Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 4 Issue 1(2012), Pages 239-246.

ON SUBORDINATION RESULTS FOR CERTAIN NEW CLASSES OF ANALYTIC FUNCTIONS DEFINED BY USING SALAGEAN OPERATOR

(COMMUNICATED BY R. K. RAINA)

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Abstract. In this paper we derive several subordination results for certain new classes of analytic functions defined by using Salagean operator.

1. Introduction

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

that are analytic and univalent in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $f(z) \in A$ be given by (1.1) and $g(z) \in A$ be given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k. \tag{1.2}$$

Definition 1 (Hadamard Product or Convolution). Given two functions f and g in the class A, where $f(z)$ is given by (1.1) and $g(z)$ is given by (1.2) the Hadamard product (or convolution) of f and g is defined (as usual) by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

We also denote by K the class of functions $f(z) \in A$ that are convex in U.
For $f(z) \in A$, Salagean [11] introduced the following differential operator:

$$D^0 f(z) = f(z),\ D^1 f(z) = zf'(z), ...,\ D^n f(z) = D(D^{n-1} f(z))(n \in \mathbb{N} = \{1, 2, ...\}).$$

We note that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

Definition 2 (Subordination Principle). For two functions f and g, analytic in U, we say that the function $f(z)$ is subordinate to $g(z)$ in U, and write $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with

2000 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Analytic, univalent, convex, convolution, subordinating factor sequence.

Submitted Jan 28, 2011. Published December 24, 2011.

$w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$ $(z \in U)$. Indeed it is known that

$$f(z) \prec g(z) \Longrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Furthermore, if the function g is univalent in U, then we have the following equivalence [8, p. 4]:

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Definition 3 [7]**.** Let $U_{m,n}(\beta, A, B)$ denote the subclass of A consisting of functions $f(z)$ of the form (1.1) and satisfy the following subordination,

$$\frac{D^m f(z)}{D^n f(z)} - \beta \left| \frac{D^m f(z)}{D^n f(z)} - 1 \right| \prec \frac{1 + Az}{1 + Bz} \tag{1.4}$$

$$(-1 \le B < A \le 1; \beta \ge 0; m \in \mathbb{N}; n \in \mathbb{N}_0, m > n; z \in U).$$

Specializing the parameters A, B, β, m and n, we obtain the following subclasses studied by various authors:

$$\begin{aligned} (i)\ U_{m,n}(\beta, 1-2\alpha, -1) &= N_{m,n}(\alpha, \beta) \\ &= \left\{ f \in A : \operatorname{Re}\left\{ \frac{D^m f(z)}{D^n f(z)} - \alpha \right\} > \beta \left| \frac{D^m f(z)}{D^n f(z)} - 1 \right| \right. \\ &\qquad \left. (0 \le \alpha < 1; \beta \ge 0; m \in \mathbb{N}; n \in \mathbb{N}_0; m > n; z \in U) \right\} \end{aligned}$$

(see Eker and Owa [4]);

$$\begin{aligned} (ii)\ U_{n+1,n}(\beta, 1-2\alpha, -1) &= S(n, \alpha, \beta) \\ &= \left\{ f \in A : \operatorname{Re}\left\{ \frac{D^{n+1} f(z)}{D^n f(z)} - \alpha \right\} > \beta \left| \frac{D^{n+1} f(z)}{D^n f(z)} - 1 \right| \right. \\ &\qquad \left. (0 \le \alpha < 1; \beta \ge 0; n \in \mathbb{N}_0; z \in U) \right\} \end{aligned}$$

(see Rosy and Murugusudaramoorthy [10] and Aouf [1]);

$$\begin{aligned} (iii)\quad U_{1,0}(\beta, 1-2\alpha, -1) &= US(\alpha, \beta) \\ &= \left\{ f \in A : \operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} > \beta \left| \frac{zf'(z)}{f(z)} - 1 \right| \right. \\ &\qquad \left. (0 \le \alpha < 1; \beta \ge 0; z \in U) \right\}, \\ U_{2,1}(\beta, 1-2\alpha, -1) &= UK(\alpha, \beta) \\ &= \left\{ f \in A : \operatorname{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} - \alpha \right\} > \beta \left| \frac{zf''(z)}{f'(z)} \right| \right. \\ &\qquad \left. (0 \le \alpha < 1; \beta \ge 0; z \in U) \right\} \end{aligned}$$

(see Shams et al. [13] and Shams and Kulkarni [12]);

$$\begin{aligned}
(iv)\ U_{1,0}(0,A,B) &= S^*(A,B) \\
&= \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \quad (-1 \le B < A \le 1; z \in U) \right\}, \\
U_{2,1}(0,A,B) &= K(A,B) \\
&= \left\{ f \in A : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz} \quad (-1 \le B < A \le 1; z \in U) \right\}
\end{aligned}$$

(see Janowski [6] and Padmanabhan and Ganesan [9]).

