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OSCILLATION CRITERIA FOR THIRD-ORDER NONLINEAR DELAY DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. By means of Riccati transformation technique, we establish some new oscillation criteria for the third-order nonlinear delay dynamic equations

$$x^{\Delta^3}(t) + p(t)x^{\gamma}(\tau(t)) = 0$$

on a time scale $\mathbb{T}$ unbounded above, here $\gamma > 0$ is a quotient of odd positive integers with p real-valued positive rd-continuous function defined on $\mathbb{T}$. Three examples are given to illustrate the main results.

1. Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger in his Ph. D. thesis [1] in order to unify continuous and discrete analysis. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to plenty of applications, among them the study of population dynamic models (see [2]).

Not only can this theory of so-called dynamic equations unify the theories of differential equations and difference equations, but also it is able to extend these classical cases to cases "in between", e.g., to so-called q-difference equations. Several authors have expounded on various aspects of this new theory; see the survey paper by Agarwal et al. [3] and references cited therein. A book on the subject of time scales, by Bohner and Peterson [2], summarizes and organizes much of the time scale calculus, see also the book by Bohner and Peterson [4] for advances in dynamic equations on time scales.

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Recently, there has been a great deal of research activity concerning the oscillation and nonoscillation of solutions of various equations on time scales, we refer the reader to the papers which deal with the first-order dynamic equations [5–7] and second-order dynamic equations [8–16]. However, there are few results dealing with the oscillation of solutions of third-order and higher-order dynamic equations [17–27].

Han et al. [16] considered the second-order Emden-Fowler delay dynamic equations

$$x^{\Delta^2}(t) + p(t)x^{\gamma}(\tau(t)) = 0,\ t \in \mathbb{T} \text{ with } \sup \mathbb{T} = \infty. \tag{1.1}$$

Using the equality

$$(u^{\gamma})^{\Delta}\left(\frac{z^2}{u^{\gamma}}\right)^{\Delta} = (z^{\Delta})^2 - (uu^{\sigma})^{\gamma}\left(\left(\frac{z}{u^{\gamma}}\right)^{\Delta}\right)^2, \tag{1.2}$$

where u and z are differentiable on a time scale $\mathbb{T}$ with $u(t) \neq 0$ for all $t \in \mathbb{T}$, they established some oscillation criteria for (1.1).

In 2007, Erbe et al. [18] investigated the third-order dynamic equations

$$x^{\Delta^3}(t) + p(t)x(t) = 0,\ t \in \mathbb{T} \text{ with } \sup \mathbb{T} = \infty, \tag{1.3}$$

where p is a positive real-valued rd-continuous function on $\mathbb{T}$, and the authors obtained some Hille and Nehari type criteria for the oscillation of (1.3).

Han et al. [25] studied the third-order nonlinear delay dynamic equations

$$\left((x^{\Delta^2}(t))^{\gamma}\right)^{\Delta} + p(t)x^{\gamma}(\tau(t)) = 0,\ t \in \mathbb{T} \text{ with } \sup \mathbb{T} = \infty, \tag{1.4}$$

and obtained some new oscillation results for (1.4).

In this paper, by using (1.2), we consider the following delay dynamic equations

$$x^{\Delta^3}(t) + p(t)x^{\gamma}(\tau(t)) = 0,\ t \in \mathbb{T} \text{ with } \sup \mathbb{T} = \infty. \tag{1.5}$$

We assume that $\gamma > 0$ is a quotient of odd positive integers, p is a positive real-valued rd-continuous function defined on $\mathbb{T}$, $\tau : \mathbb{T} \to \mathbb{T}$ is a rd-continuous function such that $\tau(t) \leq t$ and $\tau(t) \to \infty$ $(t \to \infty)$.

Since we are interested in oscillatory behavior, we assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above. We assume $t_0 \in \mathbb{T}$ and it is convenient to assume $t_0 > 0$. We define the time scale interval of the form $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$.

The paper is organized as follows: In Section 2, we apply a simple consequence of Keller's chain rule, devoted to the proof of the sufficient conditions which ensure that every solution of (1.2) is either oscillatory or has a finite limit at ∞. In Section 3, three examples are considered to illustrate the main results. In Section 4, we will give some conclusions for this paper.

