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AN ITERATION PROCESS FOR COMMON FIXED POINTS OF TWO NONSELF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we introduce an iteration process for approximating common fixed points of two nonself asymptotically nonexpansive mappings in Banach spaces. Our process contains Mann iteration process and some other processes for nonself mappings but is independent of Ishikawa iteration process. We prove some weak and strong convergence theorems for this iteration process. Our results generalize and improve some results in contemporary literature.

1. Introduction

Let E be a real Banach space with C its nonempty subset. Let $T : C \to C$ be a mapping. A point $x \in C$ is called a fixed point of T iff $Tx = x$. In this paper, $\mathbb{N}$ stands for the set of natural numbers. T is called asymptotically nonexpansive if for a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, $\|T^n x - T^n y\| \leq k_n\|x - y\|$ for all $x, y \in C$ and all $n \in \mathbb{N}$. T is called uniformly L-Lipschitzian if for some $L > 0$, $\|T^n x - T^n y\| \leq L\|x - y\|$ for all $n \in \mathbb{N}$ and all $x, y \in C$. T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Let $P : E \to C$ be a nonexpansive retraction of E into C. A nonself mapping $T : C \to E$ is called asymptotically nonexpansive (according to Chidume-Ofoedu-Zegeye [3]) if for a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, we have $\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x - y\|$ for all $x, y \in C$ and $n \in \mathbb{N}$. T is called uniformly L-Lipschitzian if for some $L > 0$, $\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|$ for all $n \in \mathbb{N}$ and all $x, y \in C$. Because these definitions depend on a given nonexpansive retraction of a space onto its subset and such retractions may not be unique, therefore from here onwards, all the nonself mappings are always considered with respect to a fixed nonexpansive retraction P.

We will also denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$ and by $F := F(T) \cap (F(S))$, the set of common fixed points of two

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mappings S and T. In what follows, we fix $x_1 \in C$ as a starting point of a process, and take $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ sequences in $(0,1)$.

We know that Mann, and Ishikawa iteration processes are defined as:

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T x_n, \quad n \in \mathbb{N} \tag{1.1}$$

and

$$\begin{cases} x_{n+1} = (1-\alpha_n) x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n) x_n + \beta_n T x_n, \quad n \in \mathbb{N} \end{cases} \tag{1.2}$$

respectively.

Agarwal-O'Regan-Sahu [1] recently introduced the iteration process:

$$\begin{cases} x_{n+1} = (1-\alpha_n) T x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)\, x_n + \beta_n T x_n, \quad n \in \mathbb{N} \end{cases} \tag{1.3}$$

They showed that their process is independent of Mann and Ishikawa and converges faster than both of these. See Proposition 3.1 [1].

Obviously the above process deals with one self mapping only. The case of two mappings in iteration processes has also remained under study since Das and Debata [5] gave and studied a two mappings scheme. Also see, for example, Takahashi and Tamura [13] and Khan and Takahashi [9]. Note that two mappings case, that is , approximating the common fixed points, has its own importance as it has a direct link with the minimization problem, see for example Takahashi [12].

Being an important generalization of the class of nonexpansive self mappings, the class of asymptotically nonexpansive self mappings was introduced by Goebel and Kirk [3] whereas the concept of asymptotically nonexpansive nonself mappings was introduced by Chidume-Ofoedu-Zegeye [3] in 2003 as the generalization of asymptotically nonexpansive self mappings. Actually they studied the iteration process:

$$x_{n+1} = P((1-\alpha_n)\, x_n + \alpha_n T\,(PT)^{n-1}\, x_n),\ n \in \mathbb{N} \tag{1.4}$$

Nonself asymptotically nonexpansive mappings have been studied by many authors, for example,Wang [14] and the references cited therein. Wang studied the process:

$$\begin{cases} x_{n+1} = P((1-\alpha_n) x_n + \alpha_n S\,(PS)^{n-1}\, y_n), \\ y_n = P((1-\beta_n) x_n + \beta_n T\,(PT)^{n-1}\, x_n), \quad n \in \mathbb{N} \end{cases} \tag{1.5}$$

We modify the iteration process of Agarwal-O'Regan-Sahu [1] to the case of two nonself asymptotically nonexpansive mappings as follows.

