

**GROWTH AND FIXED-POINTS OF MEROMORPHIC
SOLUTIONS OF HIGHER-ORDER NONHOMOGENEOUS
LINEAR DIFFERENTIAL EQUATIONS**

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ABSTRACT. In this paper, we investigate the growth and fixed points of meromorphic solutions of higher order nonhomogeneous linear differential equations with meromorphic coefficients and their derivatives. Our results extend greatly the previous results due to J. Wang and I.Laine, B. Belaïdi and A. Farissi.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notation of the Nevanlinna value distribution theory of meromorphic functions (see [21, 25]). The term “meromorphic function” will mean meromorphic in the whole complex plane \mathbb{C} . In addition, we will use notations $\rho(f)$ to denote the order of growth of a meromorphic function $f(z)$, $\lambda(f)$ to denote the exponents of convergence of the zero-sequence of a meromorphic function $f(z)$, $\bar{\lambda}(f)$ to denote the exponents of convergence of the sequence of distinct zeros of $f(z)$.

In order to give some estimates of fixed points, we recall the following definitions(see [4, 14]).

Definition 1.1. Let $z_1, z_2, \dots, (|z_j| = r_j, 0 \leq r_1 \leq r_2 \leq \dots)$ be the sequence of distinct fixed points of transcendental meromorphic function f . Then $\bar{\tau}(f)$, the exponent of convergence of the sequence of distinct fixed points of f , is defined by

$$\bar{\tau}(f) = \inf\{\tau > 0 \mid \sum_{j=1}^{\infty} |z_j|^{-\tau} < +\infty\}.$$

It is evident that $\bar{\tau}(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log \bar{N}(r, \frac{1}{f-z})}{\log r}$ and $\bar{\tau}(f) = \bar{\lambda}(f - z)$.

For the second order linear differential equation

$$f'' + e^{-z} f' + B(z) f = 0, \tag{1.1}$$

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where $B(z)$ is an entire function of finite order, it is well known that each solution f of (1.1) is an entire function. If f_1 and f_2 are any two linearly independent solutions of (1.1), then at least one of f_1, f_2 must have infinite order ([20]). Hence, “most” solutions of (1.1) will have infinite order.

Thus a natural question is: what condition on $B(z)$ will guarantee that every solution $f \neq 0$ of (1.1) will have infinite order? Frei, Ozawa, Amemiya and Langley, and Gundersen studied the question. For the case that $B(z)$ is a transcendental entire function, Gundersen [10] proved that if $\rho(B) \neq 1$, then for every solution $f \neq 0$ of (1.1) has infinite order.

For the above question, there are many results for second order linear differential equations (see for example [1, 3, 7, 9, 12, 22]). In 2002, Chen considered the problem and obtained the following result in [3].

Theorem A. *Let a, b be nonzero complex numbers and $a \neq b$, $Q(z) \neq 0$ be a nonconstant polynomial or $Q(z) = h(z)e^{bz}$, where $h(z)$ is a nonzero polynomial. Then every solution $f \neq 0$ of the equation*

$$f'' + e^{bz}f' + Q(z)f = 0$$

has infinite order.

In 2005, Chen[5] investigated the more general equation with meromorphic coefficients, and obtained the following result.

Theorem B. *Let $A_j(z) (\neq 0) (j = 0, 1)$ be meromorphic functions with $\sigma(A_j) < 1$, a, b be nonzero complex numbers and $\arg a \neq \arg b$ or $a = cb (0 < c < 1)$. Then every solution $f \neq 0$ of the equation*

$$f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = 0 \quad (1.2)$$

has infinite order.

In 2009, the author and H.X. Yi [24] improved the above result and obtained the following:

Theorem C. *Suppose that $A_j \neq 0 (j = 0, 1, \dots, k-1)$ be meromorphic functions with $\sigma(A_j) < 1 (j = 0, 1, \dots, k-1)$. Let a_0, a_1, \dots, a_{k-1} be nonzero complex constants such that for (i) $\arg a_j = \arg a_0$ and $a_j = c_j a_0 (0 < c_j < 1)$ or (ii) $\arg a_j \neq \arg a_0 (j = 0, 1, \dots, k-1)$. Then for $k \geq 2$, every transcendental meromorphic solution $f (\neq 0)$ of the equation*

$$f^{(k)} + A_{k-1}e^{a_{k-1}z}f^{(k-1)} + \dots + A_1e^{a_1z}f' + A_0e^{a_0z}f = 0. \quad (1.3)$$

has infinite order.

Consider the second-order nonhomogeneous linear differential equation

$$f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = F, \quad (1.4)$$

where a, b are complex constants and $A_j(z) \neq 0 (j = 0, 1)$ are entire functions with $\max\{\rho(A_j) (j = 0, 1), \rho(F) < 1\}$. In [16], J. Wang and I. Laine have investigated the growth of solutions of (1.4) and obtained the following.

