Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 4 Issue 1 (2012), Pages 104-116.

RANGE OF $\mathcal{D}(\mathbb{R})$ BY INTEGRAL TRANSFORMS ASSOCIATED TO THE BESSEL-STRUVE OPERATOR

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Abstract. In this paper, we establish an inversion theorem of the Weyl integral transform associated with the Bessel-Struve operator l_α , $\alpha > \frac{-1}{2}$. In the case of half integers, we give a characterization of the range of $\mathcal{D}(\mathbb{R})$ by Bessel-Struve transform and we prove a Schwartz-Paley-Wiener theorem on $\mathcal{E}'(\mathbb{R})$.

1. Introduction

In [8], Watson developed the discrete harmonic analysis associated with Bessel-Struve kernel

$$S_\lambda^\alpha(x) = j_\alpha(i\lambda x) - ih_\alpha(i\lambda x)$$

where j_α and h_α are respectively the normalized Bessel and Struve functions of index α. Those functions are given as follows :

$$j_\alpha(z) = 2^\alpha\Gamma(\alpha+1)z^{-\alpha}J_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^{+\infty}\frac{(-1)^n(z/2)^{2n}}{n!\Gamma(n+\alpha+1)}$$

and

$$h_\alpha(z) = 2^\alpha\Gamma(\alpha+1)z^{-\alpha}\mathbf{H}_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^{+\infty}\frac{(-1)^n(z/2)^{2n+1}}{\Gamma(n+\frac{3}{2})\Gamma(n+\alpha+\frac{3}{2})}$$

Watson considered "generalised Schlömilch series" which is a kind of Fourier series

$$\sum_{n=-\infty}^{+\infty} c_n(f)S_{-in}^\alpha(x)$$

where f is a suitable function and $c_n(f) \in \mathbb{C}$.
In this paper, we are interested with a kind of Fourier transform which was considered and studied by K. Trimèche in [4], called the Bessel-Struve transform, given

2000 *Mathematics Subject Classification.* 42A38, 44A05, 44A20, 46F12.

Key words and phrases. Bessel-Struve operator, Bessel-Struve transform, Intertwining operator, Weyl integral transform, Paley-Wiener theorem, Schwartz-Paley-Wiener theorem.

The authors are supported by the DGRST research project 04/UR/15-02.
Submitted May 24, 2011. Published December 7, 2011.

by

$$\mathcal{F}^{\alpha}_{BS}(f)(\lambda) = \int_{\mathbb{R}} f(x)\, S^{\alpha}_{-i\lambda}(x)\, |x|^{2\alpha+1}\, dx$$

K. Trimèche proved that this transform is related to the classical Fourier transform $\mathcal{F}$ by the relation

$$\forall\, f \in \mathcal{D}(\mathbb{R})\,, \qquad \mathcal{F}^{\alpha}_{BS}(f) = \mathcal{F} \circ W_{\alpha}(f)$$

where W_{α} is the Weyl integral transform given by

$$W_{\alpha} f(y) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})} \int_{|y|}^{+\infty} (x^2-y^2)^{\alpha-\frac{1}{2}}\, x\, f(sgn(y)x)\, dx\ , \quad y \in \mathbb{R}^{\star}$$

Furthermore, K.Trimèche [4], L. Kamoun and M. Sifi [1], looked to the Bessel-Struve operator

$$l_{\alpha} u(x) = \frac{d^2 u}{dx^2}(x) + \frac{2\alpha+1}{x}\left[\frac{du}{dx}(x) - \frac{du}{dx}(0)\right]$$

which has Bessel-Struve kernel as eigenfunction. They considered the Intertwining operator χ_{α} associated with Bessel-Struve operator on $\mathbb{R}$, given by

$$\chi_{\alpha}(f)(x) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} f(xt)\, dt\ , \quad f \in \mathcal{E}(\mathbb{R}).$$

It verifies the intertwining relation

$$l_{\alpha}\chi_{\alpha} = \chi_{\alpha}\frac{d^2}{dx^2}$$

and the duality relation with Weyl transform

$$\int_{\mathbb{R}} \chi_{\alpha} f(x)\, g(x) A(x)\, dx \ = \int_{\mathbb{R}} f(x)\, W_{\alpha} g(x) dx.$$

The outline of the content of this paper is as follows

In section 2: we give some properties of Bessel-Struve kernel and Bessel-Struve transform.

In section 3: we deal with the Weyl integral transform associated to Bessel-Struve operator. In the beginning, we consider the dual operator χ^{*}_{α} of the intertwining operator χ_{α}. This operator is related with Weyl integral associated to Bessel-Struve operator that we denote W_{α} by

$$\forall\, f \in \mathcal{D}(\mathbb{R})\,, \qquad \chi^{*}_{\alpha}\, T_{A\,f} = T_{W_{\alpha}(f)}$$

where T_f designates the distribution defined by the function f.
Next, we note that, unlike the classical case, Weyl integral transform associated to Bessel-Struve operator doesn't save the space $\mathcal{D}(\mathbb{R})$ and we characterize the range of $\mathcal{D}(\mathbb{R})$ by W_{α}. For this purpose, we introduce the space $\mathcal{K}_0$ of infinitely differentiable functions on $\mathbb{R}^*$ having bounded support and verifying a limit condition on the right and left of zero. The range of $\mathcal{D}(\mathbb{R})$ by W_{α} appears as the subspace of $\mathcal{K}_0$ which we denote $\Delta_{\alpha}(\mathbb{R})$. Furthermore we give the expression of the inverse of W_{α} denoted $V_{\alpha|\Delta_{\alpha}(\mathbb{R})}$.

In section 4 : We prove a Paley-Wiener type theorem of Bessel-Struve transform in the case $\alpha = \dfrac{1}{2}$. Finally, we prove an analogous of Schawartz-Paley-Wiener theorem associated to Bessel-Struve transform.