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = P\left((1-\alpha_n) T\,(PT)^{n-1}\, x_n + \alpha_n S\,(PS)^{n-1}\, y_n\right), \\ y_n = P\left((1-\beta_n)\, x_n + \beta_n T\,(PT)^{n-1}\, x_n\right), \quad n \in \mathbb{N} \end{cases} \tag{1.6}$$

It is also to be noted that (1.6) reduces to

- another extension of Agarwal-O'Regan-Sahu (1.3) process for one asymptotically nonexpansive mapping when $S = T$, namely

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = P\left((1-\alpha_n) T (PT)^{n-1} x_n + \alpha_n T (PT)^{n-1} y_n\right), \\ y_n = P\left((1-\beta_n) x_n + \beta_n T (PT)^{n-1} x_n\right), \quad n \in \mathbb{N} \end{cases}$$

  Not even this has been considered yet.
- Chidume-Ofoedu-Zegeye (1.4) process when $T = I$.
- Wang (1.5) process and our process are independent: neither reduces to the other. Following Agarwal-O'Regan-Sahu [1], we can say that our process is independent of Wang and converges faster than it.

Note that Agarwal-O'Regan-Sahu process (1.3) does not reduce to Mann process (1.1) but our process (1.6) does. It means that the results proved by using (1.6) not only contain the corresponding results of Agarwal-O'Regan-Sahu using (1.3) extended to nonself case but also cover the left over ones using Chidume-Ofoedu-Zegeye process (1.4). Moreover, it is able to compute common fixed points like (1.5) but at a better rate.

In this paper, we prove some weak and strong convergence theorems for two asymptotically nonexpansive mappings using (1.6).

We recall the following. Let $S = \{x \in E : \|x\| = 1\}$ and let E^* be the dual of E, that is, the space of all continuous linear functionals f on E. The space E has:

(i) Gâteaux differentiable norm if $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each x and y in S; (ii) Fréchet differentiable norm (see e.g. [1]) if for each x in S, the above limit exists and is attained uniformly for y in S and in this case, it is also well-known that

$$\langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 \leq \frac{1}{2}\|x+h\|^2 \leq \langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 + b(\|h\|) \tag{1.7}$$

for all x, h in E, where J is the Fréchet derivative of the functional $\frac{1}{2}\|.\|^2$ at $x \in X$, $\langle ., .\rangle$ is the pairing between E and E^*, and b is an increasing function defined on $[0, \infty)$ such that $\lim_{t\downarrow 0} \frac{b(t)}{t} = 0$; (iii) Opial condition [10] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$ (iv) Kadec-Klee property if for every sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $x_n \to x$ as $n \to \infty$.

Examples of Banach spaces satisfying Opial condition are Hilbert spaces and all spaces $l^p (1 < p < \infty)$. On the other hand, $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial condition. Uniformly convex Banach spaces, Banach spaces of finite dimension and reflexive locally uniform convex Banach spaces are some of the examples of reflexive Banach spaces which satisfy the Kadec-Klee property. Also note that there exist uniformly convex Banach spaces which neither satisfy the Opial condition nor do they have Fréchet differentiable norm but their duals do have the Kadec-Klee property. For example (Example 3.1, Falset et al. [6]), let us take $X_1 = \mathbb{R}^2$ with the norm denoted by $|x| = \sqrt{\|x_1\|^2 + \|x_2\|^2}$ and $X_2 = L_p[0, 1]$ with $1 < p < \infty$ and $p \neq 2$. The Cartesian product of X_1 and X_2 furnished with the l^2-norm is uniformly convex, it neither satisfies the Opial condition [6, 10] nor it has a Fréchet differentiable norm but its dual does have the Kadec-Klee property.

A mapping $T : C \to E$ is demiclosed at $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

Lemma 1. [11] *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2. *If $\{r_n\}, \{t_n\}$ and $\{s_n\}$ are sequences of nonnegative real numbers such that $r_{n+1} \leq (1 + t_n)\, r_n + s_n$, $\sum_{n=1}^{\infty} t_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$, then $\lim\limits_{n\to\infty} r_n$ exists.*

Lemma 3. [3] *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let T be a nonself asymptotically nonexpansive mapping. Then $I - T$ is demiclosed with respect to zero.*

Lemma 4. [8] *Let E be a reflexive Banach space such that E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and $q, y^* \in W = \omega_w(x_n)$(weak limit set of $\{x_n\}$). Suppose $\lim\limits_{n\to\infty} \|tx_n + (1-t)q^* - y^*\|$ exists for all $t \in [0,1]$. Then $q = y^*$.*

Lemma 5. [6] *Let C be a convex subset of a uniformly convex Banach space. Then there is a strictly increasing and continuous convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that for every Lipschitzian map $U : C \to C$ with Lipschitz constant $L \geq 1$, the following inequality holds:*

$$\|U(tx + (1-t)y) - (tUx + (1-t)Uy)\| \leq Lg^{-1}(\|x-y\| - L^{-1}\|Ux - Uy\|)$$

for all $x, y \in C$ and $t \in [0,1]$.

