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ON SOLUTIONS OF A SYSTEM OF HIGHER-ORDER NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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ZHENYU GUO, MIN LIU

Abstract. A system of higher-order nonlinear fractional differential equations is studied in this article, and some sufficient conditions for existence and uniqueness of a solution for the system is established by the nonlinear alternative of Leray-Schauder and Banach contraction principle.

1. Introduction and preliminaries

This article is concerned with the initial value problem for the following system of fractional order differential equations:

$$ {}^cD^{\rho}u(t) = f\Big(t, v^{(n)}(t), {}^cD^{\beta}v(t)\Big), \quad u^{(k)}(0) = \eta_k, \quad 0 < t \le T, \tag{1.1} $$

$$ {}^cD^{\sigma}v(t) = g\Big(t, u^{(n)}(t), {}^cD^{\alpha}u(t)\Big), \quad v^{(k)}(0) = \xi_k, \quad 0 < t \le T, \tag{1.2} $$

where ${}^cD$ denotes the Caputo fractional derivative, $f, g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ are given functions, $\rho, \sigma \in (m-1, m), \alpha, \beta \in (n-1, n), m, n \in \mathbb{N}, \rho > \beta, \sigma > \alpha$, $k = 0, 1, 2, \cdots, m-1, T > 0$, and η_k, ξ_k are suitable real constants. In this article, we consider the case that all of ρ, σ, β and α are non-integer valued.

Recently, fractional order differential equations and systems have been of great interest. For example, in 2010, Li[9] discussed the existence and uniqueness of mild solution for

$$ \begin{aligned} \frac{d^q x(t)}{dt^q} &= -Ax(t) + f\big(t, x(t), Gx(t)\big), \quad t \in [0,T], \\ x(0) + g(x) &= x_0. \end{aligned} \tag{1.3} $$

Li and Guérékata[10] studied mild solutions of the fractional integrodifferential equations as follows

$$ \frac{d^q x(t)}{dt^q} + Ax(t) = f(t, x(t)) + \int_0^t a(t-s)g(s, x(s))ds, \quad t \in [0,T], \quad x(0) = x_0. \tag{1.4} $$

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In 2011, Anguraj, Karthikeyan and Trujillo[1] investigated the existence and the uniqueness of the solution for the following fractional integrodifferential equation

$$\begin{aligned} \frac{d^q x(t)}{dt^q} &= f\Big(t, x(t), \int_0^t k\big(t,s,x(s)\big)ds, \int_0^1 h\big(t,s,x(s)\big)ds\Big), \quad t \in [0,1], \\ x(0) &= \int_0^1 g(s)x(s)ds. \end{aligned} \tag{1.5}$$

Guo and Liu[4] studied the existence of unique solutions of initial value problems of the following system of fractional order differential equations with infinite delay

$$\begin{aligned} D^\alpha y_1(t) &= f_1[t, y_{1t}, y_{2t}], \quad t \in [0,b], \\ y_1(t) &= \phi_1(t), \quad t \in (-\infty, 0], \\ D^\alpha y_2(t) &= f_2[t, y_{1t}, y_{2t}], \quad t \in [0,b], \\ y_2(t) &= \phi_2(t), \quad t \in (-\infty, 0]. \end{aligned} \tag{1.6}$$

For detailed discussion on this topic, refer to the monographs of Kilbas et al.[5], and the papers by Ahmad and Alsaedi [2], Guo and Liu [3], Kosmatov [6], Lakshmikantham and Vatsala [7], Li and Deng [8], Su [11], Goodrich [12,13], Bonilla et al. [14], Bai and Fang [15], Kobayashi [16], Wang et al. [17] and the references therein.

Applying the nonlinear alternative of Leray-Schauder, we obtain a result of existence of a solution for system (1.1)-(1.2). The uniqueness of a solution for the system is established by Banach contraction principle.

The following notations, definitions, and preliminary facts will be used throughout this paper.

Let $X = \{u : u \in C([0,T])\}$ and $Y = \{v : v \in C([0,T])\}$ be normed spaces with the sup-norm $\|u\|_X$ and $\|v\|_Y$, respectively, where $C([0,T])$ denotes the space of all continuous functions defined on $[0,T]$. Then, $(X \times Y, \|\cdots\|_{X\times Y})$ is a normed space endowed with the sup-norm given by $\|(u,v)\|_{X\times Y} := \max\{\|u\|_X, \|v\|_Y\}$.