Throughout the paper, we denote :

- $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$
- $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$
- $\dfrac{d}{dx^2} = \dfrac{1}{2x}\dfrac{d}{dx}$

2. Bessel-Struve transform

We consider the operator $l_\alpha,\ \alpha > -\dfrac{1}{2}$, defined on $\mathbb{R}$ by

$$l_\alpha u(x) = \frac{d^2u}{dx^2}(x) + \frac{2\alpha+1}{x}\left[\frac{du}{dx}(x) - \frac{du}{dx}(0)\right], \quad x \in \mathbb{R}, \tag{2.1}$$

with u is an infinitely differentiable function on $\mathbb{R}$. This operator is called Bessel-Struve operator.

For $\lambda \in \mathbb{C}$, the differential equation :

$$\begin{cases} l_\alpha u(x) = \lambda^2 u(x) \\ u(0) = 1\ ,\ u'(0) = \dfrac{\lambda\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+3/2)} \end{cases}$$

possesses a unique solution denoted S^α_λ. This eigenfunction, called the Bessel-Struve kernel, is given by :

$$S^\alpha_\lambda(x) = j_\alpha(i\lambda x) - ih_\alpha(i\lambda x) \tag{2.2}$$

The kernel S^α_λ possesses the following integral representation :

$$\forall x \in \mathbb{R},\ \forall \lambda \in \mathbb{C}, \quad S^\alpha_\lambda(x) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\ \Gamma(\alpha+\frac{1}{2})}\int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} e^{\lambda x t}dt \tag{2.3}$$

We denote by $L^1_\alpha(\mathbb{R})$, the space of measurable functions f on $\mathbb{R}$, such that

$$\|f\|_{1\,,\,\alpha} = \int_{\mathbb{R}} |f(x)|\,d\mu_\alpha(x) < +\infty\,,$$

where

$$d\mu_\alpha(x) = A(x)\,dx \quad \text{ and } \quad A(x) = |\,x|^{2\alpha+1}\ .$$

Definition 2.1. *The Bessel-Struve transform is defined on* $L^1_\alpha(\mathbb{R})$ *by*

$$\forall\,\lambda \in \mathbb{R}, \qquad \mathcal{F}^\alpha_{B,S}(f)(\lambda) = \int_{\mathbb{R}} f(x)\,S^\alpha_{-i\lambda}(x)\,d\mu_\alpha(x) \tag{2.4}$$

Proposition 2.1. The kernel S^α_λ has a unique analytic extension to $\mathbb{C}\times\mathbb{C}$. It satisfies the following properties :

- **(i):** $\forall\lambda \in \mathbb{C},\quad \forall z \in \mathbb{C}, \qquad S^\alpha_{-i\lambda}(z) = S^\alpha_{-iz}(\lambda)$
- **(ii):** $\forall\lambda \in \mathbb{C},\quad \forall z \in \mathbb{C}, \qquad S^\alpha_{-\lambda}(z) = S^\alpha_{\lambda}(-z)$
- **(iii):** $\forall n \in \mathbb{N},\ \forall\lambda \in \mathbb{R},\ \forall x \in \mathbb{R}, \qquad \left|\dfrac{d^n}{dx^n}S^\alpha_{i\lambda}(x)\right| \leq |\lambda|^n$
- **(iv):** $\forall\,x \in \mathbb{R}^*,\ \lim\limits_{\lambda\to+\infty} S^\alpha_{-i\lambda}(x) = 0$

Proof. The relation (2.2) implies directly (i) and (ii).
Applying the derivative theorem to the relation (2.3), we obtain (iii).
From the asymptotic expansion of J_α and $\mathbf{H}_\alpha$ (see [8, p.199,p.333]), and using relation (2.2), we get (iv). □

Proposition 2.2. *(K. Trimèche [4]) Let f be a function in $L^1_\alpha(\mathbb{R})$ then $\mathcal{F}^\alpha_{B,S}(f)$ belongs to $C_0(\mathbb{R})$,*
where $C_0(\mathbb{R})$ is the space of continuous functions having 0 as limit in the infinity. Furthermore,

$$\|\mathcal{F}^\alpha_{B,S}(f)\|_\infty \leq \|f\|_{1,\alpha} \tag{2.5}$$

3. Weyl integral transform

One can find an overview on the Weyl integral transform associated to Hankel transform in K. Trimèche's book [6]. Also, K. Trimèche investigates the Weyl integral transform in the framework of Chébli-Trimèche operator in [5] and Dunkl operator in [7]. In this section, we deal with Weyl integral transform associated with Bessel-Struve operator introduced by K. Trimèche in [4] . In particular, we build the range of $\mathcal{D}(\mathbb{R})$ by this integral transform and we give the expression of its inverse.

3.1. **Bessel-Struve intertwining operator and its dual.** $\mathcal{E}(\mathbb{R})$ designates the space of infinitely differentiable functions on $\mathbb{R}$.
The Bessel-Struve intertwining operator on $\mathbb{R}$ denoted χ_α, introduced by K. Trimèche in [4] is defined by:

$$\chi_\alpha(f)(x) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} f(xt)\,dt \ , \quad f \in \mathcal{E}(\mathbb{R}) \tag{3.1}$$

L. Kamoun and M. Sifi proved an inversion theorem of χ_α on $\mathcal{E}(\mathbb{R})$, [1, Theorem 1]

Remark 3.1. *We have*

$$\forall\, x \in \mathbb{R}\,,\ \forall \lambda \in \mathbb{C}\,, \qquad S^\alpha_\lambda(x) \;=\; \chi_\alpha(e^{\lambda\,\cdot})(x) \tag{3.2}$$