2. Main results

In this section we establish some new oscillation criteria for (1.5). In order to prove our main results, we will use the formula

$$((x(t))^{\gamma})^{\Delta} = \gamma \int_0^1 [hx^{\sigma}(t) + (1-h)x(t)]^{\gamma-1} x^{\Delta}(t)\mathrm{d}h, \tag{2.1}$$

where x is delta differentiable and eventually positive or eventually negative, which is a simple consequence of Keller's chain rule (see Bohner and Peterson [2, Theorem 1.90]).

Before stating our main results, we begin with the following lemmas which are crucial in the proofs of the main results.

Lemma 2.1. *Assume that x is an eventually positive solution of* (1.5). *Then there are only the following two cases for $t \geq t_1$ sufficiently large:*
(i) $x(t) > 0$, $x^{\Delta}(t) > 0$, $x^{\Delta^2}(t) > 0$ and $x^{\Delta^3}(t) < 0$ (unbounded),
or
(ii) $x(t) > 0$, $x^{\Delta}(t) < 0$, $x^{\Delta^2}(t) > 0$ and $x^{\Delta^3}(t) < 0$ (bounded).

The proof of the lemma above is a direct consequence of Kneser's theorem (see [22]) for third order derivative.

In [2, Section 1.6] the Taylor monomials $\{h_n(t,s)\}_{n=0}^{\infty}$ are defined recursively by

$$h_0(t,s) = 1, \quad h_{n+1}(t,s) = \int_s^t h_n(\tau,s)\Delta\tau, \quad t,\ s \in \mathbb{T},\ n \geq 0.$$

It follows from [2, Section 1.6] that $h_1(t,s) = t - s$ for any time scale, but simple formulas in general do not hold for $n \geq 2$.

Lemma 2.2. [18, Lemma 4] *Assume that x satisfies case (i) of Lemma 2.1. Then*

$$\liminf_{t\to\infty} \frac{tx(t)}{h_2(t,t_0)x^{\Delta}(t)} \geq 1. \tag{2.2}$$

Lemma 2.3. *Assume that x is a solution of* (1.5) *which satisfies case (i) of Lemma 2.1. If*

$$\int_{t_0}^{\infty} p(t)(h_2(\tau(t),t_0))^{\gamma}\Delta t = \infty, \tag{2.3}$$

then

$$x^{\Delta}(t) \geq tx^{\Delta^2}(t), \quad \frac{x^{\Delta}(t)}{t} \text{ is eventually nonincreasing.} \tag{2.4}$$

Proof. Let x be a solution of (1.5) such that case (i) of Lemma 2.1 holds for $t \geq t_1$. Define

$$X(t) = x^{\Delta}(t) - tx^{\Delta^2}(t).$$

Thus

$$X^{\Delta}(t) = -\sigma(t)x^{\Delta^3}(t) > 0.$$

We claim that $X(t) > 0$ eventually. Otherwise, there exists $t_2 \geq t_1$ such that $X(t) < 0$ for $t \geq t_2$. Therefore,

$$\left(\frac{x^{\Delta}(t)}{t}\right)^{\Delta} = -\frac{X(t)}{t\sigma(t)} > 0, \quad t \geq t_2,$$

which implies that $x^{\Delta}(t)/t$ is strictly increasing on $[t_2,\infty)_{\mathbb{T}}$. Pick $t_3 \geq t_2$ such that $\tau(t) \geq t_2$ for all $t \geq t_3$. Then, we have

$$\frac{x^{\Delta}(\tau(t))}{\tau(t)} \geq \frac{x^{\Delta}(t_2)}{t_2} = d > 0,$$

then $x^{\Delta}(\tau(t)) \geq d\tau(t)$ for all $t \geq t_3$. By Lemma 2.2, for any $0 < k < 1$, there exists $t_4 \geq t_3$ such that

$$\frac{x(t)}{x^{\Delta}(t)} \geq k\frac{h_2(t,t_0)}{t}, \quad t \geq t_4.$$

Hence, there exists $t_5 \geq t_4$ so that

$$x(\tau(t)) \geq k\frac{h_2(\tau(t),t_0)}{\tau(t)}x^{\Delta}(\tau(t)) \geq dk\frac{h_2(\tau(t),t_0)}{\tau(t)}\tau(t) = dkh_2(\tau(t),t_0), \quad t \geq t_5.$$

Integrating both sides of (1.5) from t_5 to t, we have

$$x^{\Delta^2}(t) - x^{\Delta^2}(t_5) + (dk)^{\gamma}\int_{t_5}^{t} p(s)(h_2(\tau(s),t_0))^{\gamma}\Delta s \leq 0,$$

which yields that

$$x^{\Delta^2}(t_5) \geq (dk)^{\gamma}\int_{t_5}^{t} p(s)(h_2(\tau(s),t_0))^{\gamma}\Delta s,$$

which contradicts (2.3). Hence, $X(t) > 0$ and $x^{\Delta}(t)/t$ is eventually nonincreasing. The proof is complete.

