Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 3 Issue 2(2011), Pages 261-268.

POSITIVE SOLUTIONS FOR A PERIODIC BOUNDARY VALUE PROBLEM WITHOUT ASSUMPTIONS OF MONOTONICITY AND CONVEXITY

(COMMUNICATED BY I.P. STAVROULAKIS)

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Abstract. In the case of not requiring the nonlinear term to be monotone or convex, we study the existence of positive solutions for second-order periodic boundary value problem by using the first eigenvalue of the corresponding linear problem and fixed point index theory. The work significantly improves and generalizes the main results of J. Graef et al. [A periodic boundary value problem with vanishing Green's function, Appl. Math. Lett. 21(2008) 176−180].

1. Introduction

In the last two decades, there has been much attention focused on questions of positive solutions for diverse nonlinear ordinary differential equation, difference equation, and functional differential equation boundary value problems, see [1], [2], and the references therein. Recently, periodic boundary value problems have deserved the attention of many researchers (see [3]-[10]). Under the positivity of the associated Green's function, the existence and multiplicity of positive solutions for the periodic boundary value problems were established by applying the nonlinear alternative of Leray-Schauder type and Krasnosel'skii fixed point theorem in [3]-[9]. In [10], Graef et al. studied the following periodic boundary value problem(PBVP)

$$\begin{gathered} u''(t)+a(t)u(t)=g(t)f(u), \quad 0\le t\le 2\pi, \\ u(0)=u(2\pi), \quad u'(0)=u'(2\pi). \end{gathered} \tag{1.1}$$

By Krasnosel'skii fixed point theorem, under the assumptions that f was either superlinear or sublinear and the associated Green's function is *nonnegative*, i.e., it may have zeros, they obtained the nonnegative solutions of the above problem. To

2000 *Mathematics Subject Classification.* 34B10, 34B15.
Key words and phrases. Periodic boundary value problem; Positive solutions; Fixed point index.

Submitted December 13, 2010. Published March 1, 2011.
F. Z. was supported by the the Changzhou University Foundation (JS201008) and National Natural Science Foundation of China (10971179).

cope with the difficulty caused by the nonnegativity of Green's function, a special cone was constructed. For convenience, we introduce the notations

$$f_0 = \lim_{u\to 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u\to +\infty} \frac{f(u)}{u}.$$

Throughout this paper, we assume that the following standing hypothesis (H):

(H) The Green's function $G(t,s)$ for (1.1) is continuous and nonnegative for all $(t,s) \in [0,2\pi]\times[0,2\pi]$, and

$$\beta = \min_{s\in[0,2\pi]} \int_0^{2\pi} G(t,s)dt > 0. \tag{1.2}$$

Remark. From Remark 2.1 in [10], condition (H) is weaker than $G(t,s) > 0$ for all $(t,s)\in[0,2\pi]\times[0,2\pi]$.

For comparison with our results below, we include a statement of the basic theorem in [10].

Theorem A *Suppose condition (H) is satisfied. In addition, assume that*

(A1) $f : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous, convex and nondecreasing.*

(A2) $g : [0,2\pi] \to \mathbb{R}^+$ *is continuous and* $\eta = \min_{t\in[0,2\pi]} g(t) > 0$*. Then in each of the following cases:*

(A3) $f_0 = +\infty,\ f_\infty = 0$ *(sublinear case),*

(A4) $f_0 = 0,\ f_\infty = +\infty$ *(superlinear case),*

the PBVP (1.1) has at least one positive solution.

In this paper, we establish the existence of positive solutions PBVP (1.1), where $g(t)$ may be singular at $t = 0$ and $t = 1$, by using the first eigenvalue of the relevant linear problem and fixed point index theory which come from Zhang−Sun [11]-[14] and Cui−Zou [15]-[16]. Here we do not impose any monotonicity and convexity conditions on f, which are essential for the technique used in [10]. Moreover, $g(t)$ is allowed to be singular at $t = 0$ and $t = 1$.

We assume the following conditions throughout:

$(H1)$ $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous.

$(H2)$ $g : (0,2\pi) \to \mathbb{R}^+$ is continuous, $0 < \eta = \min_{t\in(0,2\pi)} g(t) < +\infty$, and $\int_0^{2\pi} g(t)dt < +\infty$.

The main results of this paper are as follows.

Theorem 1.1. *Assume that conditions (H),* $(H1)$*, and* $(H2)$ *hold. Then in each of the following cases:*

$(H3)$ $f_0 > \lambda_1,\ f_\infty < \lambda_1,$

$(H4)$ $f_0 < \lambda_1,\ f_\infty > \lambda_1,$

where λ_1 *is a positive constant that will be specified later, the PBVP (1.1) has at least one positive solution.*

From Theorem 1.1 we immediately obtain the following.