Definition 3.1. *The operator χ^*_α is defined on $\mathcal{E}'(\mathbb{R})$ by*

$$< \chi^*_\alpha(T), f >=< T, \chi_\alpha f > \qquad , \quad f \in \mathcal{E}(\mathbb{R}) \tag{3.3}$$

Proposition 3.1. *χ^*_α is an isomorphism from $\mathcal{E}'(\mathbb{R})$ into itself.*

Proof. Since χ_α is an isomorphism from $\mathcal{E}(\mathbb{R})$ into itself, we deduce the result by duality. □

Proposition 3.2. *(K. Trimèche [4]) For $f \in \mathcal{D}(\mathbb{R})$, the distribution $\chi^*_\alpha T_{Af}$ is defined by the function $W_\alpha f$ having the following expression*

$$W_\alpha f(y) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_{|y|}^{+\infty} (x^2-y^2)^{\alpha-\frac{1}{2}}\, x\, f(sgn(y)x)\,dx\ , \ \ y \in \mathbb{R}^\star \tag{3.4}$$

called Weyl integral associated to Bessel-Struve operator.

Remark 3.2. *Let $f \in \mathcal{E}(\mathbb{R})$ and $g \in \mathcal{D}(\mathbb{R})$. The operator χ_α and W_α are related by the following relation*

$$\int_{\mathbb{R}} \chi_\alpha f(x)\, g(x) A(x)\, dx \;=\; \int_{\mathbb{R}} f(x)\, W_\alpha g(x) dx \tag{3.5}$$

Proposition 3.3. *(K. Trimèche [4]) We have*

$$\forall f \in \mathcal{D}(\mathbb{R})\,, \qquad \mathcal{F}^\alpha_{B,S}(f) \;=\; \mathcal{F}\circ W_\alpha(f) \tag{3.6}$$

where $\mathcal{F}$ is the classical Fourier transform defined on $L^1(\mathbb{R})$ by

$$\mathcal{F}(g)(\lambda) = \int_{\mathbb{R}} g(x) e^{-i\lambda x} dx$$

3.2. **The range of $\mathcal{D}(\mathbb{R})$ by Weyl integral transform.** The Weyl Integral transform associated with Bessel-Struve operator doesn't save the space $\mathcal{D}(\mathbb{R})$. In fact for the function given by

$$f(x) = \begin{cases} x\, e^{\frac{-1}{1-x^2}} & if \quad |x|<1 \\ 0 & else \end{cases} \quad , \quad x \in \mathbb{R},$$

we remark that $W_\alpha f$ is not continuous on 0.
In sequel, for $a>0$, we denote by $\mathcal{D}_a(\mathbb{R})$ the subspace of $\mathcal{D}(\mathbb{R})$ of functions with support included in $[-a,a]$.

Lemma 3.1. *Let $a>0$ and $f \in \mathcal{D}_a(\mathbb{R})$. Then $W_\alpha f$ is infinitely differentiable on $\mathbb{R}^\star$ and $supp\,(W_\alpha f)$ is included in $[-a,a]$. Furthermore, for all $x \in \mathbb{R}^\star$ and $n \in \mathbb{N}$,*

$$(W_\alpha f)^{(n)}(x) = \sum_{k=0}^{n} \frac{c_\alpha\, (sgn(x))^k}{x^n} \int_{|x|}^{+\infty} (y^2-x^2)^{\alpha-\frac{1}{2}} y^{k+1} f^{(k)}(y\, sgn(x))\, dy \qquad (3.7)$$

where

$$c_\alpha = \frac{2\Gamma(\alpha+1)\, C_n^k \Gamma(2\alpha+2)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})\,\Gamma(2\alpha+2-n+k)}.$$

Proof. Let $f \in \mathcal{D}_a(\mathbb{R})$. By change of variable $W_\alpha f$ can be written

$$W_\alpha f(x) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})} |x|^{2\alpha+1} \int_1^{+\infty} (t^2-1)^{\alpha-\frac{1}{2}}\, t\, f(tx)\, dt \quad , \quad x \in \mathbb{R}^\star \qquad (3.8)$$

We denote

$$\psi(x) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})} \int_1^{\frac{a}{|x|}} (t^2-1)^{\alpha-\frac{1}{2}}\, t\, f(tx)\, dt \quad , \quad x \in \mathbb{R}^*.$$

Then,

$$W_\alpha f(x) = A(x)\, \psi(x)$$

It's clear that $supp\,(\psi) \subseteq [-a,a]$. From derivative theorem and a change of variable, one obtains

$$\psi^{(k)}(x) = |x|^{-2\alpha-k-1} \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_{|x|}^{a} (y^2-x^2)^{\alpha-\frac{1}{2}}\, y^{k+1}\, f^{(k)}(sgn(x)y)\, dy, \quad x \in \mathbb{R}^*.$$

Therefore, using Leibniz formula, we get

$$W_\alpha f)^{(n)}(x) = \frac{a_\alpha}{x^n} \sum_{k=0}^{n} \frac{C_n^k (sgn(x))^k \Gamma(2\alpha+2)}{\Gamma(2\alpha+2-n+k)} \int_{|x|}^{a} (y^2-x^2)^{\alpha-\frac{1}{2}} y^{k+1} f^{(k)}(y\, sgn(x))\, dy$$

where

$$a_\alpha = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}.$$

□

We designate by $\mathcal{K}_0$ the space of functions f infinitely differentiable on $\mathbb{R}^*$ with bounded support and verifying for all $n \in \mathbb{N}$,

$$\lim_{\substack{y\to 0\\ y>0}} y^n f^{(n)}(y) \quad \text{and} \quad \lim_{\substack{y\to 0\\ y<0}} y^n f^{(n)}(y)$$

exist.