Also we note that:

$$\begin{aligned}
U_{m,n}(0,A,B) &= U(m,n;A,B) = \left\{ f(z) \in A : \frac{D^m f(z)}{D^n f(z)} \prec \frac{1+Az}{1+Bz} \right. \\
&\quad \left. (-1 \le B < A \le 1; m \in \mathbb{N}; n \in \mathbb{N}_0; m > n; z \in U) \right\}.
\end{aligned}$$

Definition 4 *(Subordination Factor Sequence). A Sequence $\{c_k\}_{k=0}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever $f(z)$ of the form (1.1) is analytic, univalent and convex in U, we have the subordination given by*

$$\sum_{k=1}^{\infty} a_k c_k z^k \prec f(z) \ (a_1 = 1; z \in U) \tag{1.5}$$

2. Main Result

Unless otherwise mentioned, we assume in the reminder of this paper that, $-1 \le B < A \le 1$, $\beta \ge 0$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m > n$ and $z \in U$.

To prove our main result we need the following lemmas.

Lemma 1. *[16]. The sequence $\{c_k\}_{k=0}^{\infty}$ is a subordinating factor sequence if and only if*

$$Re\left\{1 + 2\sum_{k=1}^{\infty} c_k z^k\right\} > 0 \quad (z \in U). \tag{2.1}$$

Now, we prove the following lemma which gives a sufficient condition for functions belonging to the class $U_{m,n}(\beta, A, B)$.

Lemma 2. *A function $f(z)$ of the form* (1.1) *is in the class $U_{m,n}(\beta, A, B)$ if*

$$\sum_{k=2}^{\infty} \left[\left(1 + \beta\left(1 + |B|\right)\right) \left(k^m - k^n\right) + \left|Bk^m - Ak^n\right| \right] |a_k| \le A - B \tag{2.2}$$

Proof. It suffices to show that

$$\left| \frac{p(z) - 1}{A - Bp(z)} \right| < 1,$$

where

$$p(z) = \frac{D^m f(z)}{D^n f(z)} - \beta \left| \frac{D^m f(z)}{D^n f(z)} - 1 \right|.$$

We have

$$\begin{aligned}\left|\frac{p(z)-1}{A-Bp(z)}\right| &= \left|\frac{D^m f(z)-\beta e^{i\theta}\left|D^m f(z)-D^n f(z)\right|-D^n f(z)}{AD^n f(z)-B\left[D^m f(z)-\beta e^{i\theta}\left|D^m f(z)-D^n f(z)\right|\right]}\right| \\ &= \left|\frac{\sum_{k=2}^{\infty}\left(k^m-k^n\right)a_k z^k-\beta e^{i\theta}\left|\sum_{k=2}^{\infty}\left(k^m-k^n\right)a_k z^k\right|}{(A-B)\,z-\left[\sum_{k=2}^{\infty}\left(Bk^m-Ak^n\right)a_k z^k-B\beta e^{i\theta}\left|\sum_{k=2}^{\infty}\left(k^m-k^n\right)a_k z^k\right|\right]}\right| \\ &\leq \frac{\sum_{k=2}^{\infty}\left(k^m-k^n\right)|a_k|\,|z|^k+\beta\sum_{k=2}^{\infty}\left(k^m-k^n\right)|a_k|\,|z|^k}{(A-B)\,|z|-\left[\sum_{k=2}^{\infty}\left|Bk^m-Ak^n\right||a_k|\,|z|^k+|B|\,\beta\sum_{k=2}^{\infty}\left(k^m-k^n\right)|a_k|\,|z|^k\right]} \\ &\leq \frac{\sum_{k=2}^{\infty}\left(k^m-k^n\right)|a_k|+\beta\sum_{k=2}^{\infty}\left(k^m-k^n\right)|a_k|}{(A-B)-\sum_{k=2}^{\infty}\left|Bk^m-Ak^n\right||a_k|-|B|\,\beta\sum_{k=2}^{\infty}\left(k^m-k^n\right)|a_k|}.\end{aligned}$$

This last expression is bounded above by 1 if

$$\sum_{k=2}^{\infty}\left[\Big(1+\beta\left(1+|B|\right)\Big)\Big(k^m-k^n\Big)+\Big|Bk^m-Ak^n\Big|\right]|a_k|\leq A-B,$$

and hence the proof is completed. ∎

Remark 1.