Corollary 1.2. *Assume that conditions (H),* $(H1)$*, and* $(H2)$ *hold. Then in each of the following cases:*

(i) $f_0 = +\infty,\ f_\infty = 0,$

(ii) $f_0 = 0,\ f_\infty = +\infty,$

the PBVP (1.1) has at least one positive solution.

Remark. Obviously, Theorem 1.1 and Corollary 1.2 extend the Theorem A.

Corollary 1.3. *Suppose conditions (H), $(H1)$ and $(H2)$ are satisfied. In addition, assume that* $0 \leq f_\infty < f_0 \leq +\infty$,

$$\lambda \in \Big(\frac{\lambda_1}{f_0}, \frac{\lambda_1}{f_\infty}\Big), \tag{1.3}$$

where λ_1 is a positive constant will be specified later. Then the singular eigenvalue periodic boundary value problem (EPBVP)

$$\begin{aligned} u''(t) + a(t)u(t) = \lambda g(t)f(u), \quad 0 \leq t \leq 2\pi, \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \end{aligned} \tag{1.4}$$

has at least one positive solution.

Corollary 1.4. *Suppose conditions (H), $(H1)$ and $(H2)$ are satisfied. In addition, assume that* $0 \leq f_0 < f_\infty \leq +\infty$,

$$\lambda \in \Big(\frac{\lambda_1}{f_\infty}, \frac{\lambda_1}{f_0}\Big), \tag{1.5}$$

where λ_1 is a positive constant that will be specified later. Then the singular EPBVP (1.4) has at least one positive solution.

2. Preliminaries and lemmas

In the Banach space $C[0, 2\pi]$ let the norm be defined by $\|u\| = \max\limits_{0 \leq t \leq 2\pi} |u(t)|$ for any $u \in C[0, 2\pi]$. We set $P = \{u \in C[0, 2\pi] \mid u(t) \geq 0,\ t \in [0, 2\pi]\}$ be a cone in $C[0, 2\pi]$. We denote by $B_r = \{u \in C[0, 2\pi] | \|u\| < r\}$ $(r > 0)$ the open ball of radius r.

The function u is said to be a positive solution of PBVP (1.1) if $u \in C[0, 2\pi] \cap C^2(0, 2\pi)$ satisfies (1.1), and $u(t) > 0$ for $t \in (0, 2\pi)$.

Define

$$(Au)(t) = \int_0^{2\pi} G(t, s)g(s)f(u(s))ds, \quad t \in [0, 2\pi], \tag{2.1}$$

$$(Tu)(t) = \int_0^{2\pi} G(t, s)g(s)u(s)ds, \quad t \in [0, 2\pi]. \tag{2.2}$$

We can verify that the nonzero fixed points of the operator A are positive solutions of the PBVP (1.1). Define the cone K in $C[0, 2\pi]$ by

$$K = \{u \in P \mid \int_0^{2\pi} u(t)dt \geq \frac{\beta}{M}\|u\|\},$$

where β is defined by (1.2) and $M = \max\limits_{t,s \in [0,2\pi]} G(t, s)$. Then K is subcone of P.

From [10] and the Arzera-Ascoli theorem, A, $T : K \to K$ defined by (2.1) and (2.2) respectively, are completely continuous operators.

By virtue of Krein$-$Rutmann theorems, we have (see [11]-[16]) the following lemma.

Lemma 2.1. *Suppose that $T : C[0, 2\pi] \to C[0, 2\pi]$ is a completely continuous linear operator and $T(P) \subset P$. If there exists $\psi \in C[0, 2\pi] \backslash \{-P\}$ and a constant $c > 0$ such that $cT\psi \geq \psi$, then the spectral radius $r(T) \neq 0$ and T has a positive eigenfunction φ_1 corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$, that is, $\varphi_1 =$*

$\lambda_1 T\varphi_1$.

Lemma 2.2. *Suppose that the condition* $(H2)$ *is satisfied; then for the operator* T *defined by (2.2), the spectral radius* $r(T) \neq 0$ *and* T *has a positive eigenfunction corresponding to its first eigenvalue* $\lambda_1 = (r(T))^{-1}$.