Corollary 3.1. *Let f be a function in $\mathcal{D}(\mathbb{R})$, we have $W_\alpha f$ belongs to $\mathcal{K}_0$.*

Proof. The result is a consequence from lemma 3.1. □

Lemma 3.2. *Let $g \in \mathcal{E}(\mathbb{R}^*)$, m and p are two integers nonnegative, we have*

$$\forall\, x \in \mathbb{R}^*, \qquad \left(\frac{d}{dx^2}\right)^p (x^m\, g(x)) = \sum_{i=0}^{p} \beta_i^p\, x^{m-2p+i}\, g^{(i)}(x) \tag{3.9}$$

where β_i^p are constants depending on i, p and m.

Proof. We will proceed by induction. The relation (3.9) is true for $p = 0$. Suppose that (3.9) is true at the order $p \geq 0$ then

$$\begin{aligned}(\frac{d}{dx^2})^{p+1}(x^m\, g(x)) &= \frac{d}{dx^2}(\sum_{i=0}^{p} \beta_i^p\, x^{m-2p+i}\, g^{(i)}(x)) \\ &= \sum_{i=0}^{p+1} \beta_i^{p+1} x^{m+i-2(p+1)}\, g^{(i)}(x)\end{aligned}$$

where

$$\beta_{p+1}^{p+1} = \frac{1}{2}\beta_p^p\,,\ \beta_0^{p+1} = \frac{1}{2}\beta_0^p(m-2p)$$

and

$$\forall 1 \leq i \leq p\,,\ \beta_i^{p+1} = \frac{1}{2}(m+i-2p)\beta_i^p + \frac{1}{2}\beta_{i-1}^p$$

□

We need the following proposition to provide the main results, of this section, which are theorem 3.1 and theorem 3.2.

Proposition 3.4. *Let f be a function in $\mathcal{K}_0$. Then the distribution $(\chi_\alpha^*)^{-1}T_f$ is defined by the function denoted $A\, V_\alpha f$, where $V_\alpha f$ has the following expression*

(i): *If $\alpha = k + \frac{1}{2}$, $k \in \mathbb{N}$*

$$V_\alpha f(x) = (-1)^{k+1} \frac{2^{2k+1} k!}{(2k+1)!} \left(\frac{d}{dx^2}\right)^{k+1} (f(x))\,, \quad x \in \mathbb{R}^\star$$

(ii): *If $\alpha = k + r$, $k \in \mathbb{N}\,,\ -\frac{1}{2} < r < \frac{1}{2}$,*

$$V_\alpha f(x) = c_1 \int_{|x|}^{+\infty} (y^2 - x^2)^{-r-\frac{1}{2}} \left(\frac{d}{dy^2}\right)^{k+1} (f)(sgn(x)y)\, y\, dy\,,\ \ x \in \mathbb{R}^\star$$

$$\text{where} \quad c_1 = \frac{(-1)^{k+1} 2\sqrt{\pi}}{\Gamma(\alpha+1)\Gamma(\frac{1}{2}-r)}$$

Proof. Let $g \in \mathcal{E}(\mathbb{R})$ then we have
$< (\chi_\alpha^\star)^{-1}\, T_f, g > = < (\chi_\alpha^{-1})^\star\, T_f, g > = < T_f, \chi_\alpha^{-1} g >$

First case $\alpha = k + \dfrac{1}{2}\,,\ k \in \mathbb{N}$ **:**

Invoking (ii) of [1, Theorem 1], we can write

$$< (\chi_\alpha^\star)^{-1}\, T_f, g > = \frac{2^{2k+1}\, k!}{(2k+1)!}\, (I_1 + I_2)$$

where

$$I_1 = \int_0^\infty f(x)\, x \left(\frac{d}{dx^2}\right)^{k+1} (x^{2k+1}\, g(x))\, dx$$

and

$$I_2 = \int_{-\infty}^{0} f(x)\, x \left(\frac{d}{dx^2}\right)^{k+1} (x^{2k+1}\, g(x))\, dx$$

By integration by parts we have, according to relation (3.9) for $p = k$ and $m = 2k+1$

$$I_1 = -\int_{0}^{+\infty} \left(\frac{d}{dx^2}\right) f(x) \left(\frac{d}{dx^2}\right)^{k} (x^{2k+1}\, g(x))\, x\, dx$$

After k integrations by parts, using relation (3.9) and the fact that $f \in \mathcal{K}_0$, we find that

$$I_1 = (-1)^{k+1} \int_{0}^{+\infty} \left(\frac{d}{dx^2}\right)^{k+1} f(x)\, g(x)\, x^{2k+2}\, dx$$

As the same we establish that

$$I_2 = (-1)^{k+1} \int_{-\infty}^{0} \left(\frac{d}{dx^2}\right)^{k+1} f(x)\, g(x)\, x^{2k+2}\, dx$$

Consequently,

$$< (\chi_\alpha^\star)^{-1}\, T_f, g >= \frac{2^{2k+1}\, k!}{(2k+1)!} (-1)^{k+1} \int_{\mathbb{R}} \left(\frac{d}{dx^2}\right)^{k+1} f(x)\, g(x)\, x^{2k+2}\, dx$$

Which proves the wanted result for $\alpha = k + \frac{1}{2}$.