Theorem D. *Let $A_j(z) \neq 0 (j = 0, 1)$ and $F(z)$ be entire functions with $\max\{\rho(A_j) (j = 0, 1), \rho(F)\} < 1$, and let a, b be complex constants that satisfy $ab \neq 0$ and $a \neq b$. Then every nontrivial solution f of equation (1.4) is of infinite order.*

Remark. *Recently, Belaidi and Farissi[2] proved Theorem D by a differential method and pointed out the solution f of equation (1.4) satisfies $\lambda(f) = \lambda(f) = \rho(f) = \infty$.*

In this paper, we will consider the higher nonhomogeneous linear differential equation

$$f^{(k)} + A_{k-1}e^{a_{k-1}z}f^{(k-1)} + \cdots + A_1e^{a_1z}f' + A_0e^{a_0z}f = F \quad (1.5)$$

where $A_j \not\equiv 0 (j = 0, 1, \dots, k-1)$ and $F \not\equiv 0$ be meromorphic functions with $\max\{\rho(A_j)(j = 0, 1, \dots, k-1), \rho(F)\} < 1$. We first prove the following result.

Theorem 1.1. *Suppose that $A_j \not\equiv 0 (j = 0, 1, \dots, k-1)$ and $F \not\equiv 0$ be meromorphic functions with $\max\{\rho(A_j)(j = 0, 1, \dots, k-1), \rho(F)\} < 1$. Let a_0, a_1, \dots, a_{k-1} be nonzero complex constants such that for (i) $\arg a_j = \arg a_0$ and $a_j = c_j a_0$ ($0 < c_j < 1$) or (ii) $\arg a_j \neq \arg a_0$ ($j = 0, 1, \dots, k-1$). Then for $k \geq 2$, every transcendental meromorphic solution $f (\not\equiv 0)$ of the equation (1.5) has infinite order and satisfies $\bar{\lambda}(f) = \lambda(f) = \rho(f) = \infty$.*

Remark. In [16], Wang and Laine consider the case of $A_j \not\equiv 0 (j = 0, 1, \dots, k-1)$ and $F \not\equiv 0$ be entire functions. Obviously, Theorem 1.1 improves Theorem D and Theorem 1.3 in [16] greatly. In fact, if $a_j z (j = 0, 1, \dots, k-1)$ is replace by $P_j(z) = a_{jn}z^n + \cdots + a_{j0} (j = 0, 1, \dots, k-1)$ and $\rho(F) < 1$ is replace by $\rho(F) < n$, we can obtain the similar result by the same method.

Since the beginning of the last four decades, a substantial number of research articles have been written to describe the fixed points of general transcendental meromorphic functions(see [27]). However, there are few studies on the fixed points of solutions of the general differential equation. In [4], Z. X. Chen first studied the problems on the fixed points of solutions of second order linear differential equations with entire coefficients. The author and Yi[24] extended some results in [5] to the case of higher order homogeneous linear differential equations with meromorphic coefficients.

The second main purpose of this paper is to study the fixed points of solution of the higher nonhomogeneous linear differential equation.

Theorem E. *Let $A_j(z), a_j, c_j$ satisfy the additional hypotheses of Theorem 1.1. If $f \not\equiv 0$ is any meromorphic solution of the equation (1.3), then f, f', f'' all have infinitely fixed points and satisfy*

$$\bar{\tau}(f) = \bar{\tau}(f') = \bar{\tau}(f'') = \infty.$$

Theorem 1.2. *Let $A_j(z), a_j, c_j$ satisfy the additional hypotheses of Theorem 1.1. If $f \not\equiv 0$ is any meromorphic solution of the equation (1.5), then f, f', f'' all have infinitely fixed points and satisfy*

$$\bar{\tau}(f) = \bar{\tau}(f') = \bar{\tau}(f'') = \infty.$$

2. LEMMAS

The linear measure of a set $E \subset [0, +\infty)$ is defined as $m(E) = \int_0^{+\infty} \chi_E(t) dt$. The logarithmic measure of a set $E \subset [1, +\infty)$ is defined by $lm(E) = \int_1^{+\infty} \chi_E(t)/t dt$, where $\chi_E(t)$ is the characteristic function of E . The upper and lower densities of E are

$$\overline{dens}E = \limsup_{r \rightarrow +\infty} \frac{m(E \cap [0, r])}{r}, \quad \underline{dens}E = \liminf_{r \rightarrow +\infty} \frac{m(E \cap [0, r])}{r}.$$

The following lemma, due to Gross [13], is important in the factorization and uniqueness theory of meromorphic functions, playing an important role in this paper as well.

Lemma 2.1 ([13, 26]). *Suppose that $f_1(z), f_2(z), \dots, f_n(z)$ ($n \geq 2$) are meromorphic functions and $g_1(z), g_2(z), \dots, g_n(z)$ are entire functions satisfying the following conditions:*

$$(i) \sum_{j=1}^n f_j(z)e^{g_j(z)} \equiv 0.$$

(ii) $g_j(z) - g_k(z)$ are not constants for $1 \leq j < k \leq n$.

