Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 4 Issue 1(2012), Pages 24-28.

QUASI-HADAMARD PRODUCT OF ANALYTIC P-VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. The authors establish certain results concerning the quasi-Hadamard product of analytic and p-valent functions with negative coefficients analogous to the results due to Vinod Kumar (J. Math. Anal. Appl. 113(1986), 230-234 and 126(1987), 70-77).

1. Introduction

Let $T(p)$ denote the class of functions of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (a_{p+n} \geq 0; p \in N = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$.

A functions $f(z)$ belonging to the class $T(p)$ is said to be in the class $F_p(\lambda, \alpha)$ if and only if

$$\mathrm{Re}\left\{(1-\lambda)\frac{f(z)}{z^p} + \lambda \frac{f'(z)}{pz^{p-1}}\right\} > \frac{\alpha}{p} \tag{1.2}$$

for some α $(0 \leq \alpha < p)$, λ $(\lambda \geq 0)$ and for all $z \in U$. The class $F_p(\lambda, \alpha)$ was studied by Lee et al. [7] and Aouf and Darwish [3].

Throughout the paper, let the functions of the form

$$f(z) = a_p z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (a_p > 0; a_{p+n} \geq 0; p \in N)\ , \tag{1.3}$$

$$f_i(z) = a_{p,i} z^p - \sum_{n=1}^{\infty} a_{p+n,i} z^{p+n} \qquad (a_{p,i} > 0; a_{p+n} \geq 0; p \in N)\ , \tag{1.4}$$

$$g(z) = b_p z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n} \qquad (b_p > 0; b_{p+n} \geq 0; p \in N)\ , \tag{1.5}$$

2000 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Analytic, p-valent, quasi-Hadamard product.

Submitted September 26, 2011. Published December 6, 2011.

and

$$g_j(z) = b_{p,j}z^p - \sum_{n=1}^{\infty} b_{p+n,j}z^{p+n} \qquad (b_{p,j} > 0; b_{p+n,j} \geq 0; p \in N) \tag{1.6}$$

be analytic and p-valent in U.

Let $F_p^*(\lambda, \alpha)$ denote the class of functions $f(z)$ of the form (1.3) and satisfying (1.2) for some $\lambda,\ \alpha$ and for all $z \in U$. Also let $G_p^*(\lambda, \alpha)$ denote the class of functions of the form (1.3) such that $\frac{zf'(z)}{p} \in F_p^*(\lambda, \alpha)$.

We note that when $a_p = 1$, the class $G_p^*(\lambda, \alpha) = G_p(\lambda, \alpha)$ was studied by Aouf [2].

Using similar arguments as given by Lee et al. [7] and Aouf and Darwish [3] we can easily prove the following analogous results for functions in the classes $F_p^*(\lambda, \alpha)$ and $G_p^*(\lambda, \alpha)$.

A function $f(z)$ defined by (1.3) belongs to the class $F_p^*(\lambda, \alpha)$ if and only if

$$\sum_{n=1}^{\infty} (p + \lambda n)a_{p+n} \leq (p - \alpha)a_p \tag{1.7}$$

and $f(z)$ defined by (1.3) belongs to the class $G_p^*(\lambda, \alpha)$ if and only if

$$\sum_{n=1}^{\infty} \left(\frac{p+n}{p}\right)(p + \lambda n)a_{p+n} \leq (p - \alpha)a_p\ . \tag{1.8}$$

We now introduce the following class of analytic and p-valent functions which plays an important role in the discussion that follows:

A function $f(z)$ defined by (1.3) belongs to the class $F_{p,k}^*(\lambda, \alpha)$ if and only if

$$\sum_{n=1}^{\infty} \left(\frac{p+n}{p}\right)^k (p + \lambda n)a_{p+n} \leq (p - \alpha)a_p\ , \tag{1.9}$$

where $0 \leq \alpha < p,\ \lambda \geq 0$ and k is any fixed nonnegative real number.