Second case $\alpha = k + r\,,\ k \in \mathbb{N}\,,\ \frac{-1}{2} < r < \frac{1}{2}$

By virtue of (i) of [1, Theorem 1] and a change of variable, we can write

$$\chi_\alpha^{-1} g(x) = \frac{2\sqrt{\pi} x}{\Gamma(\alpha+1)\Gamma(\frac{1}{2}-r)} \left(\frac{d}{dx^2}\right)^{k+1} (x^{2k+1}\, h(x))$$

where

$$h(x) = \int_0^1 (1-u^2)^{-r-\frac{1}{2}} g(xu)\, u^{2\alpha+1}\, du$$

It's clear that $h \in \mathcal{E}(\mathbb{R})$, we proceed in a similar way as in the first case, we just replace the function g by the function h and we obtain

$$< (\chi_\alpha^\star)^{-1}\, T_f, g >= c_1 \int_{\mathbb{R}} \left(\frac{d}{dx^2}\right)^{k+1} f(x)\, h(x)\, x^{2k+2}\, dx$$

Next, by a change of variable, we have

$$< (\chi_\alpha^\star)^{-1}\, T_f, g >= \frac{2\sqrt{\pi}\, (-1)^{k+1}}{\Gamma(\alpha+1)\Gamma(\frac{1}{2}-r)} (J_1 + J_2)$$

where

$$J_1 = \int_0^{+\infty} x (\frac{d}{dx^2})^{k+1} f(x) \left(\int_0^x (x^2-t^2)^{-r-\frac{1}{2}} g(t)\, |t|^{2\alpha+1}\, dt\right) dx$$

and

$$J_2 = \int_{-\infty}^0 x (\frac{d}{dx^2})^{k+1} f(x) \left(\int_0^x (x^2-t^2)^{-r-\frac{1}{2}} g(t)\, |t|^{2\alpha+1}\, dt\right) dx$$

Applying Fubini's theorem in J_1 and J_2, we obtain

$$J_1 = \int_0^{+\infty} \left(\int_t^{+\infty} (x^2-t^2)^{-r-\frac{1}{2}} x (\frac{d}{dx^2})^{k+1} f(x)\, dx\right) g(t)\, |t|^{2\alpha+1}\, dt$$

and

$$J_2 = -\int_{-\infty}^{0}\left(\int_{-\infty}^{t}(x^2-t^2)^{-r-\frac{1}{2}}\,x(\frac{d}{dx^2})^{k+1}f(x)\,dx\right)\,g(t)\,|t|^{2\alpha+1}\,dt$$

making a change of variable in J_2 and using Chasles relation , we get
$< (\chi_\alpha^\star)^{-1}\,T_f, g >=$

$$c_1\int_{\mathbb{R}}\left(\int_{|t|}^{+\infty}(x^2-t^2)^{-r-\frac{1}{2}}x(\frac{d}{dx^2})^{k+1}f(sgn(t)x)\,dx\right)\,g(t)\,|t|^{2\alpha+1}\,dt$$

Which proves the wanted result. □

Remark 3.3. *From proposition 3.4 we deduce that the operators V_α and χ_α^{-1} are related by the following relation*

$$\int_{\mathbb{R}} V_\alpha f(x)\,g(x)A(x)\,dx\;=\;\int_{\mathbb{R}} f(x)\,\chi_\alpha^{-1}g(x)\,dx \tag{3.10}$$

for all $f \in \mathcal{K}_0$ and $g \in \mathcal{E}(\mathbb{R})$.

Lemma 3.3. *Let f be in $\mathcal{D}(\mathbb{R})$. We have $W_\alpha(f) \in \mathcal{K}_0$ and $V_\alpha(W_\alpha(f)) = f$*

Proof. Using lemma 3.1, relations (3.10) and (3.5), we obtain for all $g \in \mathcal{E}(\mathbb{R})$, $f \in \mathcal{D}(\mathbb{R})$

$$\int_{\mathbb{R}} V_\alpha(W_\alpha f)(x)\,g(x)A(x)\,dx\;=\;\int_{\mathbb{R}} f(x)\,g(x)A(x)\,dx$$

Thus

$$V_\alpha(W_\alpha(f))(x)\,A(x) = f(x)\,A(x) \qquad a.e\;\; x \in \mathbb{R}$$

Since $f\,A$ and $V_\alpha \circ W_\alpha(f)\,A$ are both continuous functions on $\mathbb{R}^\star$ we have $V_\alpha \circ W_\alpha(f)(x) = f(x)$ for all x in $\mathbb{R}^\star$ therefore $V_\alpha \circ W_\alpha(f)(x) = f(x)$ for all x in $\mathbb{R}$. □

For $\alpha = k + \dfrac{1}{2}$, $k \in \mathbb{N}$, we denote by $\Delta_{a,k+\frac{1}{2}}(\mathbb{R})$ the subspace of $\mathcal{K}_0$ of functions f infinitely differentiable on $\mathbb{R}^*$ with support included in $[-a, a]$ verifying the following condition :

$$(\frac{d}{dx^2})^{k+1}f \text{ can be extended to a function belonging to } \mathcal{D}(\mathbb{R}).$$

This space is provided with the topology defined by the semi norms ρ_n where

$$\rho_n(f) = \sup_{\substack{0\le p\le n\\ x\in[-a,a]}}\left|\left((\frac{d}{dx^2})^{k+1}f\right)^{(p)}(x)\right| \qquad ,\; n \in \mathbb{N}$$

We consider , for $k \in \mathbb{N}$, the space

$$\Delta_{k+\frac{1}{2}}(\mathbb{R}) = \bigcup_{a\ge 0}\Delta_{a,k+\frac{1}{2}}(\mathbb{R})$$

endowed with the inductive limit topology.

