which, from (2.10) implies that

$$\lim_n d(x_n, x_{n+1}) = 0 \tag{2.14}$$

We now show that $\{x_n\}$ is a Cauchy sequence. Suppose that it is not. Then there exists an $\epsilon > 0$ and subsequences $\{2m(p)\}$ and $\{2n(p)\}$ such that $p < 2m(p) < 2n(p)$ with

$$d(x_{2m(p)}, x_{2n(p)}) \geq \epsilon,\ d(x_{2m(p)}, x_{2n(p)-2}) < \epsilon \tag{2.15}$$

Now

$$\begin{aligned} d(x_{2m(p)}, x_{2n(p)}) &\leq d(x_{2m(p)}, x_{2n(p)-2}) + d(x_{2n(p)-2}, x_{2n(p)-1}) + d(x_{2n(p)-1}, x_{2n(p)}) \\ &< \epsilon + d(x_{2n(p)-2}, x_{2n(p)-1}) + d(x_{2n(p)-1}, x_{2n(p)}) \end{aligned} \tag{2.16}$$

Hence

$$\lim_p \int_0^{d(x_{2m(p)}, x_{2n(p)})} \varphi(t)dt = \int_0^{\epsilon} \varphi(t)dt \tag{2.17}$$

Then by (2.13)we get

$$\begin{aligned} \int_0^{d(x_{2m(p)}, x_{2n(p)})} \varphi(t)dt &\leq h \int_0^{d(x_{2m(p)-1}, x_{2n(p)-1})} \varphi(t)dt \\ &\leq h[\int_0^{d(x_{2m(p)-1}, x_{2m(p)})} \varphi(t)dt + \int_0^{d(x_{2m(p)}, x_{2n(p)})} \varphi(t)dt \\ &\quad + \int_0^{d(x_{2n(p)-1}, x_{2n(p)})} \varphi(t)dt] \end{aligned}$$

Taking limit as $p \to \infty$ we get

$$\int_0^{\epsilon} \varphi(t)dt \leq h \int_0^{\epsilon} \varphi(t)dt$$

which is a contradiction, since $h \in (0, 1)$. Therefore, $\{x_n\}$ is Cauchy, hence convergent. Call the limit z.
From (2.9) we get

$$\begin{aligned} \int_0^{d(fz, x_{2n+2})} \varphi(t)dt &= \int_0^{d(fz, gx_{2n+1})} \varphi(t)dt \\ &\leq \alpha \int_0^{[d(z,fz)+d(x_{2n+1}, x_{2n+2})]} \varphi(t)dt + \beta \int_0^{d(z, x_{2n+1})} \varphi(t)dt \\ &\quad + \gamma \int_0^{\max\{d(z, x_{2n+2}), d(x_{2n+1}, fz)\}} \varphi(t)dt \end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\int_0^{d(fz,z)} \varphi(t)dt \leq (\alpha + \gamma) \int_0^{d(z,fz)} \varphi(t)dt$$

As $2\alpha + \beta + 2\gamma < 1$,

$$\int_0^{d(fz,z)} \varphi(t)dt = 0$$

which, from (2.10), implies that $d(fz, z) = 0$ or, $fz = z$. Similarly it can be shown that $gz = z$. So f and g have a common fixed point $z \in X$. We now show that z is the unique common fixed point of f and g. If not, then let w be another common fixed point of f and g. Then from (2.9) we have

$$\begin{aligned}\int_0^{d(z,w)} \varphi(t)dt &= \int_0^{d(fz,gw)} \varphi(t)dt \\ &\leq \alpha \int_0^{[d(z,fz)+d(w,gw)]} \varphi(t)dt + \beta \int_0^{d(z,w)} \varphi(t)dt \\ &\quad + \gamma \int_0^{\max\{d(z,gw),d(w,fz)\}} \varphi(t)dt \\ &\leq (\beta+\gamma) \int_0^{d(z,w)} \varphi(t)dt\end{aligned}$$

Since, $\beta + \gamma < 1$, this implies that

$$\int_0^{d(z,w)} \varphi(t)dt = 0$$

which, from (2.10), implies that $d(z,w) = 0$ or, $z = w$ and so the fixed point is unique. □

Acknowledgments. The authors would like to thank the referee for his comments that helped us improve this article.

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