Lemma 2.4. *Assume that x is a solution of* (1.5) *which satisfies case* (ii) *of Lemma 2.1. If*

$$\int_{t_0}^{\infty} h_2(t_0,\sigma(s))p(s)\Delta s = \infty, \tag{2.5}$$

then $\lim_{t\to\infty} x(t) = 0$.

The result above is a restriction of [23, Theorem 3.1] to (1.5).

Theorem 2.5. *Assume that* (2.3) *holds,* $\gamma \geq 1$. *If*

$$\limsup_{t\to\infty} t\int_{t}^{\infty} p(s)\left(\frac{h_2(\tau(s),t_0)}{s}\right)^{\gamma}\Delta s = \infty, \tag{2.6}$$

then every solution x of (1.5) *is either oscillatory or* $\lim_{t\to\infty} x(t)$ *exists.*

Proof. Suppose that (1.5) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. We may assume without loss of generality that $x(t) > 0$ and $x(\tau(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$, $t_1 \in [t_0,\infty)_{\mathbb{T}}$. Then by Lemma 2.1, x satisfies either case (i) or (ii).

Assume case (i) holds. Set $y(t) = x^{\Delta^2}(t)$. By (1.5), we have for $T \geq t \geq t_1$,

$$y(T) = y(t) + \int_{t}^{T} y^{\Delta}(s)\Delta s = y(t) - \int_{t}^{T} p(s)x^{\gamma}(\tau(s))\Delta s.$$

Hence

$$\int_{t}^{T} p(s)x^{\gamma}(\tau(s))\Delta s = y(t) - y(T) \leq y(t) = x^{\Delta^2}(t).$$

By (2.2) and (2.4), for any $0 < k < 1$, we have

$$\begin{aligned} x^{\Delta}(t) &\geq tx^{\Delta^2}(t) \geq t\int_{t}^{\infty} p(s)x^{\gamma}(\tau(s))\Delta s \geq t\int_{t}^{\infty} kp(s)\left(\frac{h_2(\tau(s),t_0)}{\tau(s)}x^{\Delta}(\tau(s))\right)^{\gamma}\Delta s \\ &\geq kt\int_{t}^{\infty} p(s)\left(\frac{h_2(\tau(s),t_0)}{\tau(s)}\frac{\tau(s)}{s}x^{\Delta}(s)\right)^{\gamma}\Delta s = kt\int_{t}^{\infty} p(s)\left(\frac{h_2(\tau(s),t_0)}{s}\right)^{\gamma}(x^{\Delta}(s))^{\gamma}\Delta s \\ &\geq kt(x^{\Delta}(t))^{\gamma}\int_{t}^{\infty} p(s)\left(\frac{h_2(\tau(s),t_0)}{s}\right)^{\gamma}\Delta s, \end{aligned}$$

which yields that

$$t\int_{t}^{\infty} p(s)\left(\frac{h_2(\tau(s),t_0)}{s}\right)^{\gamma}\Delta s \leq \frac{1}{k}\left(\frac{1}{x^{\Delta}(t)}\right)^{\gamma-1} \leq \frac{1}{k}\left(\frac{1}{x^{\Delta}(t_1)}\right)^{\gamma-1},$$

this contradicts (2.6). If case (ii) holds, then $\lim_{t\to\infty} x(t)$ exists. The proof is complete.

Using a Riccati transformation technique and (1.2), we establish the following results.