(iii) For $1 \leq j \leq n$, $1 \leq h < k \leq n$, $T(r, f_j) = o\{T(r, e^{g_h - g_k})\}$ ($r \rightarrow \infty, r \notin E$).

Then $f_j(z) \equiv 0$ ($j = 1, 2, \dots, n$).

We only need the following special form in Lemma 2.

Lemma 2.2 ([24]). *Suppose that $f_1(z), f_2(z), \dots, f_n(z)$ ($n \geq 2$) are meromorphic functions and $g_1(z), g_2(z), \dots, g_n(z)$ are entire functions satisfying the following conditions:*

$$(i) \sum_{j=1}^n f_j(z)e^{g_j(z)} \equiv f_{n+1}.$$

(ii) If $1 \leq j \leq n+1$, $1 \leq k \leq n$, the order of f_j is less than the order of $e^{g_k(z)}$.

If $n \geq 2$, $1 \leq j \leq n+1$, $1 \leq h < k \leq n$, and the order of $f_j(z)$ is less than the order of $e^{g_h - g_k}$.

Then $f_j(z) \equiv 0$ ($j = 1, 2, \dots, n+1$).

Lemma 2.3 ([11]). *Let f be a transcendental meromorphic function of finite order σ . Let $\varepsilon > 0$ be a constant, and k and j be integers satisfying $k > j \geq 0$. Then the following two statements hold:*

(a) *There exists a set $E_1 \subset (1, \infty)$ which has finite logarithmic measure, such that for all z satisfying $|z| \notin E_1 \cup [0, 1]$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}. \quad (2.1)$$

(b) *There exists a set $E_2 \subset [0, 2\pi)$ which has linear measure zero, such that if $\theta \in [0, 2\pi) - E_2$, then there is a constant $R = R(\theta) > 0$ such that (2.1) holds for all z satisfying $\arg z = \theta$ and $R \leq |z|$.*

Lemma 2.4. *Let $f(z) = g(z)/d(z)$, where $g(z)$ is transcendental entire, and let $d(z)$ be the canonical product (or polynomial) formed with the non-zero poles of $f(z)$. Then we have*

$$f^{(n)} = \frac{1}{d} [g^{(i)} + B_{i,i-1}g^{(k-1)} + \dots + B_{i,1}g' + B_{i,0}g],$$

$$\frac{f^{(n)}}{f} = \left[\frac{g^{(i)}}{g} + B_{i,i-1} \frac{g^{(k-1)}}{g} + \dots + B_{i,1} \frac{g'}{g} + B_{i,0} \right],$$

where $B_{i,j}$ are defined as a sum of a finite number of terms of the type

$$\sum_{(j_1 \dots j_i)} C_{jj_1 \dots j_i} \left(\frac{d'}{d}\right)^{j_1} \dots \left(\frac{d^{(i)}}{d}\right)^{j_i},$$

$C_{jj_1 \dots j_i}$ are constants, and $j + j_1 + 2j_2 + \dots + ij_i = n$.

Using mathematical induction, we can easily prove the lemma.

Lemma 2.5 ([3]). *Let $g(z)$ be a meromorphic function with $\sigma(g) = \beta < \infty$. Then for any given $\varepsilon > 0$, there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi \in [0, 2\pi) \setminus E$, then there is a constant $R = R(\psi) > 1$ such that, for all z satisfying $\arg z = \psi$ and $|z| = r > R$, we have*

$$\exp\{-r^{\beta+\varepsilon}\} \leq |g(z)| \leq \exp\{r^{\beta+\varepsilon}\}.$$

Lemma 2.6 ([15]). *Consider $g(z) = A(z)e^{az}$ where $A(z) (\neq 0)$ is a meromorphic function with $\sigma(A) = \alpha < 1$, a is a complex constant, $a = |a|e^{i\varphi}$ ($\varphi \in [0, 2\pi)$). Set $E_0 = \{\theta \in [0, 2\pi) : \cos(\varphi + \theta) = 0\}$, then E_0 is a finite set. Then for any given ε ($0 < \varepsilon < 1 - \alpha$), there is a set $E_1 \in [0, 2\pi)$ that has linear measure zero, if $z = re^{i\theta}$, $\theta \in (E_0 \cup E_1)$, then we have when r is sufficiently large:*

(i) *If $\cos(\varphi + \theta) > 0$, then*

$$\exp\{(1 - \varepsilon)r\delta(az, \theta)\} \leq |g(z)| \leq \exp\{(1 + \varepsilon)r\delta(az, \theta)\};$$

(ii) *If $\cos(\varphi + \theta) < 0$, then*

$$\exp\{(1 + \varepsilon)r\delta(az, \theta)\} \leq |g(z)| \leq \exp\{(1 - \varepsilon)r\delta(az, \theta)\};$$

where $\delta(az, \theta) = |a| \cos(\varphi + \theta)$.