2. Convergence Theorems

We start with proving some key lemmas for later use.

Lemma 6. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $P : E \to C$ be a nonexpansive retraction of E into C. Let T and S be two asymptotically nonexpansive nonself mappings of C with $\sum\limits_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}, \{\beta_n\}$ be in $[\varepsilon, 1-\varepsilon]$ for all $n \in \mathbb{N}$ and for some ε in $(0,1)$, and let $\{x_n\}$ be defined by the iteration process (1.6). If $F \neq \varnothing$, then $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = \lim\limits_{n\to\infty} \|x_n - Sx_n\| = 0$.*

Proof. Let $q \in F$. Then

$$\begin{aligned}
\|x_{n+1} - q\| &= \left\| P\left((1-\alpha_n) T (PT)^{n-1} x_n + \alpha_n S (PS)^{n-1} y_n \right) - Pq \right\| \\
&\leq (1-\alpha_n) \left\| T (PT)^{n-1} x_n - q \right\| + \alpha_n \left\| S (PS)^{n-1} y_n - q \right\| \\
&\leq (1-\alpha_n) k_n \|x_n - q\| + \alpha_n k_n \|y_n - q\| \\
&= k_n [(1-\alpha_n) \|x_n - q\| + \alpha_n \|y_n - q\|] \\
&\leq k_n \left[\begin{array}{c} (1-\alpha_n) \|x_n - q\| + \alpha_n (1-\beta_n) \|x_n - q\| \\ +\alpha_n \beta_n \left\| T (PT)^{n-1} x_n - q \right\| \end{array} \right] \\
&= k_n \left[(1 - \alpha_n + \alpha_n (1-\beta_n) + k_n \alpha_n \beta_n) \|x_n - q\| \right] \\
&\leq k_n [(1 + (k_n - 1) \alpha_n \beta_n] \|x_n - q\| \\
&\leq k_n [(1 + k_n - 1] \|x_n - q\| \\
&\leq k_n^2 \|x_n - q\| \\
&= \left[1 + \left(k_n^2 - 1 \right) \right] \|x_n - q\|
\end{aligned}$$

Thus by Lemma 2, $\lim_{n\to\infty} \|x_n - q\|$ exists. Call it c.

Now

$$\begin{aligned}
\|y_n - q\| &= \left\| P\left((1-\beta_n) x_n + \beta_n T (PT)^{n-1} x_n \right) - q \right\| \\
&\leq \beta_n \left\| T (PT)^{n-1} x_n - q \right\| + (1-\beta_n) \|x_n - q\| \\
&\leq \beta_n k_n \|x_n - q\| + (1-\beta_n) \|x_n - q\| \\
&= (1 + \beta_n (k_n - 1)) \|x_n - q\| \\
&\leq k_n \|x_n - q\|
\end{aligned}$$

implies that

$$\limsup_{n\to\infty} \|y_n - q\| \leq c. \tag{2.1}$$

Also

$$\|T (PT)^{n-1} x_n - q\| \leq k_n \|x_n - q\|$$

for all $n = 1, 2, \ldots$, so

$$\limsup_{n\to\infty} \|T (PT)^{n-1} x_n - q\| \leq c. \tag{2.2}$$

Next,

$$\|S (PS)^{n-1} y_n - q\| \leq k_n \|y_n - q\|$$

gives by (2.1) that

$$\limsup_{n\to\infty} \|S (PS)^{n-1} y_n - q\| \leq c.$$

Further,

$$\begin{aligned}
c &= \lim_{n\to\infty} \|x_{n+1} - q\| = \lim_{n\to\infty} \left\| P\left((1-\alpha_n) T(PT)^{n-1} x_n + \alpha_n S(PS)^{n-1} y_n \right) - Pq \right\| \\
&\leq \lim_{n\to\infty} \| (1-\alpha_n) \left(T(PT)^{n-1} x_n - q \right) + \alpha_n \left(S(PS)^{n-1} y_n - q \right) \| \\
&\leq \lim_{n\to\infty} \left[(1-\alpha_n) \left\| \limsup_{n\to\infty} \left(T(PT)^{n-1} x_n - q \right) \right\| + \alpha_n \left\| \limsup_{n\to\infty} \left(S(PS)^{n-1} y_n - q \right) \right\| \right] \\
&\leq \lim_{n\to\infty} [(1-\alpha_n) c + \alpha_n c] \\
&= c
\end{aligned}$$