We note that, for every nonnegative real number k, the class $F_{p,k}^*(\lambda, \alpha)$ is nonempty as the functions of the form

$$f(z) = a_p z^p - \sum_{n=1}^{\infty} \frac{(p-\alpha)a_p}{\left(\frac{p+n}{p}\right)^k (p + \lambda n)} \mu_{p+n} z^{p+n}\ , \tag{1.10}$$

where $a_p > 0,\ \mu_{p+n} \geq 0$ and $\sum\limits_{n=1}^{\infty} \mu_{p+n} \leq 1$, satisfy the inequality (1.9). It is evident that $F_{p,1}^*(\lambda, \alpha) \equiv G_p^*(\lambda, \alpha)$ and, for $k = 0$, $F_{p,0}^*(\lambda, \alpha)$ is identical to $F_p^*(\lambda, \alpha)$. Further, $F_{p,k}^*(\lambda, \alpha) \subset F_{p,h}^*(\lambda, \alpha)$ if $k > h \geq 0$, the containment being proper. Whence, for any positive integer k, we have the inclusion relation

$$F_{p,k}^*(\lambda, \alpha) \subset F_{p,k-1}^*(\lambda, \alpha) \subset ... \subset F_{p,2}^*(\lambda, \alpha) \subset G_p^*(\lambda, \alpha) \subset F_p^*(\lambda, \alpha)\ .$$

Let us define the quasi-Hadamard product of the functions $f(z)$ defined by (1.3) and $g(z)$ defined by (1.5) by

$$f * g(z) = a_p b_p z^p - \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n}\ . \tag{1.11}$$

Similarly, we can define the quasi-Hadamard product of more than two functions.

In this paper, we establish certain results concerning the quasi-Hadamard product of functions in the classes $F^*_{p,k}(\lambda,\alpha)$, $F^*_p(\lambda,\alpha)$ and $G^*_p(\lambda,\alpha)$ analogous to the results due to Kumar ([8] and [9]), Aouf et al. [4], Aouf [1], Darwish [5] and Hossen [6].

2. The main theorems

Unless otherwise mentioned we shall assume throughout the following resultes that $\lambda \geq 1, 0 \leq \alpha < p$ and $p \in N$.

Theorem 1. *Let the functions $f_i(z)$ defined by (1.4) be in the class $G^*_p(\lambda,\alpha)$ for every $i = 1, 2, ..., m$; and let the functions $g_j(z)$ defined by (1.6) be in the class $F^*_p(\lambda,\alpha)$ for every $j = 1, 2, ..., q$. Then, the quasi-Hadamard product $f_1 * f_2 * ... * f_m * g_1 * g_2 * ... * g_q(z)$ belongs to the class $F^*_{p,2m+q-1}(\lambda,\alpha)$.*

Proof. We denote quasi-Hadamard product $f_1 * f_2 * ... * f_m * g_1 * g_2 * ... * g_q(z)$ by the function $h(z)$, for the sake of convenience.

Clearly,

$$h(z) = \left\{\prod_{i=1}^{m} a_{p,i} \prod_{j=1}^{q} b_{p,j}\right\} z^p - \sum_{n=1}^{\infty}\left\{\prod_{i=1}^{m} a_{p+n,i} \prod_{j=1}^{q} b_{p+n,j}\right\} z^{p+n}. \tag{2.1}$$

To prove the theorem, we need to show that

$$\sum_{n=1}^{\infty}\left(\frac{p+n}{p}\right)^{2m+q-1}(p+\lambda n)\left\{\prod_{i=1}^{m} a_{p+n,i} \prod_{j=1}^{q} b_{p+n,j}\right\}$$
$$\leq (p-\alpha)\left\{\prod_{i=1}^{m} a_{p,i} \prod_{j=1}^{q} b_{p,j}\right\} \tag{2.2}$$