Lemma 2.7 ([8]). *Let $A_0, A_1, \dots, A_{k-1}, F \neq 0$ are finite order meromorphic function. If $f(z)$ is an infinite order meromorphic solution of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \dots + A_1f' + A_0f = F,$$

then f satisfies $\lambda(f) = \bar{\lambda}(f) = \sigma(f) = \infty$.

Lemma 2.8. *Suppose that $A_j \neq 0$ ($j = 0, 1, \dots, k-1$) and $F \neq 0$ be meromorphic functions with $\max\{\rho(A_j) (j = 0, 1, \dots, k-1), \rho(F)\} < 1$. Let a_0, a_1, \dots, a_{k-1} be nonzero complex constants such that for (i) $\arg a_j = \arg a_0$ and $a_j = c_j a_0$ ($0 < c_j < 1$) or (ii) $\arg a_j \neq \arg a_0$ ($j = 0, 1, \dots, k-1$). We denote*

$$L_f = f^{(k)} + A_{k-1}e^{a_{k-1}z}f^{(k-1)} + \dots + A_1e^{a_1z}f' + A_0e^{a_0z}f \quad (2.2)$$

If $f \neq 0$ is a finite-order entire function of (1.5), then $\rho(L_f) \geq 1$.

Proof. We suppose that $\rho(L_f) < 1$ and then we obtain a contradiction.

(i) If $\rho(f) < 1$, Then $\rho(A_j f^{(j)}) < 1$ ($j = 1, 2, \dots, k$). Equation (7) has the form

$$A_{k-1}f^{(k-1)}e^{a_{k-1}z} + \dots + A_1f'e^{a_1z} + A_0f e^{a_0z} = L_f - f^{(k)}.$$

By Lemma 2.7, we can obtain a contradiction.

(ii) If $\rho(f) \geq 1$, we rewrite (2.2) into

$$\frac{L_f}{f} = \frac{f^{(k)}}{f} + A_{k-1}e^{a_{k-1}z}\frac{f^{(k-1)}}{f} + \dots + A_1e^{a_1z}\frac{f'}{f} + A_0e^{a_0z} \quad (2.3)$$

From the equation (1.5), we know that the poles of $f(z)$ can occur only at the poles of $A_j(z)$ ($j = 0, 1, \dots, k-1$), $F(z)$. Let $f = g/d$, d be the canonical product formed with the nonzero poles of $f(z)$, with $\rho(d) = \beta \leq \alpha = \max\{\rho(A_j) : j =$

$0, 1, \dots, k-1, \rho(F)\} < 1$, g be an entire function and $1 \leq \rho(g) = \rho(f) = \rho < \infty$. Substituting $f = g/d$ into (2.3), by Lemma 2.4 we can get

$$\begin{aligned} & \frac{g^{(k)}}{g} + \frac{g^{(k-1)}}{g} [A_{k-1}e^{a_{k-1}z} + B_{k,k-1}] + \dots + \\ & \frac{g'}{g} [A_1e^{a_1z} + \sum_{i=2}^{k-1} A_i e^{a_i z} B_{i,1} + B_{k,1}] + [A_0e^{a_0z} + \sum_{i=1}^{k-1} A_i e^{a_i z} B_{i,1} + B_{k,0}] = \frac{dL_f}{g}. \end{aligned} \quad (2.4)$$

By Lemma 2.3, for any given $\varepsilon(0 < 3\varepsilon < \min\{1 - \alpha, \frac{1-c}{6}\}, c = \max\{c_j, 1 \leq j \leq k-1\})$, there exists a set $E \in [0, 2\pi)$ that has linear measure zero, such that if $\theta \in [0, 2\pi) \setminus E$, then there is a constant $R_0 = R_0(\theta) > 1$, such that for all z satisfying $\arg z = \theta$ and $|z| \geq R_0$, we have

$$\frac{g^{(j)}(z)}{g(z)} \leq |z|^{k(\rho-1+\varepsilon)}, \quad (j = 1, 2, \dots, k) \quad (2.5)$$

and

$$\frac{d^{(j)}(z)}{d(z)} \leq |z|^{k(\beta-1+\varepsilon)}, \quad (j = 1, 2, \dots, k). \quad (2.6)$$

Set $\rho(L_f) = \beta < 1$. Then, for any given $\varepsilon(0 < \varepsilon < n - \beta)$, we have for sufficiently large r

$$|L_f| \leq \exp\{r^{\beta+\varepsilon}\}. \quad (2.7)$$

From Wiman-Valiron theory(see [23]), we know that there exists a set E with finite logarithmic measure such that for a point z satisfying $|z| = r \notin E$ and $|g(z)| = M(r, g)$, we have

$$v_g(r) < [\log \mu_g(r)]^2, \quad (2.8)$$

where $\mu_g(r)$ is a maximum term of g . By Cauchy's inequality, we have $\mu_g(r) \leq M(r, g)$. This and (2.8) yield

$$v_g(r) < [\log |g(r)|]^2, \quad (r \notin E).$$

By f is transcendental function we know that $v_g(r) \rightarrow \infty$. Then for sufficiently large $|z| = r$ we have $|g(z)| = M(r, g) \geq 1$, then

$$\left| \frac{dL_f}{g} \right| \leq |dL_f| \leq \exp\{r^{\beta+\varepsilon}\}. \quad (2.9)$$