Proof. It is obvious that there is $t_1 \in (0, 2\pi)$ such that $G(t_1, t_1)g(t_1) > 0$. Thus there exists $[a_1, b_1] \subset (0, 2\pi)$ such that $t_1 \in (a_1, b_1)$ and $G(t, s)g(s) > 0, \forall\, t, s \in [a_1, b_1]$. Take $\psi \in C[0, 2\pi]$ such that $\psi(t) \geq 0$, $\forall\, t \in [0, 2\pi]$, $\psi(t_1) > 0$ and $\psi(t) = 0$, $\forall\, t \notin [a_1, b_1]$. Then for $t \in [a_1, b_1]$

$$(T\psi)(t) = \int_0^{2\pi} G(t, s)g(s)\psi(s)ds \geq \int_{a_1}^{b_1} G(t, s)g(s)\psi(s)ds > 0.$$

So there exists a constant $c > 0$ such that $c(T\psi)(t) \geq \psi(t), \forall\, t \in [0, 2\pi]$. From Lemma 2.1, we know that the spectral radius $r(T) \neq 0$ and T has a positive eigenfunction corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$.

The following lemmas are needed in our argument.

Lemma 2.3. ([17]) *Let* E *be Banach space,* P *be a cone in* E*, and* $\Omega(P)$ *be a bounded open set in* P*. Suppose that* $A : \overline{\Omega(P)} \to P$ *is a completely continuous operator. If there exists* $u_0 \in P\backslash\{\theta\}$ *such that*

$$u - Au \neq \mu u_0, \quad \forall\, u \in \partial\Omega(P),\ \mu \geq 0,$$

then the fixed point index $i(A, \Omega(P), P) = 0$.

Lemma 2.4. ([17]) *Let* E *be Banach space,* P *be a cone in* E*, and* $\Omega(P)$ *be a bounded open set in* P *with* $\theta \in \Omega(P)$*. Suppose that* $A : \overline{\Omega(P)} \to P$ *is a completely continuous operator. If*

$$Au \neq \mu u, \quad \forall\, u \in \partial\Omega(P),\ \mu \geq 1,$$

then the fixed point index $i(A, \Omega(P), P) = 1$.

3. Proof of the main results

Proof of Theorem 1.1. We show respectively that the operator A defined by (2.1) has a nonzero fixed point in two cases that $(H3)$ is satisfied and $(H4)$ is satisfied.

Case (i). It follows from $(H3)$ that there exists $r_1 > 0$ such that

$$f(u) \geq \lambda_1 u, \quad \forall\ 0 \leq u \leq r_1. \tag{3.1}$$

Let u^* be the positive eigenfunction of T corresponding to λ_1, thus $u^* = \lambda_1 T u^*$.

For every $u \in \partial B_{r_1} \cap K$, it follows from (3.1) that

$$(Au)(t) \geq \lambda_1 \int_0^{2\pi} G(t, s)g(s)u(s)ds = \lambda_1 (Tu)(t), \quad t \in [0, 2\pi]. \tag{3.2}$$

We may suppose that A has no fixed point on $\partial B_{r_1} \cap K$ (otherwise, the proof is finished). Now we show that

$$u - Au \neq \tau u^*, \quad \forall\ u \in \partial B_{r_1} \cap K, \tau \geq 0. \tag{3.3}$$

Suppose to the contrary that there exist $u_1 \in \partial B_{r_1} \cap K$ and $\tau_1 \geq 0$ such that $u_1 - Au_1 = \tau_1 u^*$. Hence $\tau_1 > 0$ and

$$u_1 = Au_1 + \tau_1 u^* \geq \tau_1 u^*.$$

Put

$$\tau^* = \sup\{\tau \mid u_1 \geq \tau u^*\}. \tag{3.4}$$

It is easy to see that $\tau^* \geq \tau_1 > 0$ and $u_1 \geq \tau^* u^*$. We find from $T(K) \subset K$ that

$$\lambda_1 T u_1 \geq \tau^* \lambda_1 T u^* = \tau^* u^*.$$

Therefore by (3.2), we have

$$u_1 = Au_1 + \tau_1 u^* \geq \lambda_1 T u_1 + \tau_1 u^* \geq \tau^* u^* + \tau_1 u^* = (\tau^* + \tau_1) u^*,$$

which contradicts the definition of τ^*. Hence (3.3) is true and we have from Lemma 2.3 that

$$i(A, B_{r_1} \cap K, K) = 0. \tag{3.5}$$

It follows from $(H3)$ that there exist $0 < \sigma < 1$ and $r_2 > r_1$ such that

$$f(u) \leq \sigma \lambda_1 u, \ \forall\ u \geq r_2. \tag{3.6}$$

Let $T_1 u = \sigma \lambda_1 T u,\ u \in C[0, 2\pi]$; then $T_1 : C[0, 2\pi] \to C[0, 2\pi]$ is a bounded linear operator and $T_1(K) \subset K$. Denote

$$M^* = M \sup_{u \in \overline{B}_{r_2} \cap K} \int_0^{2\pi} g(s) f(u(s)) ds. \tag{3.7}$$

It is clear that $M^* < +\infty$. Let

$$W = \{u \in K \mid u = \mu A u,\ 0 \leq \mu \leq 1\}. \tag{3.8}$$

In the following, we prove that W is bounded.