Definition 1.1. *For a function $f \in C^m([0,T]), m \in \mathbb{N}$, where $C^m([0,T])$ denotes the space of all continuous functions with mth order derivative, the Caputo derivative of fractional order $\alpha \in (m-1, m)$ is defined by*

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s)ds. \tag{1.7}$$

Definition 1.2. *The Riemann-Liouville fractional integral of order α, inversion of D^α, is defined by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds. \tag{1.8}$$

Lemma 1.3. [8] *If $\alpha \in (m-1,m), m \in \mathbb{N}, f \in C^m([0,T])$ and $g \in C^1([0,T])$, then*

$$(1)\ {}^cD^\alpha I^\alpha g(t) = g(t);$$

$$(2)\ I^\alpha({}^cD^\alpha) f(t) = f(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} f^{(k)}(0).$$

Lemma 1.4. [6] *If $m-1 < \alpha < \beta < m$ and $f \in C^m([0,T])$, then for all $k \in \{1, 2, \cdots, m-1\}$ and for all $t \in [0,T]$, the following relations hold:*

$${}^cD^{\beta-m+k} f^{m-k}(t) = {}^c D^\beta f(t), \tag{1.9}$$

$$ {}^cD^{\beta-\alpha}\ {}^cD^{\alpha}f(t) = {}^cD^{\beta}f(t). \tag{1.10}$$

Theorem 1.5. *(the nonlinear alternative of Leray-Schauder) Let X be a normed linear space, $S \subset X$ be a convex set, U be open in S with $0 \in U$, and $F : \overline{U} \to S$ be a continuous and compact mapping. Then either the mapping F has a fixed point in $\overline{U}$ or there exist $n \in \partial U$ and $\lambda \in (0,1)$ with $n = \lambda F n$.*

Now list the following hypotheses for convenience:

(H1) $f : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ are continuously differentiable function with $f(0,0,0) = 0$ and $f(t,0,0) \neq 0$ on a compact subinterval of $(0,T]$;

(H2) $g : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ are continuously differentiable function with $g(0,0,0) = 0$ and $g(t,0,0) \neq 0$ on a compact subinterval of $(0,T]$;

(H3) there exist nonnegative functions $a_1, a_2, a_3, b_1, b_2, b_3 \in C([0,T])$ such that

$$\begin{aligned} |f(t,x,y)| &\leq a_1(t) + a_2(t)|x| + a_3(t)|y|, \quad t \in [0,T], \\ |g(t,x,y)| &\leq b_1(t) + b_2(t)|x| + b_3(t)|y|, \quad t \in [0,T]; \end{aligned} \tag{1.11}$$

(H4) there exist nonnegative functions $l_1, l_2, l_3, l_4 \in C([0,T])$ such that

$$\begin{aligned} |f(t,x_1,y_1) - f(t,x_2,y_2)| &\leq l_1(t)|x_1 - x_2| + l_2(t)|y_1 - y_2|, \quad t \in [0,T], \\ |g(t,x_1,y_1) - g(t,x_2,y_2)| &\leq l_3(t)|x_1 - x_2| + l_4(t)|y_1 - y_2|, \quad t \in [0,T]. \end{aligned} \tag{1.12}$$

2. Existence and uniqueness of a solution

In this section, the theorems of existence and uniqueness of a solution for system (1.1)-(1.2) will be given.

Lemma 2.1. *Let (H1)-(H2) hold and $n-1 < \alpha, \beta < n \leq m-1 < \rho, \sigma < m$. Then, a function $u \in C^m([0,T])$ is a solution of the initial value problem (1.1) if and only if*

$$u(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!}\eta_k + \int_0^t \frac{(t-s)^{n-1}}{\Gamma(n)} w_1(s)ds,\ \ 0 < t \leq 1, \tag{2.1}$$

where $w_1(t) = u^{(n)}(t) \in C^{m-n}([0,T])$ with $u^{(n+i)}(t) = w_1^{(i)}(t),\ 0 \leq i \leq m-n-1$ is a solution of the integral equation

$$w_1(t) = \sum_{i=0}^{m-n-1} \frac{t^i}{i!}\eta_{n+i} + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} f\Big(s, w_2(s), \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} w_2(\tau)d\tau\Big)ds, \tag{2.2}$$