Theorem 2.6. *Assume that* (2.3) *holds,* $\gamma \ge 1$. *Furthermore, suppose that there exists a function* $z \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that for some* $0 < k < 1$ *and for all constants* $M > 0$, *one has*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[p(s)\xi(s)(z(\sigma(s)))^2 - M^{\gamma-1}(z^{\Delta}(s))^2\right]\Delta s = \infty, \tag{2.7}$$

where $\xi(t) = k^{\gamma}\left(h_2(\tau(t),t_0)/\sigma(t)\right)^{\gamma}$. *Then every solution* x *of* (1.5) *is either oscillatory or* $\lim_{t\to\infty} x(t)$ *exists.*

Proof. Suppose that (1.5) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. We may assume without loss of generality that $x(t) > 0$ and $x(\tau(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$, $t_1 \in [t_0,\infty)_{\mathbb{T}}$. Then by Lemma 2.1, x satisfies either case (i) or (ii).

Assume x satisfies case (i). Define the function ω by

$$\omega = \frac{z^2 x^{\Delta^2}}{(x^{\Delta})^{\gamma}}. \tag{2.8}$$

Then $\omega(t) > 0$. Using the product rule, we have

$$\omega^{\Delta} = \left(\frac{z^2}{(x^{\Delta})^{\gamma}}\right)^{\sigma} x^{\Delta^3} + x^{\Delta^2}\left(\frac{z^2}{(x^{\Delta})^{\gamma}}\right)^{\Delta}.$$

So, letting $u = x^{\Delta}$, from (1.5) and (1.2) we get

$$\omega^{\Delta} = -p(z^{\sigma})^2\left(\frac{x\circ\tau}{x^{\Delta\sigma}}\right)^{\gamma} + \frac{x^{\Delta^2}}{((x^{\Delta})^{\gamma})^{\Delta}}(z^{\Delta})^2 - \frac{x^{\Delta^2}}{((x^{\Delta})^{\gamma})^{\Delta}}(x^{\Delta}x^{\Delta\sigma})^{\gamma}\left(\left(\frac{z}{(x^{\Delta})^{\gamma}}\right)^{\Delta}\right)^2. \tag{2.9}$$

In view of (2.2) and (2.4), for any $0 < k < 1$, we find

$$\begin{aligned}\frac{x^{\gamma}(\tau(t))}{(x^{\Delta\sigma}(t))^{\gamma}} &= \frac{x^{\gamma}(\tau(t))}{(x^{\Delta}(\tau(t)))^{\gamma}}\frac{(x^{\Delta}(\tau(t)))^{\gamma}}{(x^{\Delta\sigma}(t))^{\gamma}}\\ &\ge \left(k\frac{h_2(\tau(t),t_0)}{\tau(t)}\right)^{\gamma}\left(\frac{\tau(t)}{\sigma(t)}\right)^{\gamma} = k^{\gamma}\left(\frac{h_2(\tau(t),t_0)}{\sigma(t)}\right)^{\gamma}.\end{aligned} \tag{2.10}$$

On the other hand, it is follows (2.1) that

$$\begin{aligned}((x^{\Delta}(t))^{\gamma})^{\Delta} &= \gamma\int_0^1 [hx^{\Delta\sigma}(t) + (1-h)x^{\Delta}(t)]^{\gamma-1}x^{\Delta^2}(t)\mathrm{d}h\\ &\ge \gamma(x^{\Delta}(t))^{\gamma-1}x^{\Delta^2}(t) \ge M^{1-\gamma}x^{\Delta^2}(t),\end{aligned} \tag{2.11}$$

where $M = (\gamma^{1/(\gamma-1)}x^{\Delta}(t_1))^{-1}$, if $\gamma > 1$. If $\gamma = 1$, we choose $M = 1$.

Thus, from (2.9), (2.10) and (2.11), we see that

$$\omega^{\Delta} \le -p(z^{\sigma})^2\xi + M^{\gamma-1}(z^{\Delta})^2. \tag{2.12}$$

Therefore,

$$\int_{t_1}^{t}\left[p(s)\xi(s)(z(\sigma(s)))^2 - M^{\gamma-1}(z^{\Delta}(s))^2\right]\Delta s \le \omega(t_1),$$

which contradicts (2.7).

If case (ii) holds, then $\lim_{t\to\infty} x(t)$ exists. This completes the proof.

From Theorem 2.6, we can obtain different conditions for oscillation of all solutions of (1.5) with different choices of z.

For example, let $z = \sqrt{t},\ z \equiv 1$. Now Theorem 2.6 yields the following results.