For any $u \in W$, set $\widetilde{u}(t) = \min\{u(t), r_2\}$ and denote $E(t) = \{t \in [0, 2\pi] \mid u(t) > r_2\}$. Then

$$\begin{aligned}
u(t) = &\ \mu(Au)(t) \leq \int_0^{2\pi} G(t,s) g(s) f(u(s)) ds \\
= &\ \int_{E(t)} G(t,s) g(s) f(u(s)) ds + \int_{[0,2\pi]\setminus E(t)} G(t,s) g(s) f(u(s)) ds \\
\leq &\ \sigma \lambda_1 \int_0^{2\pi} G(t,s) g(s) u(s) ds + M \int_0^{2\pi} g(s) f(\widetilde{u}(s)) ds \\
\leq &\ (T_1 u)(t) + M^*, \quad t \in [0, 2\pi].
\end{aligned}$$

Thus $((I - T_1)u)(t) \leq M^*,\ t \in [0, 2\pi]$. Since λ_1 is the first eigenvalue of T and $0 < \sigma < 1$, the first eigenvalue of T_1, $(r(T_1))^{-1} > 1$. Therefore, the inverse operator $(I - T_1)^{-1}$ exists and

$$(I - T_1)^{-1} = I + T_1 + T_1^2 + \cdots + T_1^n + \cdots.$$

It follows from $T_1(K) \subset K$ that $(I - T_1)^{-1}(K) \subset K$. So we know that $u(t) \leq (I - T_1)^{-1} M^*,\ t \in [0, 2\pi]$ and W is bounded.

Select $r_3 > \max\{r_2,\ \sup W\}$. Then from the homotopy invariance property of fixed point index we have

$$i(A,\ B_{r_3} \cap K,\ K) = i(\theta,\ B_{r_3} \cap K,\ K) = 1. \tag{3.9}$$

By (3.5) and (3.9), we have that

$$i(A,\ (B_{r_3} \cap K) \setminus (\overline{B}_{r_1} \cap K),\ K) = i(A,\ B_{r_3} \cap K,\ K) - i(A,\ B_{r_1} \cap K,\ K) = 1.$$

Then A has at least one fixed point on $(B_{r_3} \cap K)\backslash(\overline{B}_{r_1} \cap K)$. This means that singular PBVP (1.1) has at least one positive solution.

Case (ii). It follows from $(H4)$ that there exists $\varepsilon > 0$ such that $f(u) \geq \lambda_1(\varepsilon + \frac{1}{\lambda_1\beta\eta})u$ when u is sufficiently large. We know from $(H1)$ that there exists $b_1 \geq 0$ such that

$$f(u) \geq \lambda_1(\varepsilon + \frac{1}{\lambda_1\beta\eta})u - b_1, \quad \forall\ 0 \leq u < +\infty. \tag{3.10}$$

Take

$$R > \max\Big\{1, \frac{2\pi b_1 M^2 \int_0^{2\pi} g(s)ds}{\lambda_1\varepsilon\beta^2\eta}\Big\}.$$

If there exist $u_2 \in \partial B_R \cap K$, $\mu_2 \geq 0$, such that $u_2 - Au_2 = \mu_2\psi^*$, then $u_2 = Au_2 + \mu_2\psi^* \geq Au_2$. Integrating this inequality from 0 to 2π and using (1.2), (3.10), $(H2)$, we have