and a function $v \in C^m([0,T])$ is a solution of the initial value problem (1.2) if and only if

$$v(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!}\xi_k + \int_0^t \frac{(t-s)^{n-1}}{\Gamma(n)} w_2(s)ds,\ \ 0 < t \leq 1, \tag{2.3}$$

where $w_2(t) = v^{(n)}(t) \in C^{m-n}([0,T])$ with $v^{(n+i)}(t) = w_2^{(i)}(t)$ is a solution of the integral equation

$$w_2(t) = \sum_{i=0}^{m-n-1} \frac{t^i}{i!}\xi_{n+i} + \int_0^t \frac{(t-s)^{\sigma-n-1}}{\Gamma(\sigma-n)} g\Big(s, w_1(s), \int_0^s \frac{(s-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} w_1(\tau)d\tau\Big)ds. \tag{2.4}$$

Proof. Since the two parts of the Lemma is similar, we only give the proof of the first part briefly. Lemma 1.4 ensures that

$$ {}^{c}D^{\rho-n}u^{(n)}(t) = {}^{c}D^{\rho}u(t) = f\Big(t, v^{(n)}(t), {}^{c}D^{\beta}v(t)\Big). \tag{2.5} $$

By Definition 1.1, we obtain

$$ {}^{c}D^{\rho-n}u^{(n)}(t) = f\Big(t, v^{(n)}(t), \int_0^t \frac{(t-s)^{n-\beta-1}}{\Gamma(n-\beta)} v^{(n)}(s)ds\Big). \tag{2.6} $$

It follows from Definition 1.2, Lemma 1.3 (2) and the substitutions $u^{(n)}(t) = w_1(t), v^{(n)}(t) = w_2(t)$ that

$$ \begin{aligned} w_1(t) = u^{(n)}(t) &= \sum_{i=0}^{m-n-1} \frac{t^i}{i!} u^{(n+i)}(0) + I^{\rho-n}({}^{c}D^{\rho-n}u^{(n)}(t)) \\ &= \sum_{i=0}^{m-n-1} \frac{t^i}{i!} w_1^{(i)}(0) \\ &\quad + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} f\Big(s, v^{(n)}(s), \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} v^{(n)}(\tau)d\tau\Big)ds \\ &= \sum_{i=0}^{m-n-1} \frac{t^i}{i!} \eta_{n+i} \\ &\quad + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} f\Big(s, w_2(s), \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} w_2(\tau)d\tau\Big)ds. \end{aligned} \tag{2.7} $$

Conversely, suppose that $w_1 \in C^{m-n}([0,T])$ is a solution of (2.2). Then,

$$ \begin{aligned} u^{(n)}(t) = w_1(t) &= \sum_{i=0}^{m-n-1} \frac{t^i}{i!} \eta_{n+i} \\ &\quad + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} f\Big(s, w_2(s), \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} w_2(\tau)d\tau\Big)ds \\ &= \sum_{i=0}^{m-n-1} \frac{t^i}{i!} \eta_{n+i} + I^{\rho-n} f\Big(t, v^{(n)}(t), {}^{c}D^{\beta}v(t)\Big). \end{aligned} \tag{2.8} $$

Since $\rho - n \in (m-n-1, m-n)$, by Lemma 1.3 (1) and Lemma 1.4, we have

$$ \begin{aligned} {}^{c}D^{\rho}u(t) &= {}^{c}D^{\rho-n}u^{(n)}(t) \\ &= {}^{c}D^{\rho-n}\Big(\sum_{i=0}^{m-n-1} \frac{t^i}{i!} \eta_{n+i}\Big) + {}^{c}D^{\rho-n} I^{\rho-n} f\Big(t, v^{(n)}(t), {}^{c}D^{\beta}v(t)\Big) \\ &= f\Big(t, v^{(n)}(t), {}^{c}D^{\beta}v(t)\Big),\ 0 < t \le 1. \end{aligned} \tag{2.9} $$