Corollary 2.7. *Assume that* (2.3) *holds,* $\gamma \geq 1$. *If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(p(s)\sigma(s)\xi(s) - \frac{M^{\gamma-1}}{(\sqrt{s}+\sqrt{\sigma(s)})^2} \right) \Delta s = \infty \tag{2.13}$$

for some $0 < k < 1$ *and for all constants* $M > 0$, *where* ξ *is as in Theorem 2.6, then every solution* x *of* (1.5) *is either oscillatory or* $\lim_{t\to\infty} x(t)$ *exists.*

Corollary 2.8. *Assume that* (2.3) *holds,* $\gamma \geq 1$. *If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} p(s) \left(\frac{h_2(\tau(s), t_0)}{\sigma(s)} \right)^{\gamma} \Delta s = \infty, \tag{2.14}$$

then every solution x *of* (1.5) *is either oscillatory or* $\lim_{t\to\infty} x(t)$ *exists.*

Sometimes the following criteria is easier to check than the one given in Corollary 2.7, but it follows easily from Corollary 2.7 as we always have $\sigma(t) \geq t$ for all $t \in \mathbb{T}$.

Corollary 2.9. *Assume that* (2.3) *holds,* $\gamma \geq 1$. *If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(p(s)\sigma(s)\xi(s) - \frac{M^{\gamma-1}}{4s} \right) \Delta s = \infty \tag{2.15}$$

for some $0 < k < 1$ *and for all constants* $M > 0$, *where* ξ *is as in Theorem 2.6, then every solution* x *of* (1.5) *is either oscillatory or* $\lim_{t\to\infty} x(t)$ *exists.*

Theorem 2.10. *Assume that* (2.3) *holds,* $\gamma \leq 1$. *Furthermore, assume that there exists a positive function* $z \in C^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ *such that for some* $0 < k < 1$ *and for all constants* $K > 0$, *one has*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[p(s)\xi(s)(z(\sigma(s)))^2 - K^{\gamma-1}(\sigma(s))^{1-\gamma}(z^{\Delta}(s))^2 \right] \Delta s = \infty, \tag{2.16}$$

where ξ *is as defined as in Theorem 2.6. Then every solution* x *of* (1.5) *is either oscillatory or* $\lim_{t\to\infty} x(t)$ *exists.*

Proof. Suppose that (1.5) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. We may assume without loss of generality that $x(t) > 0$ and $x(\tau(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$, $t_1 \in [t_0,\infty)_{\mathbb{T}}$. Then by Lemma 2.1, x satisfies either case (i) or (ii).

Assume x satisfies case (i). Define the function ω as (2.8). We proceed as in the proof of Theorem 2.6 and we get (2.9) and (2.10). Note that $\gamma \leq 1$, by (2.1) we have

$$\begin{aligned} ((x^{\Delta}(t))^{\gamma})^{\Delta} &= \gamma \int_0^1 [h x^{\Delta\sigma}(t) + (1-h)x^{\Delta}(t)]^{\gamma-1} x^{\Delta^2}(t) \mathrm{d}h \\ &\geq \gamma (x^{\Delta\sigma}(t))^{\gamma-1} x^{\Delta^2}(t), \end{aligned}$$

from (2.4), there exists a constant $L > 0$ such that $x^{\Delta}(t) \leq Lt$, so

$$((x^{\Delta}(t))^{\gamma})^{\Delta} \geq \frac{(\sigma(t))^{\gamma-1}}{K^{\gamma-1}} x^{\Delta^2}(t), \tag{2.17}$$

where we put $K = (\gamma^{1/(\gamma-1)}L)^{-1}$, if $\gamma < 1$. If $\gamma = 1$, we choose $K = 1$.

Hence, from (2.9), (2.10) and (2.17), we see that

$$\omega^{\Delta} \leq -p(z^{\sigma})^2\xi + K^{\gamma-1}\sigma^{1-\gamma}(z^{\Delta})^2. \tag{2.18}$$

Therefore,

$$\int_{t_1}^{t} \left[p(s)\xi(s)(z(\sigma(s)))^2 - K^{\gamma-1}(\sigma(s))^{1-\gamma}(z^{\Delta}(s))^2\right] \Delta s \leq \omega(t_1),$$

which contradicts (2.16).

If case (ii) holds, then $\lim_{t\to\infty} x(t)$ exists. This completes the proof.

From Theorem 2.10, we can obtain different conditions for oscillation of all solutions of (1.5) with different choices of z.