gives that

$$\lim_{n\to\infty} \|\,(1-\alpha_n)\left(T(PT)^{n-1}x_n - q\right) + \alpha_n\left(S(PS)^{n-1}y_n - q\right)\| = c. \tag{2.3}$$

Applying Lemma 1, we obtain

$$\lim_{n\to\infty} \left\|S(PS)^{n-1}y_n - T(PT)^{n-1}x_n\right\| = 0. \tag{2.4}$$

$$\begin{aligned} \|x_{n+1} - q\| &= \left\|P\left((1-\alpha_n)T(PT)^{n-1}x_n + \alpha_n S(PS)^{n-1}y_n\right) - Pq\right\| \\ &\le \left\|(1-\alpha_n)T(PT)^{n-1}x_n + \alpha_n S(PS)^{n-1}y_n - q\right\| \\ &\le \left\|T(PT)^{n-1}x_n - q\right\| + \alpha_n\left\|S(PS)^{n-1}y_n - T(PT)^{n-1}x_n\right\| \end{aligned}$$

yields that

$$c \le \liminf_{n\to\infty} \left\|T(PT)^{n-1}x_n - q\right\|$$

so that (2.2) gives $\lim_{n\to\infty}\left\|T(PT)^{n-1}x_n - q\right\| = c$.

In turn,

$$\begin{aligned} \left\|T(PT)^{n-1}x_n - q\right\| &\le \left\|T(PT)^{n-1}x_n - S(PS)^{n-1}y_n\right\| + \|S(PS)^{n-1}y_n - q\| \\ &\le \left\|T(PT)^{n-1}x_n - S(PS)^{n-1}y_n\right\| + k_n\|y_n - q\| \end{aligned}$$

implies

$$c \le \liminf_{n\to\infty} \|y_n - q\|. \tag{2.5}$$

By (2.1) and (2.5), we obtain

$$\lim_{n\to\infty} \|y_n - q\| = c.$$

Moreover,

$$\begin{aligned} c &= \lim_{n\to\infty}\|y_n - q\| = \lim_{n\to\infty}\left\|P\left((1-\beta_n)x_n + \beta_n T(PT)^{n-1}x_n\right) - Pq\right\| \\ &\le \lim_{n\to\infty}\|\,(1-\beta_n)(x_n - q) + \beta_n\left(T(PT)^{n-1}x_n - q\right)\| \\ &\le \lim_{n\to\infty}\left[(1-\beta_n)\left\|\limsup_{n\to\infty}(x_n - q)\right\| + \beta_n\left\|\limsup_{n\to\infty}\left(T(PT)^{n-1}x_n - q\right)\right\|\right] \\ &\le \lim_{n\to\infty}\left[(1-\beta_n)c + \beta_n c\right] \\ &= c \end{aligned}$$

gives by Lemma 1 that

$$\lim_{n\to\infty}\left\|T(PT)^{n-1}x_n - x_n\right\| = 0. \tag{2.6}$$

Now $x_n \in C$, the range of P, therefore $Px_n = x_n$ for all $n \in \mathbb{N}$ and so

$$\begin{aligned} \|y_n - x_n\| &= \left\|P\left(\beta_n(T(PT)^{n-1}x_n + (1-\beta_n)x_n\right) - Px_n\right\| \\ &\le \left\|\beta_n T(PT)^{n-1}x_n + (1-\beta_n)x_n - x_n\right\| \\ &\le \beta_n\left\|T(PT)^{n-1}x_n - x_n\right\|. \end{aligned}$$