Differentiating (2.2), we get

$$ \begin{aligned} w_1^{(k)} = &\sum_{i=0}^{m-n-k-1} \frac{t^i}{i!} \eta_{n+i+k} + \prod_{j=1}^{k} (\rho-n-j) \int_0^t \frac{(t-s)^{\rho-n-1-k}}{\Gamma(\rho-n)} \\ &f\Big(s, w_2(s), \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} w_2(\tau)d\tau\Big)ds \end{aligned} \tag{2.10} $$

for each $k = 0, 1, \cdots, m-n-1$. As $\rho - n - 1 - k \in (-1, m-n-1)$, the second term in (2.10) goes to zero as $t \to 0$. Thus, we have

$$u^{(n+k)}(0) = w_1^{(k)}(0) = \eta_{n+k},\ k = 0, 1, \cdots, m-n-1, \tag{2.11}$$

which means that $u^{(k)}(0) = \eta_k, k = 0, 1, \cdots, m-1$. Clearly, $w_1^{(m-n)} = u^{(m)} \in C([0,T])$. Therefore, u is a solution of (1.1). □

For the sake of simplicity, Lemma 2.1 can be rewritten as

Lemma 2.2. *Let $f, g : [0,T]\times\mathbb{R} \to \mathbb{R}$ be continuous functions. Then $(u,v) \in X \times Y$ is a solution of (1.1)-(1.2) if and only if $(u,v) \in X \times Y$ is a solution of (2.1)-(2.4).*

Theorem 2.3. *Assume (H1)-(H3) hold, and*

$$\begin{aligned}
B_1 &= \sup_{t\in[0,T]} \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)}\Big(a_2(s) + \frac{s^{n-\beta}}{\Gamma(n-\beta+1)}a_3(s)\Big)ds < 1,\\
B_2 &= \sup_{t\in[0,T]} \int_0^t \frac{(t-s)^{\sigma-n-1}}{\Gamma(\sigma-n)}\Big(b_2(s) + \frac{s^{n-\alpha}}{\Gamma(n-\alpha+1)}b_3(s)\Big)ds < 1,\\
0 < C_1 &= \sup_{t\in[0,T]} \Big(|\eta(t)| + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)}a_1(s)ds\Big) < +\infty,\\
0 < C_2 &= \sup_{t\in[0,T]} \Big(|\xi(t)| + \int_0^t \frac{(t-s)^{\sigma-n-1}}{\Gamma(\sigma-n)}b_1(s)ds\Big) < +\infty,
\end{aligned} \tag{2.12}$$

where

$$\eta(t) = \sum_{i=0}^{m-n-1} \frac{t^i}{i!}\eta_{n+i}, \qquad \xi(t) = \sum_{i=0}^{m-n-1} \frac{t^i}{i!}\xi_{n+i}. \tag{2.13}$$

Then the system of integral equations (2.1)-(2.4) has a solution.

Proof. Define a mapping $F : X \times Y \to X \times Y$ and a ball U in the normed space $X \times Y$ by

$$F(w_1, w_2)(t) = (F_1 w_2(t), F_2 w_1(t)), \tag{2.13}$$

and

$$U = \{(w_1(t), w_2(t)) : (w_1(t), w_2(t)) \in X \times Y, \|(w_1(t), w_2(t))\|_{X\times Y} < R, t \in [0,T]\}, \tag{2.14}$$

where

$$\begin{aligned}
F_1 w_2(t) &= \eta(t) + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} f\Big(s, w_2(s), \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)}w_2(\tau)d\tau\Big)ds,\\
F_2 w_1(t) &= \xi(t) + \int_0^t \frac{(t-s)^{\sigma-n-1}}{\Gamma(\sigma-n)} g\Big(s, w_1(s), \int_0^s \frac{(s-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)}w_1(\tau)d\tau\Big)ds,
\end{aligned} \tag{2.15}$$

and

$$R = \frac{C}{1-B},\ B = \max\{B_1, B_2\},\ C = \max\{C_1, C_2\}. \tag{2.16}$$

Clearly, by (H1) and (H2), F is well defined and continuous. Let $(w_1, w_2) \in \overline{U}$. Then $\|(w_1, w_2)\|_{X\times Y} \le R$, and