Setting $z = re^{i\theta}$, then

$$\operatorname{Re}\{a_j z\} = \delta(a_j z, \theta)r, \quad \operatorname{Re}\{a_0 z\} = \delta(a_0 z, \theta)r. \quad (2.10)$$

Case 1 Suppose first that $\arg a_j \neq \arg a_0$ ($j = 1, 2, \dots, k-1$). In view of Lemma 2.6 and (2.10), it is easy to see for the above ε there is a ray $\arg z = \theta$ such that $\theta \in [0, 2\pi) \setminus (E_1 \cup E_2 \cup E_0)$ (where E_2 and E_0 are defined as in Lemma 2.6, $E_1 \cup E_2 \cup E_0$ is of linear measure zero) satisfying $\delta(a_j z, \theta) < 0$, $c_j \delta(a_0 z, \theta) > 0$, and for a sufficiently large r , we have

$$|A_0(re^{i\theta})e^{a_0 r e^{i\theta}} f(re^{i\theta})| \geq \exp\{(1 - \varepsilon)\delta(a_0 z, \theta)r\}, \quad (2.11)$$

$$|A_j(re^{i\theta})e^{a_j r e^{i\theta}}| \leq \exp\{(1 - \varepsilon)\delta(a_j z, \theta)r\} \quad (j = 1, \dots, k-1), \quad (2.12)$$

By (2.6), (2.11) and (2.12), we have

$$|A_{k-1}e^{a_{k-1}z} + B_{k,k-1}| \leq \exp\{(1 - \varepsilon)\delta(a_j z, \theta)r\} + M r^{k(\beta-1+\varepsilon)}, \dots, \quad (2.13)$$

$$\left| A_1 e^{a_1 z} + \sum_{i=2}^{k-1} A_i e^{a_i z} B_{i,1} + B_{k,1} \right| \leq \exp\{(1-\varepsilon)\delta(a_j z, \theta)r\} + M r^{k(\beta-1+\varepsilon)}, \quad (2.14)$$

and

$$\left| A_0 e^{a_0 z} + \sum_{i=1}^{k-1} A_i e^{a_i z} B_{i,1} + B_{k,0} \right| \geq \exp\{(1-\varepsilon)\delta(a_0 z, \theta)r\}(1-o(1)), \quad (2.15)$$

where $M > 0$ is a constant, it can be different in different occurrences.

By (2.4), (2.5) and (2.13)-(2.15), we have

$$\begin{aligned} \exp\{(1-\varepsilon)\delta(a_0 z, \theta)r\}(1-o(1)) &\leq \left| A_0 e^{a_0 z} + \sum_{i=1}^{k-1} A_i e^{a_i z} B_{i,1} + B_{k,0} \right| \\ &\leq \left| \frac{g^{(k)}(z)}{g(z)} \right| + \left| \frac{g^{(k-1)}(z)}{g(z)} (A_{k-1} e^{a_{k-1} z} + B_{k,k-1}) \right| + \dots \\ &\quad + \left| \frac{g'(z)}{g(z)} (A_1 e^{a_1 z} + \sum_{i=2}^{k-1} A_i e^{a_i z} B_{i,1} + B_{k,1}) \right| + \left| \frac{dL_f}{g} \right| \\ &\leq r^{k(\sigma-1+\varepsilon)} + r^{(k-1)(\sigma-1+\varepsilon)} [\exp\{(1-\varepsilon)\delta(a_j z, \theta)r_j\} + M r^{k(\beta-1+\varepsilon)}] + \dots \\ &\quad + r^{(\sigma-1+\varepsilon)} [\exp\{(1-\varepsilon)\delta(a_j z, \theta)r_j\} + M r^{k(\beta-1+\varepsilon)}] + \exp\{r^{\beta+\varepsilon}\} \\ &\leq r^M + \exp\{r^{\beta+\varepsilon}\}. \end{aligned}$$

This is a contradiction with $\beta + \varepsilon < 1$. Hence $\rho(L_f) \geq 1$.