Hence by (2.6),

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{2.7}$$

Also note that

$$\begin{aligned}\|x_{n+1}-x_n\| &= \left\|P\left((1-\alpha_n)T(PT)^{n-1}x_n+\alpha_n S(PS)^{n-1}y_n\right)-Px_n\right\| \\ &\le \left\|(1-\alpha_n)T(PT)^{n-1}x_n+\alpha_n S(PS)^{n-1}y_n-x_n\right\| \\ &\le \left\|T(PT)^{n-1}x_n-x_n\right\|+\alpha_n\left\|S(PS)^{n-1}y_n-T(PT)^{n-1}x_n\right\| \\ &\to 0 \text{ as } n\to\infty, \end{aligned} \tag{2.8}$$

so that

$$\begin{aligned}\|x_{n+1}-y_n\| &\le \|x_{n+1}-x_n\|+\|y_n-x_n\| \\ &\to 0 \text{ as } n\to\infty.\end{aligned} \tag{2.9}$$

Furthermore, from

$$\begin{aligned}\left\|x_{n+1}-T(PT)^{n-1}y_n\right\| &\le \|x_{n+1}-x_n\|+\left\|x_n-T(PT)^{n-1}x_n\right\|+\left\|T(PT)^{n-1}x_n-T(PT)^{n-1}y_n\right\| \\ &\le \|x_{n+1}-x_n\|+\left\|x_n-T(PT)^{n-1}x_n\right\|+k_n\|x_n-y_n\|\end{aligned}$$

we find that

$$\lim_{n\to\infty}\left\|x_{n+1}-T(PT)^{n-1}y_n\right\|=0. \tag{2.10}$$

Now we shall make use of the fact that every asymptotically nonexpansive mapping is uniformly L-Lipschitzian combined with (2.6), (2.9) and (2.10) to reach at

$$\begin{aligned}\|x_n-Tx_n\| &\le \left\|x_n-T(PT)^{n-1}x_n\right\|+\left\|T(PT)^{n-1}x_n-T(PT)^{n-1}y_{n-1}\right\| \\ &\quad+\left\|T(PT)^{n-1}y_{n-1}-Tx_n\right\| \\ &\le \left\|x_n-T(PT)^{n-1}x_n\right\|+k_n\|x_n-y_{n-1}\| \\ &\quad+L\left(\left\|T(PT)^{n-2}y_{n-1}-x_n\right\|\right)\end{aligned}$$

so that

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{2.11}$$

To prove that $\lim_{n\to\infty}\|x_n-Sx_n\|=0$, first note that

$$\begin{aligned}\|S(PS)^{n-1}x_n-x_n\| &\le \left\|S(PS)^{n-1}x_n-S(PS)^{n-1}y_n\right\|+\left\|S(PS)^{n-1}y_n-T(PT)^{n-1}x_n\right\| \\ &\quad+\left\|T(PT)^{n-1}x_n-x_n\right\| \\ &\le k_n\|x_n-y_n\|+\left\|S(PS)^{n-1}y_n-T(PT)^{n-1}x_n\right\|+\left\|T(PT)^{n-1}x_n-x_n\right\|.\end{aligned}$$

so that by (2.7), (2.4) and (2.6),

$$\lim_{n\to\infty}\|S(PS)^{n-1}x_n-x_n\|=0. \tag{2.12}$$

Next note that

$$\|x_{n+1}-S(PS)^{n-1}x_n\|\le\|x_{n+1}-x_n\|+\|x_n-S(PS)^{n-1}x_n\|$$

gives by (2.8) and (2.12)

$$\lim_{n\to\infty}\|x_{n+1}-S(PS)^{n-1}x_n\|=0. \tag{2.13}$$

Again making use of the fact that every asymptotically nonexpansive mapping is L-Lipschitzian, we have

$$\begin{aligned}\|x_{n+1}-Sx_{n+1}\| &\le \|x_{n+1}-S(PS)^n x_{n+1}\|+\|S(PS)^n x_{n+1}-S(PS)^n x_n\| \\ &\quad+\|S(PS)^n x_n-Sx_{n+1}\| \\ &\le \|x_{n+1}-S(PS)^n x_{n+1}\|+k_{n+1}\|x_{n+1}-x_n\| \\ &\quad+L\|S(PS)^{n-1}x_n-x_{n+1}\|.\end{aligned}$$

That us by (2.12) and (2.13) , we have

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{2.14}$$

This completes the proof of the lemma.

Lemma 7. *For any* $p_1,\ p_2 \in F$, $\lim_{n\to\infty} \|tx_n + (1-t)p_1 - p_2\|$ *exists for all* $t \in [0,1]$ *under the conditions of Lemma* 6.