Lemma 3.4. *For all f in $\mathcal{D}_a(\mathbb{R})$ we have*

(i): $\forall\, x \in \mathbb{R}^*, \qquad [W_{\frac{1}{2}}f]'(x) = \,-x\,f(x)$

(ii): $\forall\, \alpha > \frac{1}{2}$, $\forall\, x \in \mathbb{R}^*, \qquad [W_\alpha f]'(x) = -2\,\alpha x W_{\alpha-1}f(x)$

Proof. We get (i) of lemma 3.4 using relation (3.4) and derivation theorem. Now, we take $\alpha > \dfrac{1}{2}$, by lemma 3.1 $supp(W_\alpha f) \subset [-a,a]$.
Let $\varphi \in \mathcal{D}(\,(0,+\infty)\,)$ then we have

$$\begin{aligned} < [W_\alpha f]', \varphi > &= \; - < W_\alpha f, \varphi' > \\ &= -a_\alpha \int_0^a \int_y^a (x^2-y^2)^{\alpha-\frac{1}{2}} x\, f(x)\, dx\; \varphi'(y)\, dy \end{aligned}$$

Using Fubini's theorem, an integration by parts and relation (3.4), we obtain

$$< [W_\alpha f]', \varphi > = -2\alpha \int_0^a y\, W_{\alpha-1} f(y)\, \varphi(y)\, dy = < -2y\, W_{\alpha-1} f, \varphi >$$

This proves that the derivative of the distribution $W_\alpha f$ is the distribution defined by the function $-2\alpha x\, W_{\alpha-1}$ on $(0,+\infty)$. The theorem III in [3, p.54] allows us to say that the derivative on $(0,+\infty)$ of the function $W_\alpha f$ is the function $-2\alpha\, x W_{\alpha-1} f$. In the same way we obtain that the derivative on $(-\infty,0)$ of the function $W_\alpha f$ is the function $-2\alpha\, x W_{\alpha-1} f$ and (ii) of lemma 3.4 yields. □

Theorem 3.1. *The operator $W_{k+\frac{1}{2}}$ is a topological isomorphism from $\mathcal{D}_a(\mathbb{R})$ into $\Delta_{a,k+\frac{1}{2}}(\mathbb{R})$ and its inverse is $V_{k+\frac{1}{2}|\Delta_{a,k+\frac{1}{2}}(\mathbb{R})}$.*

Proof. We will proceed by induction.
According to (i) of lemma 3.4 we have $W_{\frac{1}{2}}(\mathcal{D}_a(\mathbb{R})) \subset \Delta_{a,\frac{1}{2}}(\mathbb{R})$. Let f be a function in $\mathcal{D}_a(\mathbb{R})$ then according to (ii) of lemma 3.4, the induction hypothesis and lemma 3.1 we conclude that $W_{k+\frac{1}{2}} f \in \Delta_{a,k+\frac{1}{2}}(\mathbb{R})$.
In the other hand, using proposition 3.4 and the fact that $g \in \Delta_{a,k+\frac{1}{2}}(\mathbb{R})$, we get

$$W_{k+\frac{1}{2}}(V_{k+\frac{1}{2}})(g) = g.$$

From proposition 3.4 and lemma 3.3, we have

$$\rho_n(W_{k+\frac{1}{2}} f) = C \sup_{\substack{0 \le p \le n \\ x \in [-a,a]}} |f^{(p)}(x)|$$

which proves that $W_{k+\frac{1}{2}}$ is a topological isomorphism from $\mathcal{D}_a(\mathbb{R})$ into $\mathcal{D}_{a,k+\frac{1}{2}}(\mathbb{R})$ and its inverse is given by $V_{k+\frac{1}{2}|\Delta_{a,k+\frac{1}{2}}(\mathbb{R})}$. □

For $k \in \mathbb{N}$ we take $\alpha = k + r\,,\ r \in]\frac{-1}{2}, \frac{1}{2}[$.

We denote by $\Delta_{a,k+r}(\mathbb{R})$ the subspace of $\mathcal{K}_0$ of functions f infinitely differentiable on $\mathbb{R}^*$ with support included in $[-a,a]$ verifying the following condition : $(\dfrac{d}{dx^2})^{k+1}\left(\displaystyle\int_1^{+\infty} (t^2-1)^{-r-\frac{1}{2}} f(xt)\, t^{-2k-1} dt\right)$ can be extended to a function belonging to $|x|^{2r-1}\,\mathcal{D}(\mathbb{R})$

This space is provided with the topology defined by the semi norms q_n where

$$q_n(f) = \sup_{\substack{0 \le p \le n \\ x \in [-a,a]}} \left| D^p \left(|x|^{-2r+1} (\frac{d}{dx^2})^{k+1} \left(\int_1^{+\infty} (t^2-1)^{-r-\frac{1}{2}} f(xt)\, t^{-2k-1} dt \right)\right)\right|$$

We consider, for $k \in \mathbb{N}$, the space

$$\Delta_{k+r}(\mathbb{R}) = \bigcup_{a \ge 0} \Delta_{a,k+r}(\mathbb{R})$$

endowed with the inductive limit topology.

Lemma 3.5. *We have for all f in $\Delta_{k+r}(\mathbb{R})$,*

$$V_{k+r}(f) \in \mathcal{D}(\mathbb{R}) \quad \text{and} \quad W_{k+r}(V_{k+r}(f)) = f$$

Proof. Let $f \in \Delta_{k+r}(\mathbb{R})$, using lemma 3.2 and the linearity of integral sign, we obtain

$$V_{k+r}(f)(y) == |y|^{-2r+1}(\frac{d}{dy^2})^{k+1}\left(\int_1^{+\infty}(t^2-1)^{-r-\frac{1}{2}}f(yt)\,t^{-2k-1}dt\right)$$

Then $V_{k+r}(f) \in \mathcal{D}(\mathbb{R})$. From relations (3.5) and (3.10), we have, for all $g \in \mathcal{E}(\mathbb{R})$,

$$\int_{\mathbb{R}} W_{k+r}(V_{k+r}(f))(x)\,g(x)\,dx = \int_{\mathbb{R}} f(x)\,g(x)\,dx$$

Therefore

$$W_{k+r}(V_{k+r}(f))(x) = f(x)\ , \qquad a.e.\ x \in \mathbb{R}$$

Since $W_{k+r}(V_{k+r}(f))$ and f are both continuous functions on $\mathbb{R}^*$, we get

$$\forall\ x \in \mathbb{R}^*\ , \qquad W_{k+r}(V_{k+r}(f))(x) = f(x)$$

□

Theorem 3.2. *W_{k+r} is a topological isomorphism from $\mathcal{D}_a(\mathbb{R})$ into $\Delta_{a,k+r}(\mathbb{R})$ and its inverse is $V_{k+r|\Delta_{a,k+r}(\mathbb{R})}$*