Since $f_i(z) \in G^*_p(\lambda,\alpha)$, we have

$$\sum_{n=1}^{\infty}\left(\frac{p+n}{p}\right)(p+\lambda n)a_{p+n,i} \leq (p-\alpha)a_{p,i}\ , \tag{2.3}$$

for every $i = 1, 2, ..., m$. Therefore

$$\left(\frac{p+n}{p}\right)(p+\lambda n)a_{p+n,i} \leq (p-\alpha)a_{p,i}\ ,$$

or

$$a_{p+n,i} \leq \frac{(p-\alpha)}{\left(\frac{p+n}{p}\right)(p+\lambda n)}a_{p,i}\ ,$$

for every $i = 1, 2, ..., m$. The right-hand expression of this last inequality is not greater than $\frac{a_{p,i}}{(\frac{p+n}{p})^2}$. Hence

$$a_{p+n,i} \leq \frac{a_{p,i}}{\left(\frac{p+n}{p}\right)^2}\ , \tag{2.4}$$

for every $i = 1, 2, ..., q$. Similarly, for $g_j(z) \in F^*_p(\lambda,\alpha)$, we have

$$\sum_{n=1}^{\infty}(p+\lambda n)b_{p+n,j} \leq (p-\alpha)b_{p,j} \tag{2.5}$$

for every $j = 1, 2, ..., q$. Whence we obtain

$$b_{p+n,j} \leq \frac{b_{p,j}}{\left(\frac{p+n}{p}\right)} \ , \tag{2.6}$$

for every $j = 1, 2, ..., q$.

Using (2.4) for for every $i = 1, 2, ..., m$, (2.6) for $j = 1, 2, ..., q-1$, and (2.5) for $j = q$, we get

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left[\left(\frac{p+n}{p}\right)^{2m+q-1} (p+\lambda n) \left\{ \prod_{i=1}^{m} a_{p+n,i} \prod_{j=1}^{q} b_{p+n,j} \right\} \right] \\
\leq \ & \sum_{n=1}^{\infty} \left[\left(\frac{p+n}{p}\right)^{2m+q-1} (p+\lambda n) \left(\frac{p+n}{p}\right)^{-2m} \left(\frac{p+n}{p}\right)^{-(q-1)} \right. \cdot \\
& \left. \cdot \left(\prod_{i=1}^{m} a_{p,i} \prod_{j=1}^{q-1} b_{p,j} \right) \right] b_{p+n,q} \\
= \ & \sum_{n=1}^{\infty} [(p+\lambda n) b_{p+n,q}] \left(\prod_{i=1}^{m} a_{p,i} \prod_{j=1}^{q-1} b_{p,j} \right) \\
\leq \ & (p-\alpha) \left(\prod_{i=1}^{m} a_{p,i} \prod_{j=1}^{q} b_{p,j} \right) \ .
\end{aligned}$$

Hence $h(z) \in F^{*}_{p,2m+q-1}(\lambda, \alpha)$. This completes the proof of Theorem 1.

We note that the required estimate can be also obtained by using (2.4) for $i = 1, 2, ..., m-1$, (2.6) for $j = 1, 2, ..., q$ and (2.3) for $i = m$.

Now we discuss the applications of Theorem 1. Taking into account the quasi-Hadamard product of functions $f_1(z), f_2(z), ..., f_m(z)$ only, in the proof of Theorem 1, and using (2.4) for $i = 1, 2, ..., m-1$, and (2.3) for $i = m$, we are led to

Corollary 1. *Let the functions $f_i(z)$ defined by (1.4) belongs to the class $G^{*}_{p}(\lambda, \alpha)$ for every $i = 1, 2, ..., m$. Then the quasi-Hadamard product $f_1 * f_2 * ... * f_m(z)$ belongs to the class $F^{*}_{p,2m-1}(\lambda, \alpha)$.*

Next, taking into account the quasi-Hadamard product of the functions $g_1(z), g_2(z), ..., g_q(z)$ only, in the proof of Theorem 1, and using (2.6) for $j = 1, 2, ..., q-1$, and (2.5) for $j = q$, we are led to

Corollary 2. *Let the functions $g_j(z)$ defined by (1.6) belongs to the class $F^{*}_{p}(\lambda, \alpha)$ for every $j = 1, 2, ..., q$. Then the quasi-Hadamard product $g_1 * g_2 * ... * g_q(z)$ belongs to the class $F^{*}_{p,q-1}(\lambda, \alpha)$.*

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