$$
\begin{aligned}
&\|F_1 w_2\|_X \\
&= \sup_{t\in[0,T]} \Big|\eta(t) + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} f\Big(s, w_2(s), \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} w_2(\tau)d\tau\Big)ds\Big| \\
&\le \sup_{t\in[0,T]} \Big(|\eta(t)| + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} \Big|f\Big(s, w_2(s), \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} w_2(\tau)d\tau\Big)\Big|ds\Big) \\
&\le \sup_{t\in[0,T]} \Big(|\eta(t)| + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} \Big(a_1(s) + a_2(s)|w_2(s)| \\
&\qquad + a_3(s)\int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} |w_2(\tau)|d\tau\Big)ds\Big) \\
&\le \sup_{t\in[0,T]} \Big(|\eta(t)| + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} a_1(s)ds\Big) \\
&\quad + \sup_{t\in[0,T]} \Big(\int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} \Big(a_2(s) + a_3(s)\int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} d\tau\Big)ds\Big)\|w_2\|_Y \\
&\le \sup_{t\in[0,T]} \Big(|\eta(t)| + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} a_1(s)ds\Big) \\
&\quad + \sup_{t\in[0,T]} \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} \Big(a_2(s) + \frac{s^{n-\beta}}{\Gamma(n-\beta+1)} a_3(s)\Big)ds\|w_2\|_Y \\
&= C_1 + B_1\|w_2\|_Y \le C + BR = R.
\end{aligned}
\tag{2.17}
$$

Similarly, we have

$$
\|F_2 w_1\|_Y \le C_2 + B_2\|w_1\|_X \le C + BR = R. \tag{2.18}
$$

Therefore, $\|F(w_1, w_2)\|_{X\times Y} \le R$, which implies that $F(w_1, w_2) \in \overline{U}$. In order to show that F is completely continuous (continuous and compact), put

$$
\begin{aligned}
M_f &= \max_{t\in[0,T]} \Big|f\Big(t, w_2(t), \int_0^t \frac{(t-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} w_2(\tau)d\tau\Big)\Big|, \\
M_g &= \max_{t\in[0,T]} \Big|g\Big(t, w_1(t), \int_0^t \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} w_1(\tau)d\tau\Big)\Big|.
\end{aligned}
\tag{2.19}
$$

For $(w_1, w_2) \in U$ and $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, we obtain

$$
\begin{aligned}
&|F_1 w_2(t_2) - F_1 w_2(t_1)| \\
&\quad = \Big|\eta(t_2) - \eta(t_1) + \int_0^{t_2} \frac{(t_2-s)^{\rho-n-1}}{\Gamma(\rho-n)} f\Big(s, w_2(s), \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} w_2(\tau)d\tau\Big) \\
&\qquad - \int_0^{t_1} \frac{(t_1-s)^{\rho-n-1}}{\Gamma(\rho-n)} f\Big(s, w_2(s), \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} w_2(\tau)d\tau\Big)ds\Big| \\
&\quad \le |\eta(t_2) - \eta(t_1)| + M_f\Big|\int_0^{t_2} \frac{(t_2-s)^{\rho-n-1}}{\Gamma(\rho-n)} ds - \int_0^{t_1} \frac{(t_1-s)^{\rho-n-1}}{\Gamma(\rho-n)} ds\Big| \\
&\quad \le |\eta(t_2) - \eta(t_1)| + \frac{M_f}{\Gamma(\rho-n+1)} |t_2^{\rho-n} - t_1^{\rho-n}|,
\end{aligned}
\tag{2.20}
$$

and, in a similar manner,

$$|F_2w_1(t_2) - F_2w_1(t_1)| \le |\xi(t_2) - \xi(t_1)| + \frac{M_g}{\Gamma(\sigma - n + 1)}|t_2^{\sigma-n} - t_1^{\sigma-n}|. \tag{2.21}$$

It follows from the uniform continuity of functions $t^k, t^{\rho-n}$ and $t^{\sigma-n}$ on $[0,T]$ that FU is an equicontinuous set. Moreover, it is uniformly bounded as $FU \subset U$. Hence, F is a completely continuous mapping.