(i) The result obtained by Lemma 2 correct the result obtained by Li and Tang [7, Theorem 1];
(ii) Putting $A=1-2\alpha\ (0\leq\alpha<1)$, and $B=-1$ in Lemma 2, we correct the result obtained by Eker and Owa [4, Theorem 2.1];
(iii) Putting $A=1-2\alpha\ (0\leq\alpha<1)$, $B=-1$ and $m=n+1(n\in\mathbb{N}_0)$, we obtain the result obtained by Rosy and Murugusudaramoorthy [10, Theorem 2].

Let $U^*_{m,n}(\beta,A,B)$ denote the class of $f(z)\in A$ whose coefficients satisfy the condition (2.2). We note that $U^*_{m,n}(\beta,A,B)\subseteq U_{m,n}(\beta,A,B)$.

Employing the technique used earlier by Attiya [3] and Srivastava and Attiya [14], we prove:

Theorem 3. *Let $f(z)\in U^*_{m,n}(\beta,A,B)$. Then*

$$\frac{\left(1+\beta(1+|B|)\right)\left(2^m-2^n\right)+\left|B2^m-A2^n\right|}{2\left[\left(1+\beta(1+|B|)\right)\left(2^m-2^n\right)+\left|B2^m-A2^n\right|+(A-B)\right]}\left(f*h\right)(z)\prec h(z)\ \ (z\in U), \tag{2.3}$$

for every function h in K, and

$$\operatorname{Re}\{f(z)\}>-\frac{\left(1+\beta(1+|B|)\right)\left(2^m-2^n\right)+\left|B2^m-A2^n\right|+(A-B)}{\left(1+\beta(1+|B|)\right)\left(2^m-2^n\right)+\left|B2^m-A2^n\right|}\ \ (z\in U). \tag{2.4}$$

The constant factor $\frac{(1+\beta(1+|B|))(2^m-2^n)+|B2^m-A2^n|}{2[(1+\beta(1+|B|))(2^m-2^n)+|B2^m-A2^n|+(A-B)]}$ in the subordination result (2.3) cannot be replaced by a larger one.

Proof. Let $f(z)\in U^*_{m,n}(\beta,A,B)$ and let $h(z)=z+\sum_{k=2}^{\infty}c_k z^k\in K$. Then we have

$$\begin{aligned}&\frac{\left(1+\beta(1+|B|)\right)\left(2^m-2^n\right)+\left|B2^m-A2^n\right|}{2\left[\left(1+\beta(1+|B|)\right)\left(2^m-2^n\right)+\left|B2^m-A2^n\right|+(A-B)\right]}\left(f*h\right)(z) \\ &=\frac{\left(1+\beta(1+|B|)\right)\left(2^m-2^n\right)+\left|B2^m-A2^n\right|}{2\left[\left(1+\beta(1+|B|)\right)\left(2^m-2^n\right)+\left|B2^m-A2^n\right|+(A-B)\right]}\left(z+\sum_{k=2}^{\infty}a_k c_k z^k\right).\end{aligned} \tag{2.5}$$

Thus, by Definition 4, the subordination result (2.3) will hold true if the sequence

$$\left\{ \frac{(1+\beta(1+|B|))\,(2^m-2^n)+|B2^m-A2^n|}{2\left[(1+\beta(1+|B|))\,(2^m-2^n)+|B2^m-A2^n|+(A-B)\right]} a_k \right\}_{k=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1, this is equivalent to the following inequality:

$$\operatorname{Re}\left\{ 1+\sum_{k=1}^{\infty} \frac{(1+\beta(1+|B|))\,(2^m-2^n)+|B2^m-A2^n|}{(1+\beta(1+|B|))\,(2^m-2^n)+|B2^m-A2^n|+(A-B)} a_k z^k \right\} > 0 \ (z \in U). \tag{2.6}$$