Proof. By Lemma 6, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$ and therefore $\{x_n\}$ is bounded. Thus there exists a real number $r > 0$ such that $\{x_n\} \subseteq D \equiv \overline{B_r(0)} \cap C$, so that D is a closed convex nonempty subset of C. Put

$$g_n(t) = \|tx_n + (1-t)p_1 - p_2\|$$

for all $t \in [0,1]$. Then $\lim_{n\to\infty} g_n(0) = \|p_1 - p_2\|$ and $\lim_{n\to\infty} g_n(1) = \lim_{n\to\infty} \|x_n - p_2\|$ exist. Let $t \in (0,1)$.

Define $B_n : D \to D$ by:

$$\begin{aligned} B_n x &= P\left((1-\alpha_n)T(PT)^{n-1}x + \alpha_n S(PS)^{n-1}A_n x\right) \\ A_n x &= P\left((1-\beta_n)\, x + \beta_n T(PT)^{n-1}x\right) \end{aligned}$$

Then $B_n x_n = x_{n+1}$, $B_n p = p$ for all $p \in F$. Also

$$\begin{aligned} \|A_n x - A_n y\| &= \left\|P((1-\beta_n)\, x + \beta_n T(PT)^{n-1}x) - P\left((1-\beta_n)\, y + \beta_n T(PT)^{n-1}y\right)\right\| \\ &\leq \left\|(1-\beta_n)\,(x-y) + \beta_n (T(PT)^{n-1}x - T(PT)^{n-1}y)\right\| \\ &= (1-\beta_n)\,\|x-y\| + \beta_n k_n \|x-y\| \\ &\leq k_n\,\|x-y\| \end{aligned}$$

and

$$\begin{aligned} \|B_n x - B_n y\| &= \left\| \begin{array}{c} P[(1-\alpha_n)T(PT)^{n-1}x + \alpha_n S(PS)^{n-1}A_n x] \\ -P[(1-\alpha_n)T(PT)^{n-1}y + \alpha_n S(PS)^{n-1}A_n y)] \end{array} \right\| \\ &\leq \left\| \begin{array}{c} [(1-\alpha_n)\left(T(PT)^{n-1}x - T(PT)^{n-1}y\right) \\ +\alpha_n (S(PS)^{n-1}A_n x - S(PS)^{n-1}A_n y)] \end{array} \right\| \\ &\leq (1-\alpha_n)k_n\|x-y\| + \alpha_n k_n\,\|A_n x - A_n y\| \\ &\leq (1-\alpha_n)k_n^2\|x-y\| + \alpha_n k_n^2\,\|x-y\| \\ &= k_n^2\|x-y\|. \end{aligned}$$

Set

$$R_{n,m} = B_{n+m-1}B_{n+m-2}...B_n,\ m \geq 1$$

and

$$v_{n,m} = \|R_{n,m}\,(tx_n + (1-t)p_1) - (tR_{n,m}x_n + (1-t)p_1)\|\,.$$

Then $R_{n,m}x_n = x_{n+m}$ and $R_{n,m}p = p$ for all $p \in F$. Also

$$\begin{aligned}
\|R_{n,m}x - R_{n,m}y\| &\leq \|B_{n+m-1}B_{n+m-2}...B_n x - B_{n+m-1}B_{n+m-2}...B_n y\| \\
&\leq k_{n+m-1}^2 \|B_{n+m-2}...B_n x - B_{n+m-2}...B_n y\| \\
&\leq k_{n+m-1}^2 k_{n+m-2}^2 \|B_{n+m-3}...B_n x - B_{n+m-3}...B_n y\| \\
&\ \ \vdots \\
&\leq \left(\prod_{j=n}^{n+m-1} k_j^2\right) \|x - y\|
\end{aligned}$$

Applying Lemma 5 with $x = x_n$, $y = p_1, U = R_{n,m}$ and using the facts that $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} \|x_n - w\|$ exists for all $w \in F$, we obtain $v_{n,m} \to 0$ as $n \to \infty$ and for all $m \geq 1$.

Finally, from the inequality

$$\begin{aligned}
g_{n+m}(t) &= \|tR_{n,m}x_n + (1-t)p_1 - p_2\| \\
&\leq v_{n,m} + \|R_{n,m}(tx_n + (1-t)p_1) - p_2\| \\
&\leq v_{n,m} + \prod_{j=n}^{n+m-1} k_j^2 g_n(t),
\end{aligned}$$

it follows that

$$\begin{aligned}
\limsup_{n\to\infty} g_n(t) &\leq \limsup_{n,m\to\infty} v_{n,m} + \liminf_{n\to\infty} g_n(t) \\
&= \liminf_{n\to\infty} g_n(t)
\end{aligned}$$

That is,

$$\limsup_{n\to\infty} g_n(t) \leq \liminf_{n\to\infty} g_n(t).$$

so that $\lim_{n\to\infty} \|tx_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0,1]$.