For example, let $z = \sqrt{t},\ z \equiv 1$. Now Theorem 2.10 yields the following results.

Corollary 2.11. *Assume that* (2.3) *holds,* $\gamma \leq 1$. *If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(p(s)\sigma(s)\xi(s) - \frac{K^{\gamma-1}(\sigma(s))^{1-\gamma}}{(\sqrt{s} + \sqrt{\sigma(s)})^2} \right) \Delta s = \infty \tag{2.19}$$

for some $0 < k < 1$ *and for all constants* $K > 0$, *where* ξ *is as in Theorem 2.6, then every solution* x *of* (1.5) *is either oscillatory or* $\lim_{t\to\infty} x(t)$ *exists.*

Corollary 2.12. *Assume that* (2.3) *holds,* $\gamma \leq 1$. *If* (2.14) *holds, then every solution* x *of* (1.5) *is either oscillatory or* $\lim_{t\to\infty} x(t)$ *exists.*

Sometimes the following criteria is easier to check than the one given in Corollary 2.11, but it follows easily from Corollary 2.11 as we always have $\sigma(t) \geq t$ for all $t \in \mathbb{T}$.

Corollary 2.13. *Assume that* (2.3) *holds,* $\gamma \leq 1$. *If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(p(s)\sigma(s)\xi(s) - (\sigma(s))^{1-\gamma}\frac{K^{\gamma-1}}{4s} \right) \Delta s = \infty \tag{2.20}$$

for some $0 < k < 1$ *and for all constants* $K > 0$, *where* ξ *is as in Theorem 2.6, then every solution* x *of* (1.5) *is either oscillatory or* $\lim_{t\to\infty} x(t)$ *exists.*

From Lemma 2.4, we have the following results.

Theorem 2.14. *Assume that* (2.3) *and* (2.5) *hold,* $\gamma \geq 1$. *If one of the conditions* (2.6) *and* (2.7) *holds, then every solution of* (1.5) *oscillates or tends to zero.*

Theorem 2.15. *Assume that* (2.3), (2.5) *and* (2.16) *hold,* $\gamma \leq 1$. *Then every solution of* (1.5) *oscillates or converges to zero.*

3. Examples

In this section we give the following examples to illustrate our main results.

Example 3.1 Consider the third order delay dynamic equations on time scales

$$x^{\Delta^3}(t) + \frac{\beta}{t}\left(\frac{\sigma(t)}{h_2(\tau(t), t_0)}\right)^{\gamma} x^{\gamma}(\tau(t)) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{3.1}$$

where $\beta > 0$, $\gamma \geq 1$ is a quotient of odd positive integers.

Let $p(t) = \beta\left(\sigma(t)/h_2(\tau(t), t_0)\right)^{\gamma}/t$. It is easy to see that all the conditions of Corollary 2.8 hold. Hence, every solution x of (3.1) is either oscillatory or $\lim_{t\to\infty} x(t)$ exists.

Example 3.2 Consider the third order delay dynamic equations on time scales

$$x^{\Delta^3}(t) + \frac{\beta}{t(h_2(\tau(t), t_0))^\gamma} x^\gamma(\tau(t)) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{3.2}$$

where $\beta > 0$, $\gamma \leq 1$ is a quotient of odd positive integers.

Let $p(t) = \beta/(t(h_2(\tau(t), t_0))^\gamma)$. It is easy to see that all the conditions of Corollary 2.12 hold. Therefore, every solution x of (3.2) is either oscillatory or $\lim_{t\to\infty} x(t)$ exists.

Example 3.3 Consider the third order delay differential equation

$$x'''(t) + (e^{2t-6})x^3(t-2) = 0, \quad t \in [t_0, \infty), \tag{3.3}$$

where $\gamma = 3$, $\tau(t) = t - 2$.

For $\mathbb{T} = \mathbb{R}$, we have $h_2(\tau(t), t_0) = (t - 2 - t_0)^2/2$. Let $p(t) = e^{2t-6}$. It is easy to see that all the conditions of Theorem 2.14 hold. Thus, every solution x of (3.3) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$. For example, $x(t) = e^{-t}$ is a solution of (3.3).

4. Conclusions

In this paper, we consider the oscillatory behavior of the third-order nonlinear delay dynamic equations (1.5), the method is different from [25–27], and these results are new.

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