Now, since

$$\Psi(k) = (1+\beta\,(1+|B|))\,(k^m-k^n)+|Bk^m-Ak^n|$$

is an increasing function of k ($k \geq 2$), we have

$$\begin{aligned}
& \operatorname{Re}\left\{ 1+\sum_{k=1}^{\infty} \frac{(1+\beta(1+|B|))\,(2^m-2^n)+|B2^m-A2^n|}{(1+\beta(1+|B|))\,(2^m-2^n)+|B2^m-A2^n|+(A-B)} a_k z^k \right\} \\
= \ & \operatorname{Re}\left\{ 1+ \frac{(1+\beta(1+|B|))\,(2^m-2^n)+|B2^m-A2^n|}{(1+\beta(1+|B|))\,(2^m-2^n)+|B2^m-A2^n|+(A-B)} z + \right. \\
& \left. \frac{1}{(1+\beta(1+|B|))(2^m-2^n)+|B2^m-A2^n|+(A-B)} \sum_{k=2}^{\infty} \left[(1+\beta(1+|B|))\,(2^m-2^n)+|B2^m-A2^n|\right] a_k z^k \right\} \\
\geq \ & 1 - \frac{(1+\beta(1+|B|))\,(2^m-2^n)+|B2^m-A2^n|}{(1+\beta(1+|B|))\,(2^m-2^n)+|B2^m-A2^n|+(A-B)} r - \\
& \frac{1}{(1+\beta(1+|B|))(2^m-2^n)+|B2^m-A2^n|+(A-B)} \sum_{k=2}^{\infty} \left[(1+\beta\,(1+|B|))\,(k^m-k^n)+|Bk^m-Ak^n|\right] |a_k|\, r^k \\
> \ & 1 - \frac{(1+\beta(1+|B|))(2^m-2^n)+|B2^m-A2^n|}{(1+\beta(1+|B|))(2^m-2^n)+|B2^m-A2^n|+(A-B)} r - \frac{A-B}{(1+\beta(1+|B|))(2^m-2^n)+|B2^m-A2^n|+(A-B)} r \\
= \ & 1-r > 0 \ (|z| = r < 1),
\end{aligned}$$

where we have also made use of assertion (2.2) of Lemma 2. Thus (2.6) holds true in U, this proves the inequality (2.3). The inequality (2.4) follows from (2.3) by taking the convex function $h(z) = \dfrac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k$. To prove the sharpness of the constant $\frac{(1+\beta(1+|B|))(2^m-2^n)+|B2^m-A2^n|}{2[(1+\beta(1+|B|))(2^m-2^n)+|B2^m-A2^n|+(A-B)]}$, we consider the function $f_0(z) \in U^*_{m,n}(\beta, A, B)$ given by

$$f_0(z) = z - \frac{A-B}{(1+\beta(1+|B|))\,(2^m-2^n)+|B2^m-A2^n|} z^2. \tag{2.7}$$

Thus from (2.3), we have

$$\frac{(1+\beta(1+|B|))\,(2^m-2^n)+|B2^m-A2^n|}{2\left[(1+\beta(1+|B|))\,(2^m-2^n)+|B2^m-A2^n|+(A-B)\right]} f_0(z) \prec \frac{z}{1-z} \ (z \in U). \tag{2.8}$$

Moreover, it can easily be verified for the function $f_0(z)$ given by (2.7) that

$$\min_{|z|\leq r} \left\{ \operatorname{Re} \frac{(1+\beta(1+|B|))\,(2^m-2^n)+|B2^m-A2^n|}{2\left[(1+\beta(1+|B|))\,(2^m-2^n)+|B2^m-A2^n|+(A-B)\right]} f_0(z) \right\} = -\frac{1}{2}. \tag{2.9}$$

This shows that the constant $\frac{(1+\beta(1+|B|))(2^m-2^n)+|B2^m-A2^n|}{2[(1+\beta(1+|B|))(2^m-2^n)+|B2^m-A2^n|+(A-B)]}$ is the best possible. This completes the proof of Theorem 1. ∎

Remark 2.