Lemma 8. *Assume that the conditions of Lemma 6 are satisfied. Then, for any $p_1, p_2 \in F$, $\lim_{n\to\infty} \langle x_n, J(p_1 - p_2)\rangle$ exists; in particular, $\langle p - q, J(p_1 - p_2)\rangle = 0$ for all $p, q \in \omega_w(x_n)$.*

Proof. Take $x = p_1 - p_2$ with $p_1 \neq p_2$ and $h = t(x_n - p_1)$ in the inequality (1.7) to get:

$$\begin{aligned}
\frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, J(p_1 - p_2)\rangle &\leq \frac{1}{2}\|tx_n + (1-t)p_1 - p_2\|^2 \\
&\leq \frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, J(p_1 - p_2)\rangle \\
&\quad + b(t\|x_n - p_1\|).
\end{aligned}$$

As $\sup_{n\geq 1} \|x_n - p_1\| \leq M'$ for some $M' > 0$, it follows that

$$\begin{aligned}
\frac{1}{2}\|p_1 - p_2\|^2 + t\limsup_{n\to\infty} \langle x_n - p_1, J(p_1 - p_2)\rangle &\leq \frac{1}{2}\lim_{n\to\infty} \|tx_n + (1-t)p_1 - p_2\|^2 \\
&\leq \frac{1}{2}\|p_1 - p_2\|^2 + b(tM') \\
&\quad + t\liminf_{n\to\infty} \langle x_n - p_1, J(p_1 - p_2)\rangle.
\end{aligned}$$

That is,

$$\limsup_{n\to\infty} \langle x_n - p_1, J(p_1 - p_2)\rangle \le \liminf_{n\to\infty} \langle x_n - p_1, J(p_1 - p_2)\rangle + \frac{b(tM')}{tM'} M'.$$

If $t \to 0$, then $\lim_{n\to\infty} \langle x_n - p_1, J(p_1 - p_2)\rangle$ exists for all $p_1, p_2 \in F$; in particular, we have $\langle p - q, J(p_1 - p_2)\rangle = 0$ for all $p, q \in \omega_w(x_n)$.

2.1. **Weak Convergence.** We now give our weak convergence theorem.

Theorem 9. *Let E be a uniformly convex Banach space and let C, T, S and $\{x_n\}$ be taken as in Lemma 6. Assume that (a) E satisfies Opial's condition or (b)E has a Fréchet differentiable norm or (c) dual E^* of E satisfies Kadec-Klee property. If $F \neq \phi$ then $\{x_n\}$ converges weakly to a point of F.*

Proof. Let $p \in F$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists as proved in Lemma 6. We prove that $\{x_n\}$ has a unique weak subsequential limit in F. Let u and v be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemma 6, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $I - T$ is demiclosed with respect to zero by Lemma 3, therefore we obtain $Tu = u$. Similarly, $Su = u$. Again in the same fashion, we can prove that $v \in F$. Next, we prove the uniqueness. To this end, first assume (a) is true. If u and v are distinct, then by Opial condition,

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - u\| &= \lim_{n_i\to\infty} \|x_{n_i} - u\| \\
&< \lim_{n_i\to\infty} \|x_{n_i} - v\| \\
&= \lim_{n\to\infty} \|x_n - v\| \\
&= \lim_{n_j\to\infty} \|x_{n_j} - v\| \\
&< \lim_{n_j\to\infty} \|x_{n_j} - u\| \\
&= \lim_{n\to\infty} \|x_n - u\|.
\end{aligned}$$

This is a contradiction so $u = v$. Next assume (b). By Lemma 8, $\langle p - q, J(p_1 - p_2)\rangle = 0$ for all $p, q \in \omega_w(x_n)$. Therefore $\|u - v\|^2 = \langle u - v, J(u - v)\rangle = 0$ implies $u = v$. Finally, say (c) is true. Since $\lim_{n\to\infty} \|tx_n + (1-t)u - v\|$ exists for all $t \in [0, 1]$ by Lemma 7, therefore $u = v$ by Lemma 4. Consequently, $\{x_n\}$ converges weakly to a point of F and this completes the proof.