Now to consider the following eigenvalue problem

$$(w_1, w_2) = \lambda F(w_1, w_2) = (\lambda F_1 w_2, \lambda F_2 w_1),\ \lambda \in (0,1). \tag{2.22}$$

Assume that (w_1, w_2) is a solution of (2.22) for $\lambda \in (0,1)$. Then,

$$\begin{aligned}
&\|w_1\|_X \\
&= \sup_{t\in[0,T]} |\lambda F_1 w_2(t)| \\
&=\lambda \sup_{t\in[0,T]} \Big|\eta(t) + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} f\Big(s, w_2(s), \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} w_2(\tau)d\tau\Big)ds\Big| \\
&\le\lambda \sup_{t\in[0,T]} \Big(|\eta(t)| + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} \Big|f\Big(s, w_2(s), \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} w_2(\tau)d\tau\Big)\Big|ds\Big) \\
&\le\lambda(C + B\|w_2\|_Y),
\end{aligned} \tag{2.23}$$

and, similarly,

$$\|w_2\|_Y = \sup_{t\in[0,T]} |\lambda F_2 w_1(t)| \le \lambda(C + B\|w_1\|_X). \tag{2.24}$$

(2.23) and (2.24) guarantee that $(w_1, w_2) \notin \partial U$. Therefore, by Theorem 1.5, there exists a fixed point (w_{10}, w_{20}) in $\overline{U}$ such that $\|(w_{10}, w_{20})\|_{X\times Y} \le R$, which completes the proof. $\square$

It follows from Lemma 2.1 and Theorem 2.3 that the solution (u_0, v_0) of (1.1)-(1.2) is given by

$$\begin{aligned}
u_0(t) &= \sum_{k=0}^{n-1} \frac{t^k}{k!}\eta_k + \int_0^t \frac{(t-s)^{n-1}}{\Gamma(n)} w_{10}(s)ds, \\
v_0(t) &= \sum_{k=0}^{n-1} \frac{t^k}{k!}\xi_k + \int_0^t \frac{(t-s)^{n-1}}{\Gamma(n)} w_{20}(s)ds,
\end{aligned} \tag{2.25}$$

where

$$\begin{aligned}
w_{10}(t) = &\sum_{i=0}^{m-n-1} \frac{t^i}{i!}\eta_{n+i} \\
&+ \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} f\Big(s, w_{20}(s), \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} w_{20}(\tau)d\tau\Big)ds, \\
w_{20}(t) = &\sum_{i=0}^{m-n-1} \frac{t^i}{i!}\xi_{n+i} \\
&+ \int_0^t \frac{(t-s)^{\sigma-n-1}}{\Gamma(\sigma-n)} g\Big(s, w_{10}(s), \int_0^s \frac{(s-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} w_{10}(\tau)d\tau\Big)ds.
\end{aligned} \tag{2.26}$$

Theorem 2.4. *Assume (H1), (H2) and (H4) hold, and*

$$\begin{aligned} D_1 &= \sup_{t\in[0,T]} \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)}\Big(l_1(s)+\frac{s^{n-\beta}}{\Gamma(n-\beta+1)}l_2(s)\Big)ds<1,\\ D_2 &= \sup_{t\in[0,T]} \int_0^t \frac{(t-s)^{\sigma-n-1}}{\Gamma(\sigma-n)}\Big(l_3(s)+\frac{s^{n-\alpha}}{\Gamma(n-\alpha+1)}l_4(s)\Big)ds<1,\\ 0&<\sup_{t\in[0,T]}\Big(\int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)}|f(s,0,0)|ds\Big)<+\infty,\\ 0&<\sup_{t\in[0,T]}\Big(\int_0^t \frac{(t-s)^{\sigma-n-1}}{\Gamma(\sigma-n)}|g(s,0,0)|ds\Big)<+\infty \end{aligned} \tag{2.27}$$

Then the system of integral equations (2.1)-(2.4) has a unique solution.

Proof. Define the mapping F and the ball U as those in the proof of Theorem 2.3, where

$$R=\frac{1}{1-D_1}\sup_{t\in[0,T]}\Big(\int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)}|f(s,0,0)|ds\Big). \tag{2.28}$$

Then F is well defined and continuous. For $(w_1,w_2)\in\overline{U}$, we obtain

$$\begin{aligned} \|F_1w_2\|_X \le& \|F_1w_2-F_10\|_X+\|F_10\|_X\\ \le& \sup_{t\in[0,T]} \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)}\Big(l_1(s)+\frac{s^{n-\beta}}{\Gamma(n-\beta+1)}l_2(s)\Big)ds\|w_2\|_Y\\ &+\sup_{t\in[0,T]}\Big(\int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)}|f(s,0,0)|ds\Big)\\ \le& D_1R+(1-D_1)R\le R. \end{aligned} \tag{2.29}$$

Similarly, $\|F_2w_1\|_Y\le R$. Therefore, $F\overline{U}\subset\overline{U}$.