$$\begin{aligned}
\int_0^{2\pi} u_2(t)dt &\geq \int_0^{2\pi} Au_2(t)dt = \int_0^{2\pi}\int_0^{2\pi} G(t,s)g(s)f(u_2(s))dsdt \\
&\geq \lambda_1(\varepsilon + \frac{1}{\lambda_1\beta\eta})\int_0^{2\pi}\int_0^{2\pi} G(t,s)g(s)u_2(s)dsdt - b_1\int_0^{2\pi} g(s)\int_0^{2\pi} G(t,s)dtds \\
&= \lambda_1(\varepsilon + \frac{1}{\lambda_1\beta\eta})\int_0^{2\pi} g(s)u_2(s)\int_0^{2\pi} G(t,s)dtds - b_1\int_0^{2\pi} g(s)\int_0^{2\pi} G(t,s)dtds \\
&\geq \lambda_1(\varepsilon + \frac{1}{\lambda_1\beta\eta})\beta\eta\int_0^{2\pi} u_2(s)ds - 2\pi b_1 M\int_0^{2\pi} g(s)ds \\
&\geq (\lambda_1\varepsilon\beta\eta + 1)\int_0^{2\pi} u_2(s)ds - 2\pi b_1 M\int_0^{2\pi} g(s)ds.
\end{aligned}$$

By the definition of K, we get $\int_0^{2\pi} u_2(t)dt \geq \frac{\beta}{M}\|u_2\| = \frac{\beta}{M}R$. Therefore it follows that

$$R \leq \frac{2\pi b_1 M^2 \int_0^{2\pi} g(s)ds}{\lambda_1\varepsilon\beta^2\eta},$$

which is a contradiction with the choice of R. Hence hypotheses of Lemma 2.3 hold. Therefore we have

$$i(A,\ B_R \cap K,\ K) = 0. \tag{3.11}$$

It follows from $(H4)$ that there exists $0 < r < 1$ such that

$$f(u) \leq \lambda_1 u, \quad \forall\ 0 \leq u \leq r. \tag{3.12}$$

Define $T_2 u = \lambda_1 Tu, u \in C[0, 2\pi]$. Hence $T_2 : C[0, 2\pi] \to C[0, 2\pi]$ is a bounded linear completely continuous operator and

$$T_2(K) \subset K, \quad r(T_2) = 1.$$

For every $u \in \partial B_r \cap K$, it follows from (3.12) that

$$\begin{aligned}(Au)(t) &= \int_0^{2\pi} G(t,s)g(s)f(u(s))ds \\ &\le \lambda_1 \int_0^{2\pi} G(t,s)g(s)u(s)ds \\ &= (T_2u)(t), t \in [0,2\pi],\end{aligned}$$

hence $Au \le T_2u, \ \forall\ u \in \ \partial B_r \cap K$. We may also assume that A has no fixed point on $\partial B_r \cap K$ (otherwise, the proof is finished).

Now we show that

$$Au \ne \mu u, \quad \forall\ u \in \partial B_r \cap K, \ \mu \ge 1. \tag{3.13}$$

If otherwise, there exist $u_3 \in \partial B_r \cap K$ and $\mu_3 \ge 1$ such that $Au_3 = \mu_3 u_3$. Thus $\mu_3 > 1$ and $\mu_3 u_3 = Au_3 \le T_2 u_3$. By induction, we have $\mu_3^n u_3 \le T_2^n u_3 (n = 1, 2, \ldots)$. Then

$$\mu_3^n u_3 \le T_2^n u_3 \le \|T_2^n\| \|u_3\|,$$

and taking the maximum over $[0, 2\pi]$ gives $\mu_3^n \le \|T_2^n\|$. In view of Gelfand's formula, we have

$$r(T_2) = \lim_{n\to\infty} \sqrt[n]{\|T_2^n\|} \ge \lim_{n\to\infty} \sqrt[n]{\mu_3^n} = \mu_3 > 1,$$

which is a contradiction. Hence (3.13) is true and by Lemma 2.4, we have

$$i(A, B_r \cap K, K) = 1. \tag{3.14}$$

By (3.11) and (3.14) we have

$$i(A, (B_R \cap K)\backslash(\overline{B_r} \cap K), K) = i(A, (B_R \cap K, K) - i(A, B_r \cap K, K) = -1.$$

Thus A has at least one fixed point on $(B_R \cap K)\backslash(\overline{B}_r \cap K)$. This means that the singular PBVP (1.1) has at least one positive solution.

Proof of Corollary 1.3 By (1.3), we know that

$$\lim_{u\to 0^+} \frac{\lambda f(u)}{u} > \lambda_1, \quad \lim_{u\to +\infty} \frac{\lambda f(u)}{u} < \lambda_1.$$

So Corollary 1.3 holds from Theorem 1.1.

Proof of Corollary 1.4 By (1.5), we know that

$$\lim_{u\to +\infty} \frac{\lambda f(u)}{u} > \lambda_1, \quad \lim_{u\to 0^+} \frac{\lambda f(u)}{u} < \lambda_1.$$

So Corollary 1.4 holds from Theorem 1.1.

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