Case 2 Suppose that $\arg a_j = \arg a_0$, and $a_j = c_j a_0 (0 < c_j < 1)$; then $\delta(a_j z, \theta) = c_j \delta(a_0 z, \theta)$, $\operatorname{Re}\{a_j z\} = c_j \operatorname{Re}\{a_0 z\}$. Using the same argument as above, we know that (2.5), (2.6) hold. Moreover, there is a ray $\arg z = \theta$ satisfying $\delta(a_j z, \theta) = c_j \delta(a_0 z, \theta) > 0$, then for a sufficiently large r , we have (2.11) and

$$\left| A_j (r e^{i\theta}) e^{a_j r e^{i\theta}} \right| \leq \exp\{(1+\varepsilon)c_j \delta r(a_0 z, \theta)\} \quad (j = 1, \dots, k-1), \quad (2.16)$$

By (2.6), (2.11) and (2.16), we have

$$\left| A_{k-1} e^{a_{k-1} z} + B_{k,k-1} \right| \leq \exp\{(1+\varepsilon)c_{k-1} \delta(a_0 z, \theta)r\}, \dots, \quad (2.17)$$

$$\left| A_1 e^{a_1 z} + \sum_{i=2}^{k-1} A_i e^{a_i z} B_{i,1} + B_{k,1} \right| \leq \exp\{(1+\varepsilon)c_i \delta(a_0 z, \theta)r\}, \quad (2.18)$$

and

$$\left| A_0 e^{a_0 z} + \sum_{i=1}^{k-1} A_i e^{a_i z} B_{i,1} + B_{k,0} \right| \geq \exp\{(1-\varepsilon)\delta(a_0 z, \theta)r\}(1-o(1)). \quad (2.19)$$

By (2.4), (2.5) and (2.17)-(2.19), we have

$$\begin{aligned}
\exp\{(1-\varepsilon)\delta(a_0z, \theta)r\}(1-o(1)) &\leq \left| A_0e^{a_0z} + \sum_{i=1}^{k-1} A_i e^{a_i z} B_{i,1} + B_{k,0} \right| \\
&\leq \left| \frac{g^{(k)}(z)}{g(z)} \right| + \left| \frac{g^{(k-1)}(z)}{g(z)} (A_{k-1}e^{a_{k-1}z} + B_{k,k-1}) \right| + \cdots \\
&\quad + \left| \frac{g'(z)}{g(z)} (A_1e^{a_1z} + \sum_{i=2}^{k-1} A_i e^{a_i z} B_{i,1} + B_{k,1}) \right| + \left| \frac{dL_f}{g} \right| \\
&\leq r^{k(\sigma-1+\varepsilon)} + r^{(k-1)(\sigma-1+\varepsilon)} \exp\{(1+\varepsilon)c_{k-1}\delta(a_0z, \theta)r\}(1+o(1)) + \cdots \\
&\quad + r^{(\sigma-1+\varepsilon)} \exp\{(1+\varepsilon)c_i\delta(a_0z, \theta)r\}(1+o(1)) + \exp\{r^{\beta+\varepsilon}\} \\
&\leq Mr^{k(\sigma-1+\varepsilon)} \exp\{(1+\varepsilon)c\delta(a_0z, \theta)r\}(1+o(1)).
\end{aligned}$$

From this and $3\varepsilon < \frac{1-c}{6}$, we get

$$\exp\left\{\frac{1-c}{2}r\delta(a_0z, \theta)\right\} \leq Mr^{k(\sigma-1+\varepsilon)}.$$

It is a contradiction. Hence $\rho(L_f) \geq 1$. The proof of Lemma 2.8 is completed. \square

3. PROOF OF THEOREM 1.1

Assume that $f(\not\equiv 0)$ is a meromorphic function of (1.5). We first prove that f is of infinite order. We suppose the contrary $\rho(f) < \infty$. By Lemma 2.8, we have $n \leq \rho(L_f) = \rho(F) < n$ and this is a contradiction. Hence every solution f of equation (1.5) is of infinite order. By Lemma 2.7, every solution f of equation (1.5) satisfies $\bar{\lambda}(f) = \lambda(f) = \rho(f) = \infty$. The proof of Theorem 1.1 is completed. \square

4. PROOF OF THEOREM 1.2

Assume $f(\not\equiv 0)$ is a meromorphic function of (1.5); then $\rho(f) = \infty$ by Theorem 1.1. Set $g_0(z) = f(z) - z$, then z is a fixed point of $f(z)$ if and only if $g_0(z) = 0$. $g_0(z)$ is a meromorphic function and $\rho(g_0) = \rho(f) = \infty$. Substituting $f = g_0 + z$ into (1.5), we have

$$g_0^{(k)} + A_{k-1}e^{a_{k-1}z}g_0^{(k-1)} + \cdots + A_1e^{a_1z}g_0' + A_0e^{a_0z}g_0 = -A_1e^{a_1z} - zA_0e^{a_0z} + F. \quad (4.1)$$

We can rewrite (4.1) as the following form:

$$g_0^{(k)} + h_{0,k-1}g_0^{(k-1)} + \cdots + h_{0,1}g_0' + h_{0,0}g_0 = -h_{0,1} - zh_{0,0} + F.$$

Obviously, $h_0 = -[h_{1,0} + zh_{0,0}] = -A_1e^{a_1z} - zA_0e^{a_0z} \not\equiv -F$. Otherwise, it contradicts with Lemma 2.2. Therefore, $-h_{0,1} - zh_{0,0} + F \not\equiv 0$. Here we just consider the meromorphic solutions of infinite order satisfying $g_0 = f - z$, by Lemma 2.7 we know that $\bar{\lambda}(g_0) = \bar{\tau}(f) = \infty$ holds.