For $(w_1,w_2),(w_1',w_2')\in\overline{U}$, we have

$$\begin{aligned} &\|F_1w_2-F_1w_2'\|_X\\ &\le \sup_{t\in[0,T]}|F_1w_2(t)-F_1w_2'(t)|\\ &\le \sup_{t\in[0,T]} \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)}\Big(l_1(s)+\frac{s^{n-\beta}}{\Gamma(n-\beta+1)}l_2(s)\Big)ds\|w_2-w_2'\|_Y\\ &=D_1\|w_2-w_2'\|_Y, \end{aligned} \tag{2.30}$$

and, similarly,

$$\|F_2w_1-F_2w_1'\|_X\le D_2\|w_1-w_1'\|_X. \tag{2.31}$$

Noting that $D_1<1, D_2<1$, F is a contractive mapping. It follows from Banach contraction principle that F has a unique fixed point $(w_{10}',w_{20}')\in\overline{U}$, which is a solution of integral equations (2.1)-(2.4). This completes the proof. □

3. Example

Consider the following coupled system of fractional differential equations:

$$\begin{aligned} {}^cD^{11/5}u(t) &= \frac{t}{2} + \frac{t}{3}v''(t) + \frac{t^{4/5}}{4}\ {}^cD^{9/5}v(t),\ 0 < t \le 1,\\ u(0) &= 0,\quad u'(0) = 1,\quad u''(0) = 2,\\ {}^cD^{11/4}v(t) &= t + \frac{1}{2}u''(t) + \frac{t^{-3/4}}{3}\ {}^cD^{5/4}u(t),\ 0 < t \le 1,\\ v(0) &= 3,\quad v'(0) = 4,\quad v''(0) = 5. \end{aligned} \tag{3.1}$$

Here $T = 1, n = 2, m = 3, \rho = 11/5, \sigma = 11/4, \beta = 9/5, \alpha = 5/4, \eta_1 = 0, \eta_2 = 1, \eta_3 = 2, \xi_1 = 3, \xi_2 = 4$, and $\xi_3 = 5$. Obviously, the hypotheses (H1)-(H3) are satisfied with $a_1(t) = t/2, a_2(t) = t/3, a_3(t) = t^{4/5}/4, b_1(t) = t, b_2(t) = 1/2, b_3(t) = t^{-3/4}/3$. In this case

$$\begin{aligned} B_1 =& \frac{1}{\Gamma(1/5)} \sup_{t\in[0,1]} \int_0^t (t-s)^{-4/5}\Big(\frac{s}{3} + \frac{s}{4\Gamma(6/5)}\Big)ds\\ =& \frac{1}{\Gamma(1/5)}\Big(\frac{1}{3} + \frac{1}{4\Gamma(6/5)}\Big)\frac{15}{4} < 1,\\ B_2 =& \frac{1}{\Gamma(3/4)} \sup_{t\in[0,1]} \int_0^t (t-s)^{-1/4}\Big(\frac{1}{2} + \frac{1}{3\Gamma(7/4)}\Big)ds\\ =& \frac{1}{\Gamma(3/4)}\Big(\frac{1}{2} + \frac{1}{3\Gamma(7/4)}\Big)\frac{4}{3} < 1,\\ 0 < C_1 =& \sup_{t\in[0,1]} \Big(2 + \frac{1}{\Gamma(1/5)}\int_0^t (t-s)^{-4/5}\cdot\frac{s}{2}ds\Big)\\ =& 2 + \frac{1}{\Gamma(1/5)}\cdot\frac{15}{8} < +\infty,\\ 0 < C_2 =& \sup_{t\in[0,1]} \Big(5 + \frac{1}{\Gamma(3/4)}\int_0^t (t-s)^{-1/4}\cdot s ds\Big)\\ =& 5 + \frac{1}{\Gamma(3/4)}\cdot\frac{-8}{3} < +\infty. \end{aligned} \tag{3.2}$$

Thus, all the conditions of Theorem 2.3 are satisfied, and there exists a solution of system (3.1).

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School of Sciences, Liaoning Shihua University, Postal 113001, Fushun, Liaoning, China

E-mail address: guozy@163.com

E-mail address: min_liu@yeah.net