Although the following can be obtained as a corollary from our above theorem by putting $S = T$, yet it is new in itself.

Corollary 10. *Let E be a uniformly convex Banach space and let C, T be taken as in Lemma 6 and and $\{x_n\}$ as*

$$\begin{cases}
x_1 = x \in C, \\
x_{n+1} = P\left((1-\alpha_n)T\,(PT)^{n-1}\,x_n + \alpha_n T\,(PT)^{n-1}\,y_n\right), \\
y_n = P\left((1-\beta_n)\,x_n + \beta_n T\,(PT)^{n-1}\,x_n\right), \quad n \in \mathbb{N}
\end{cases}$$

Assume that (a) E satisfies Opial condition or (b) E has a Fréchet differentiable norm or (c) dual E^* of E satisfies Kadec-Klee property. If $F(T) \neq \phi$ then $\{x_n\}$ converges weakly to a point of $F(T)$.

Corollary 11. *Let E be a uniformly convex Banach space and let C, T be taken as in Lemma 6 and and $\{x_n\}$ as*

$$x_{n+1} = P((1-\alpha_n)\, x_n + \alpha_n T\, (PT)^{n-1}\, x_n),\ n \in \mathbb{N}.$$

Assume that (a) E satisfies Opial condition or (b) E has a Fréchet differentiable norm or (c) dual E^* of E satisfies Kadec-Klee property. If $F(T) \neq \phi$ then $\{x_n\}$ converges weakly to a point of $F(T)$.

2.2. **Strong Convergence.** Following [7], we say that two mappings $S, T : C \to E$, where C is a subset of a normed space E, are said to satisfy the Condition (A') if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0,\ f(r) > 0$ for all $r \in (0,\infty)$ such that either $\|x - Sx\| \geq f(d(x,F))$ or $\|x - Tx\| \geq f(d(x,F))$ for all $x \in C$ where $d(x,F) = \inf\{\|x-p\| : p \in F\}$.

Theorem 12. *Let E be a real Banach space and let $C, T, S, F, \{x_n\}$ be taken as in Theorem 6. Then $\{x_n\}$ converges to a point of F if and only if $\liminf_{n\to\infty} d(x_n,F) = 0$ where $d(x,F) = \inf\{\|x-p\| : p \in F\}$.*

Proof. Necessity is obvious. Suppose that $\liminf_{n\to\infty} d(x_n,F) = 0$. As proved in Theorem 6, $\lim_{n\to\infty} \|x_n - w\|$ exists for all $w \in F$, therefore $\lim_{n\to\infty} d(x_n,F)$ exists. But by hypothesis, $\liminf_{n\to\infty} d(x_n,F) = 0$, therefore we have $\lim_{n\to\infty} d(x_n,F) = 0$. On the lines similar to [7], it can be proved that $\lim_{n\to\infty} d(x_n,F) = 0$. This gives that $d(q,F) = 0$ and so $q \in F$.

Applying Theorem 12, we obtain a strong convergence of the scheme (1.6) under the Condition (A') as follows.

Theorem 13. *Let E be a real Banach space and let $C, T, S, F, \{x_n\}$ be taken as in Lemma 6. If T, S satisfy the Condition (A') then $\{x_n\}$ converges strongly to a common fixed point of T and S .*

Proof. We proved in Theorem 6 that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{2.15}$$

From the Condition (A') and (2.15), we get

$$\lim_{n\to\infty} f(d(x_n,F)) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0,$$

or

$$\lim_{n\to\infty} f(d(x_n,F)) \leq \lim_{n\to\infty} \|x_n - Sx_n\| = 0,$$

In both the cases,

$$\lim_{n\to\infty} f(d(x_n,F)) = 0.$$

Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0,\ f(r) > 0$ for all $r \in (0,\infty)$, therefore we have

$$\lim_{n\to\infty} d(x_n,F) = 0.$$

Now all the conditions of Theorem 12 are satisfied, therefore by its conclusion $\{x_n\}$ converges strongly to a point of F.

Remark 1. *Corollaries like Corollary 10 and Corollary 11 can now be obtained in this case as well.*

Remark 2. *The case of nonexapnsive mappings now follows as a corollary from our above results.*

Remark 3. *Theorems of this paper can also be proved with error terms.*

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