Now we consider the fixed points of $f'(z)$.

Let $g_1(z) = f' - z$, then z is a fixed point of $f'(z)$ if and only if $g_1(z) = 0$. $g_1(z)$ is a meromorphic function and $\rho(g_1) = \rho(f') = \rho(f) = \infty$. Differentiating both

sides of the equation (1.5), we have

$$\begin{aligned} & f^{(k+1)} + A_{k-1}e^{a_{k-1}z}f^{(k)} + [(A_{k-1}e^{a_{k-1}z})' + A_{k-2}e^{a_{k-2}z}]f^{(k-1)} \\ & + \cdots + [(A_3e^{a_3z})' + A_2e^{a_2z}]f''' + [(A_2e^{a_2z})' + A_1e^{a_1z}]f'' \\ & + [(A_1e^{a_1z})' + A_0e^{a_0z}]f' + (A_0e^{a_0z})'f = F'. \end{aligned} \quad (4.2)$$

By (1.5), we have

$$f = -\frac{1}{A_0e^{a_0z}} [f^{(k)} + A_{k-1}e^{a_{k-1}z}f^{(k-1)} + \cdots + A_2e^{a_2z}f'' + A_1e^{a_1z}f' - F]. \quad (4.3)$$

Substituting (4.3) into (4.2), we have

$$\begin{aligned} & f^{(k+1)} + [A_{k-1}e^{a_{k-1}z} - \frac{(A_0e^{a_0z})'}{A_0e^{a_0z}}]f^{(k)} + [(A_{k-1}e^{a_{k-1}z})' + A_{k-2}e^{a_{k-2}z} - \\ & \frac{(A_0e^{a_0z})'}{A_0e^{a_0z}}]A_{k-1}e^{a_{k-1}z}f^{(k-1)} + \cdots + [(A_3e^{a_3z})' + A_2e^{a_2z} - \frac{(A_0e^{a_0z})'}{A_0e^{a_0z}}]A_3e^{a_3z}f''' \\ & + [(A_2e^{a_2z})' + A_1e^{a_1z} - \frac{(A_0e^{a_0z})'}{A_0e^{a_0z}}]A_2e^{a_2z}f'' \\ & + [(A_1e^{a_1z})' + A_0e^{a_0z} - \frac{(A_0e^{a_0z})'}{A_0e^{a_0z}}]A_1e^{a_1z}f' = -\frac{(A_0e^{a_0z})'}{A_0e^{a_0z}}F + F'. \end{aligned} \quad (4.4)$$

We can denote the equation by the following form:

$$f^{(k+1)} + h_{1,k-1}f^{(k)} + h_{1,k-2}f^{(k-1)} + \cdots + h_{1,2}f''' + h_{1,1}f'' + h_{1,0}f' = H_1, \quad (4.5)$$

where $h_{1,j}$ ($j = 0, 1, \dots, k-1$) is the meromorphic functions defined by the equation (4.4) and $H_1 = -\frac{(A_0e^{a_0z})'}{A_0e^{a_0z}}F + F' = -(\frac{A_0'}{A_0} + a_0)F + F'$ with $\rho(H_1) < 1$. Substituting $f' = g_1 + z$, $f'' = g_1' + 1$, $f^{(j+1)} = g_1^{(j)}$, ($2 \leq j \leq k$) into (4.5), we get

$$g_1^{(k)} + h_{1,k-1}g_1^{(k-1)} + \cdots + h_{1,1}g_1' + h_{1,0}g_1 = h_1, \quad (4.6)$$

where

$$\begin{aligned} h_1 &= -(h_{1,1} + zh_{1,0}) + H_1 \\ &= -\left[(A_2' + a_2A_2 - \frac{A_0'}{A_0}A_2 + a_0A_2)e^{a_2z} + (A_1 + zA_1' + za_1A_1 - \right. \\ & \quad \left. zA_1\frac{A_0'}{A_0} - za_0A_1)e^{a_1z} + zA_0e^{a_0z} \right] + H_1. \end{aligned}$$

We claim $h_1 \neq 0$. Since a_2, a_1, a_0 are different each other, if $h_1 \equiv 0$ by Lemma 2.2, we conclude that $A_0 \equiv 0$, a contradiction. Therefore, $h_1 \neq 0$. Applying Lemma 2.7 to (4.6) above, we obtain $\bar{\lambda}(g_1) = \bar{\lambda}(f' - z) = \bar{\tau}(f') = \sigma(g_1) = \sigma(f) = \infty$.

Now we prove that $\bar{\tau}(f'') = \bar{\lambda}(f'' - z) = \infty$. Set $g_2(z) = f'' - z$. Using the same argument as above, we need to prove only that $\bar{\lambda}(g_2) = \infty$.