Proof. Let $f \in \mathcal{D}_a(\mathbb{R})$, proposition 3.1 and lemma 3.3, allows us to prove that $W_{k+r}(f) \in \Delta_{a,k+r}(\mathbb{R})$.
Furthermore, from lemma 3.5, lemma 3.3 and the fact that

$$q_n(W_{k+r}(f)) = Cp_n(f)$$

one can deduce that W_{k+r} is a topological isomorphism from $\mathcal{D}_a(\mathbb{R})$ into $\Delta_{a,k+r}(\mathbb{R})$ and $V_{k+r|\Delta_{a,k+r}(\mathbb{R})}$ is its inverse. □
The following theorem is a consequence from theorem 3.1 and theorem 3.2 .

Theorem 3.3. *W_α is a topological isomorphism from $\mathcal{D}(\mathbb{R})$ into $\Delta_\alpha(\mathbb{R})$ and its inverse is given by $V_{\alpha|\Delta_\alpha(\mathbb{R})}$*

4. Paley Wiener type theorem associated to Bessel-Struve transform

In this section we shall try to characterize the range of $\mathcal{D}(\mathbb{R})$ by Bessel-Struve transform.

4.1. **Range of $\mathcal{D}(\mathbb{R})$ by Bessel-Struve transform for half integers.** Let $a > 0$, $\mathcal{H}_a$ designates the space of entire functions f verifying :

$$\forall n \in \mathbb{N}\,,\ \exists\, c_n > 0\,;\ \forall z \in \mathbb{C}\,, \qquad (1+|z|^2)^n|f(z)|e^{-a\,Im(z)}\ <\ c_n$$

and

$$\mathcal{H} = \bigcup_{a>0}\mathcal{H}_a$$

We introduce the space $\Lambda_{a,\frac{1}{2}}$ the space of entire functions g verifying

$$\exists\, h \in \mathcal{H}_a\ \forall\ z \in \mathbb{C}^* \qquad g(z) = \frac{h'(z)-h'(0)}{z} \tag{4.1}$$

and we denote $\Lambda_{\frac{1}{2}} = \bigcup_{a>0} \Lambda_{a,\frac{1}{2}}$

Theorem 4.1. *We have*

$$\mathcal{F}_{BS}^{\frac{1}{2}}(\mathcal{D}(\mathbb{R})) = \Lambda_{\frac{1}{2}}$$

Proof. Let $f \in \mathcal{D}(\mathbb{R})$. From relation (3.6) and by integration by parts, we have

$$-iz\,\mathcal{F}_{BS}^{\frac{1}{2}}(f)(z) = -c + \mathcal{F}(xf)(z)$$

where

$$c = \lim_{\substack{x\to 0\\ x>0}} W_{\frac{1}{2}} f(x) - \lim_{\substack{x\to 0\\ x<0}} W_{\frac{1}{2}} f(x)$$

Since $\mathcal{F}(x\,f)(z) = i[\mathcal{F}(f)]'(z)$, we get $c = i[\mathcal{F}(f)]'(0)$, for $z = 0$.
Therefore

$$\mathcal{F}_{BS}^{\frac{1}{2}}(f)(z) = \frac{[\mathcal{F}(-f)]'(z) - [\mathcal{F}(-f)]'(0)}{z}$$

which proves that

$$\mathcal{F}_{BS}^{\frac{1}{2}}(\mathcal{D}(\mathbb{R})) \subset \Lambda_{\frac{1}{2}}$$

Now let g be an entire function verifying relation (4.1). From classical Paley-Wiener theorem and relation (4.1), we have

$$\exists\, f \in \mathcal{D}_a(\mathbb{R}) \text{ such that } \frac{[\mathcal{F}(f)]'(z) - [\mathcal{F}(f)]'(0)}{z} = g(z)$$

Therefore, for $\lambda \neq 0$

$$\begin{aligned} g(\lambda) &= -\frac{i}{\lambda}(\mathcal{F}(t\,f)(\lambda) - \mathcal{F}(t\,f)(0)) \\ &= \int_{\mathbb{R}} f(t)\Big(\frac{-\sin(\lambda\,t)}{\lambda\,t} + i\frac{1-\cos(\lambda\,t)}{\lambda\,t}\Big)t^2\,dt \\ &= -\int_{\mathbb{R}} f(t) S_{\frac{1}{2}}(-i\,\lambda\,t)\,t^2\,dt \\ &= \mathcal{F}_{BS}^{\frac{1}{2}}(-f)(\lambda) \end{aligned}$$

□

By induction, we can build the range of $\mathcal{D}(\mathbb{R})$ by $\mathcal{F}_{B,S}^{k+\frac{1}{2}}$ from theorem 4.1 and the following proposition.