(i) Taking $A = 1 - 2\alpha$ $(0 \leq \alpha < 1)$, and $B = -1$ in Theorem 1, we correct the result obtained by Srivastava and Eker [15, Theorem 1];
(ii) Taking $A = 1 - 2\alpha$ $(0 \leq \alpha < 1)$, $B = -1$ and $m = n + 1 (n \in \mathbb{N}_0)$, in Theorem 1, we obtain the result obtained by Aouf et al. [2, Corollary 4];
(iii) Taking $A = 1 - 2\alpha$ $(0 \leq \alpha < 1)$, $B = -1$, $m = 1$ and $n = 0$ in Theorem 1, we obtain the result obtained by Frasin [5, Corollary 2.2];
(iv) Taking $A = 1 - 2\alpha$ $(0 \leq \alpha < 1)$, $B = -1$, $m = 2$ and $n = 1$ in Theorem 1, we obtain the result obtained by Frasin [5, Corollary 2.5];
(v) Taking $A = 1 - 2\alpha$ $(0 \leq \alpha < 1)$, $B = -1$, $\beta = 0$, $m = 1$ and $n = 0$ in Theorem 1, we obtain the result obtained by Frasin [5, Corollary 2.3];
(vi) Taking $A = 1 - 2\alpha$ $(0 \leq \alpha < 1)$, $B = -1$, $\beta = 0$, $m = 2$ and $n = 1$ in Theorem 1, we obtain the result obtained by Frasin [5, Corollary 2.6];
(vii) Taking $A = 1$, $B = -1$, $\beta = 0$, $m = 1$ and $n = 0$ in Theorem 1, we obtain the result obtained by Frasin [5, Corollary 2.4];
(viii) Taking $A = 1$, $B = -1$, $\beta = 0$, $m = 2$ and $n = 1$ in Theorem 1, we obtain the result obtained by Frasin [5, Corollary 2.7];
(ix) Taking $A = 1 - 2\alpha$ $(0 \leq \alpha < 1)$, $B = -1$, $\beta = 1$, $m = 1$ and $n = 0$ in Theorem 1, we obtain the result obtained by Aouf et al. [2, Corollary 1];
(x) Taking $A = 1 - 2\alpha$ $(0 \leq \alpha < 1)$, $B = -1$, $\beta = 1$, $m = 2$ and $n = 1$ in Theorem 1, we obtain the result obtained by Aouf et al. [2, Corollary 2];
(xi) Taking $A = 1$, $B = -1$, $m = 2$ and $n = 1$ in Theorem 1, we obtain the result obtained by Aouf et al. [2, Corollary 3];

Also, we establish subordination results for the associated subclasses $S^{**}(A, B)$, $K^*(A, B)$ and $U^*(m, n; A, B)$, whose coefficients satisfy the inequality (2.2) in the special cases as mentioned.

Putting $\beta = 0$, $m = 1$ and $n = 0$ in Theorem 1, we have

Corollary 4. *Let the function $f(z)$ defined by* (1.1) *be in the class $S^{**}(A, B)$ and suppose that $h(z) \in K$. Then*

$$\frac{1 + |2B - A|}{2\left[1 + |2B - A| + (A - B)\right]} (f * h)(z) \prec h(z) \quad (z \in U), \tag{2.10}$$

and

$$\operatorname{Re}\{f(z)\} > -\frac{1 + |2B - A| + (A - B)}{1 + |2B - A|} \quad (z \in U). \tag{2.11}$$

The constant factor $\frac{1+|2B-A|}{2[1+|2B-A|+(A-B)]}$ in the subordination result (2.10) *cannot be replaced by a larger one.*

Putting $\beta = 0$, $m = 2$ and $n = 1$ in Theorem 1, we have

Corollary 5. *Let the function $f(z)$ defined by* (1.1) *be in the class $K^*(A, B)$ and suppose that $h(z) \in K$. Then*

$$\frac{1 + |2B - A|}{2 + 2|2B - A| + (A - B)} (f * h)(z) \prec h(z) \quad (z \in U), \tag{2.12}$$

and

$$\operatorname{Re}\{f(z)\} > -\frac{2 + 2|2B - A| + (A - B)}{2 + 2|2B - A|} \quad (z \in U). \tag{2.13}$$

The constant factor $\frac{1+|2B-A|}{2+2|2B-A|+(A-B)}$ in the subordination result (2.12) *cannot be replaced by a larger one.*

Putting $\beta = 0$ in Theorem 1, we have

Corollary 6. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $U^*(m, n; A, B)$ *and suppose that* $h(z) \in K$. *Then*

$$\frac{(2^m - 2^n) + |B2^m - A2^n|}{2\left[(2^m - 2^n) + |B2^m - A2^n| + (A - B)\right]} (f * h)(z) \prec h(z) \quad (z \in U), \tag{2.14}$$

and

$$\operatorname{Re}\{f(z)\} > -\frac{(2^m - 2^n) + |B2^m - A2^n| + (A - B)}{(2^m - 2^n) + |B2^m - A2^n|} \ (z \in U)\ . \tag{2.15}$$

The constant factor $\frac{(2^m - 2^n)+|B2^m - A2^n|}{2[(2^m - 2^n)+|B2^m - A2^n|+(A-B)]}$ *in the subordination result* (2.14) *cannot be replaced by a larger one.*

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