We differentiate both sides of (4.5), and obtain

$$\begin{aligned} & f^{(k+2)} + h_{1,k-1}f^{(k+1)} + [h_{1,k-1}' + h_{1,k-2}]f^{(k)} + \cdots \\ & + [h_{1,1}' + h_{1,0}]f'' + h_{1,0}'f' = H_1'. \end{aligned} \quad (4.7)$$

By (4.5) and (4.7), we have

$$\begin{aligned} & f^{(k+2)} + [h_{1,k-1} - \frac{h'_{1,0}}{h_{1,0}}]f^{(k+1)} + [h'_{1,k-1} + h_{1,k-2} - \frac{h'_{1,0}}{h_{1,0}}h_{1,k-1}]f^{(k)} + \cdots \\ & + [h'_{1,2} + h_{1,1} - \frac{h'_{1,0}}{h_{1,0}}h_{1,2}]f''' + [h'_{1,1} + h_{1,0} - \frac{h'_{1,0}}{h_{1,0}}h_{1,1}]f'' = \frac{h'_{1,0}}{h_{1,0}}H_1 + H'_1. \end{aligned} \quad (4.8)$$

We can write (4.7) to the following form

$$f^{(k+2)} + h_{2,k-1}f^{(k+1)} + h_{2,k-2}f^{(k)} + \cdots + h_{2,1}f''' + h_{2,0}f'' = H_2, \quad (4.9)$$

where $h_{2,j}$ are meromorphic functions with $\rho(h_{2,j}) \leq 1 (j = 0, 1, \dots, k-1)$, and

$$\begin{aligned} h_{2,1} &= h'_{1,2} + h_{1,1} - \frac{h'_{1,0}}{h_{1,0}}h_{1,2}, \\ h_{2,0} &= h'_{1,1} + h_{1,0} - \frac{h'_{1,0}}{h_{1,0}}h_{1,1}, \end{aligned} \quad (4.10)$$

where

$$\begin{aligned} h_{1,2} &= (A_3e^{a_1z})' + A_2e^{a_0z} - \frac{(A_0e^{a_0z})'}{A_0e^{a_0z}}A_3e^{a_3z}, \\ h_{1,1} &= (A_2e^{a_1z})' + A_1e^{a_0z} - \frac{(A_0e^{a_0z})'}{A_0e^{a_0z}}A_2e^{a_2z}, \\ h_{1,0} &= (A_1e^{a_1z})' + A_0e^{a_0z} - \frac{(A_0e^{a_0z})'}{A_0e^{a_0z}}A_1e^{a_1z}, \\ &= \left[A'_1 + (a_1 - a_0 - \frac{A'_0}{A_0})A_1 \right] e^{a_1z} + A_0e^{a_0z} \\ H_2 &= \frac{h'_{1,0}}{h_{1,0}}H_1 + H'_1. \end{aligned} \quad (4.11)$$

Substituting $f'' = g_2 + z$, $f''' = g'_2 + 1$, $f^{(j+2)} = g_2^{(j)}$, ($2 \leq j \leq k$) into (4.9), we get

$$g_2^{(k)} + h_{2,k-1}g_2^{(k-1)} + \cdots + h_{2,1}g'_2 + h_{2,0}g_2 = -(h_{2,1} + zh_{2,0}) + H_2. \quad (4.12)$$

We claim $H_2 - h_{2,1} - zh_{2,0} \neq 0$. By (4.10), (4.11) we know $H_2 - h_{2,1} - zh_{2,0}$ can write into the following form

$$h_2 = H_2 - h_{2,1} - zh_{2,0} = \frac{h'_{1,0}H_1 + H'_1h_{1,0} - h_{2,1}h_{1,0} - zh_{2,0}h_{1,0}}{h_{1,0}} = \frac{\varphi(z)}{h_{1,0}},$$

where $\varphi(z) = -zh_{1,0}^2 - zh'_{1,0}h_{1,1} - zh_{1,0}h'_{1,1} - h'_{1,0}h_{1,2} - h_{1,1}h_{1,0} - h'_{1,2}h_{1,0} + h_{1,0}H'_1 + h'_{1,0}H_1 = -zA_0^2e^{2a_0z} + \sum_{\gamma \in \Lambda} D_\gamma e^{\gamma z}$, where D_γ are meromorphic functions

in A_1, A_2, A_3, F and their derivatives, whose order less than 1. The index set Λ denotes the sums of $a_i, a_j (0 \leq i, j \leq 3)$, except for $2a_0$. Obviously, the differences of every sum are not the constant which satisfies the condition (ii) and (iii) in Lemma 2.1. Similarly with the above, if $\varphi \equiv 0$, by Lemma 2.1, there must be $A_0 \equiv 0$, it is a contradiction. Then applying Lemma 2.2 to (4.12), we have $\bar{\lambda}(g_2) = \bar{\lambda}(f'' - z) = \bar{\tau}(f'') = \infty$.

This proves the theorem. \square

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