Proposition 4.1. *For $\alpha > \frac{1}{2}$, the following assertions are equivalent*

- **(i):** $g = \mathcal{F}_{BS}^{\alpha}(f)$ where $f \in \mathcal{D}_a(\mathbb{R})$
- **(ii):** *g is extented to an entire function $\tilde{g}$ verifying*

$$\exists\, h \in \mathcal{F}_{BS}^{\alpha-1}(\mathcal{D}_a(\mathbb{R}))\ ;\ \ \forall\, z \in \mathbb{C} \qquad \tilde{g}(z) = 2\alpha\,\frac{h'(z) - h'(0)}{z} \tag{4.2}$$

Proof. Let $f \in \mathcal{D}(\mathbb{R})$ and $z \in \mathbb{C}$. We proceed in a similar way as in theorem 4.1 and we obtain

$$iz\mathcal{F}_{BS}^{\alpha}(f)(z) - c = \mathcal{F}([W_\alpha f]')(z)$$

where

$$c = \lim_{\substack{x\to 0\\ x>0}} W_\alpha f(x) - \lim_{\substack{x\to 0\\ x<0}} W_\alpha f(x)$$

Furthermore, using (ii) of lemma 3.4 and analysity theorem, we get

$$\mathcal{F}([W_\alpha f]')(z) = -2i\,\alpha[\mathcal{F}(W_{\alpha-1} f)]'(z).$$

From relation (3.6) and the fact that $c = [\mathcal{F}^{\alpha}_{BS}(f)]'(0)$, we conclude that $\mathcal{F}^{\alpha}_{BS}(f)$ verifies relation (4.2).
Now let g be an entire function verifying relation (4.2).
Then

$$\exists\, f \in \mathcal{D}_a(\mathbb{R}) \;\; such\, that \;\; \frac{2\alpha([\mathcal{F}^{\alpha-1}_{BS}(f)]'(z) - [\mathcal{F}^{\alpha-1}_{BS}(f)]'(0))}{z} \;=\; g(z)$$

Finally, from relation (3.6) and by integration by parts we get

$$g(z) = i\mathcal{F}^{\alpha}_{BS}(f)(z)$$

□

4.2. **Schwartz Paley Wiener theorem.** In this subsection we will prove a Paley Wiener theorem in distributions space with bounded support.

Definition 4.1. *The Fourier Bessel-Struve Transformis defined on $\mathcal{E}'(\mathbb{R})$ by*

$$\forall\, T \in \mathcal{E}'(\mathbb{R})\,,\; \mathcal{F}^{\alpha}_{B,S}(T)(\lambda) \;=< T, S^{\alpha}_{-i\lambda} > \tag{4.3}$$

Proposition 4.2. *For all $T \in \mathcal{E}'(\mathbb{R})$,*

$$\mathcal{F}^{\alpha}_{B,S}(T) \;=\; \mathcal{F} \circ \chi^{\star}_{\alpha}(T) \tag{4.4}$$

Proof. We get the result using relations (4.3), (3.2) and (3.3). □

Lemma 4.1. *Let $T \in \mathcal{E}'(\mathbb{R})$, then*

$$supp(T) \subseteq [-b, b] \Longleftrightarrow supp(\chi^{\star}_{\alpha}(T)) \subseteq [-b, b]$$

Proof. Let $T \in \mathcal{E}'(\mathbb{R})$ such that $supp(T)$ included in $[-b, b]$. For $\varphi \in \mathcal{D}(\mathbb{R})$ with support in $[-b, b]^c$ we have $\chi_\alpha\varphi$ have the support included in $[-b, b]^c$ therefore from relation (3.3),we get that $\chi^{\star}_{\alpha}(T)$ have the support in $[-b, b]$
Now we consider a distribution T such that $supp\,(\chi^{\star}_{\alpha}(T))$ included in $[-b, b]$. For $\varphi \in \mathcal{D}(\mathbb{R})$ with support in $[-b, b]^c$ we have

$$< T, \varphi >=< (\chi^{-1}_{\alpha})^{\star} \circ \chi^{\star}_{\alpha}(T), \varphi >=< \chi^{\star}_{\alpha}(T), \chi^{-1}_{\alpha}\varphi >$$

Using [1, Theorem 1] $supp\,(\chi^{-1}_{\alpha}\varphi)$ included in $[-b, b]$ so $< T, \varphi >= 0$ which completes the proof. □

Theorem 4.2. *Let $b > 0$ and $f \in \mathcal{E}(\mathbb{R})$.There is an equivalence between the two following assertions*

1. *There exists a distribution $T \in \mathcal{E}'(\mathbb{R})$ with support included in $[-b, b]$ such that $f \;=\; \mathcal{F}_{B,S}(T)$*
2. *f is extended to an analytic function $\tilde{f}$ on $\mathbb{C}$ such that*

$$\exists m \in \mathbb{N}\,,\; \exists c > 0\,,\; \forall z \in \mathbb{C}\;\; |\tilde{f}(z)| \;\le\; c\,(1 + |z|^2)^{\frac{m}{2}}\, e^{b(Im(z))} \tag{4.5}$$

Proof. The theorem is a consequence from Lemma 4.1, proposition 4.2, the classical Paley-Wiener Schwartz (one can see [2]) and proposition 3.1. □

References

[1] L. Kamoun and M. Sifi, Bessel-Struve Intertwining Operator and Generalized Taylor Series on the Real Line, Integral Transforms and Spec. Funct., Vol.16, January 2005, 39-55.

[2] V. Kohan, Distributions Analyse de Fourier Opérateurs aux Dérivées Partielles, TomeII, Librairie Vuiebert 1972

[3] L. Schwartz, Théorie des Distributions, Hermann Paris 1966

[4] K. Trimèche, A New Harmonic Analysis Related to The Dunkl Operators Theory, Preprint Faculty of Sciences of Tunis, 2003.

[5] K. Trimèche, Inversion Formulas for the Dunkl Intertwining Operator and Its Dual on Spaces of Functions and Distributions, SIGMA Symmetry Integrability Geom. Methods Appl. 4 , 2008, 087, 22 p.

[6] K. Trimèche, Generalised Harmonic Analysis and Wavelets Packets, Gordon and Breach Science Publishers, 2001

[7] K. Trimèche, Transformation Intégrale de Weyl et Théorème de Paley Wiener Associés à un Oprateur Différentiel Singulier sur $(0, +\infty)$, J.Math.Pures Appli. 60,(1981), 51-98

[8] G.N. Watson, A Treatise on the Theory of Bessel Functions, Camb. Univ. Press, Cambridge, 1966.

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