Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 4 Issue 1 (2012), Pages 29-61.

HARMONIC ANALYSIS AND UNCERTAINTY PRINCIPLES FOR INTEGRAL TRANSFORMS GENERALIZING THE SPHERICAL MEAN OPERATOR

(COMMUNICATED BY HUSEIN BOR)

KHALED HLEILI, SLIM OMRI, LAKHDAR TANNECH RACHDI

Abstract. For $m, n \in \mathbb{N}$; $m \geqslant n \geqslant 1$, we define an integral transform $\mathscr{R}_{m,n}$ that generalizes the spherical mean operator. We establish many harmonic analysis results for the Fourier transform $\mathscr{F}_{m,n}$ connected with $\mathscr{R}_{m,n}$. Next, we establish inversion formulas for the operator $\mathscr{R}_{m,n}$ and its dual ${}^t\mathscr{R}_{m,n}$. Finally, we prove some uncertainty principles related to the Fourier transform $\mathscr{F}_{m,n}$.

1. Introduction

The spherical mean operator $\mathscr{R}$ is defined, for a function f on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable, as

$$\mathscr{R}(f)(r,x) = \int_{S^n} f((0,x) + r\omega) d\sigma_n(\omega); \quad (r,x) \in \mathbb{R} \times \mathbb{R}^n.$$

where S^n is the unit sphere: $S^n = \{\omega \in \mathbb{R} \times \mathbb{R}^n \ ; \ |\omega| = 1\}$ and σ_n is the surface measure on S^n normalized to have total measure one.

The dual of the spherical mean operator ${}^t\mathscr{R}$ is defined by

$${}^t\mathscr{R}(g)(r,x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} g(\sqrt{r^2 + |x-y|^2}, y) dy,$$

where dy is the Lebesgue measure on $\mathbb{R}^n$.

The spherical mean operator $\mathscr{R}$ and its dual ${}^t\mathscr{R}$ play an important role and have many applications, for example, in image processing of so-called synthetic aperture radar (SAR) data [13, 14], or in the linearized inverse scattering problem in acoustics [10]. Many aspects of such operator have been studied [2, 7, 16, 19, 20, 23, 24, 26].

In [3] Baccar, Ben Hamadi and Rachdi defined and studied the Riemann-Liouville operator $\mathscr{R}_\alpha$ which generalizes the spherical mean operator in dimension two, and

2000 *Mathematics Subject Classification.* 42B10, 42B35, 43A32.

Key words and phrases. Integral transform; Fourier transform; Inversion formula; Uncertainty principles.

Submitted November 2, 2011. Published December 11, 2011.

in the same work, the authors established several inversions formula connected with the operator $\mathscr{R}_\alpha$. In [15] Hleili, Omri and Rachdi proved many importants uncertainty principles for the same operator $\mathscr{R}_\alpha$.

Our purpose in this work is to define and study a class of integral transforms which generalizes the spherical mean operator $\mathscr{R}$ in dimension n, and to establish several uncertainty principles for this class of integral transforms.

Namely, for every integers $m \geqslant n \geqslant 1$, we define the integral transform $\mathscr{R}_{m,n}$ by

$$\mathscr{R}_{m,n}(f)(r,x) = \begin{cases} \dfrac{2\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m-n}{2})\Gamma(\frac{n+1}{2})} \displaystyle\int_0^1 \int_{S^n} f((0,x)+rt\omega) & \\ \qquad\qquad \times (1-t^2)^{\frac{m-n}{2}-1} t^n dt d\sigma_n(\omega), & \text{if } m > n, \\ \displaystyle\int_{S^n} f((0,x)+r\omega) d\sigma_n(\omega), & \text{if } m = n, \end{cases}$$

where f is a continuous function on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable.

The dual operator ${}^t\mathscr{R}_{m,n}$ is defined by

$${}^t\mathscr{R}_{m,n}(f)(s,y) = \begin{cases} \dfrac{1}{2^{\frac{m}{2}}\Gamma(\frac{m-n}{2})\pi^{\frac{n}{2}}} \displaystyle\int\int_{s^2+|z|^2<r^2} f(r,z+y) & \\ \qquad\qquad \times (r^2-s^2-|z|^2)^{\frac{m-n}{2}-1} r dr dz & \text{if } m > n, \\ \dfrac{1}{(2\pi)^{\frac{n}{2}}} \displaystyle\int_{\mathbb{R}^n} f(\sqrt{s^2+|x-y|^2}, x) dx & \text{if } m = n. \end{cases}$$

We associate to the operator $\mathscr{R}_{m,n}$, the Fourier transform $\mathscr{F}_{m,n}$ defined by

$$\forall (\mu,\lambda) \in \Upsilon \ ; \ \mathscr{F}_{m,n}(f)(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)\varphi_{\mu,\lambda}(r,x)\, d\nu_{m,n}(r,x),$$

where

- $\varphi_{\mu,\lambda}(r,x) = \mathscr{R}_{m,n}\left(\cos(\mu.)e^{-i\langle\lambda|.\rangle}\right)(r,x) = j_{\frac{m-1}{2}}\left(r\sqrt{\mu^2+\lambda^2}\right)e^{-i\langle\lambda|x\rangle}$, and $j_{\frac{m-1}{2}}$, is the modified Bessel function of the first kind and index $\dfrac{m-1}{2}$.
- $d\nu_{m,n}$ is the measure defined on $[0,+\infty[\times\mathbb{R}^n$, by

$$d\nu_{m,n}(r,x) = \frac{r^m dr}{2^{\frac{m-1}{2}}\Gamma(\frac{m+1}{2})} \otimes \frac{dx}{(2\pi)^{\frac{n}{2}}}.$$

- Υ is the set given by

$$\Upsilon = \mathbb{R}\times\mathbb{R}^n \cup \left\{(it,x) \mid (t,x)\in\mathbb{R}\times\mathbb{R}^n \ , \ |t| \leqslant |x|\right\}.$$

Then we have established the harmonic analysis related to the Fourier transform $\mathscr{F}_{m,n}$. Next, we define and study the fractional powers of the Bessel operator $\ell_\alpha = \dfrac{\partial^2}{\partial r^2} + \dfrac{2\alpha+1}{r}\dfrac{\partial}{\partial r}$, $\alpha \geqslant -\frac{1}{2}$, and the Laplacian operator $\Delta = \dfrac{\partial^2}{\partial r^2} + \displaystyle\sum_{i=1}^n \dfrac{\partial^2}{\partial x_i^2}$.

Using these fractional powers, we determine some subspaces of the schwartz space $S_e(\mathbb{R}\times\mathbb{R}^n)$ (the space of infinitely differentiable functions on $\mathbb{R}\times\mathbb{R}^n$ rapidly decreasing together with every their derivatives and even with respect to the first variable) where $\mathscr{R}_{m,n}$ and ${}^t\mathscr{R}_{m,n}$ are topological isomorphisms and we give the inverse isomorphisms, more precisely we have the following inversion formulas

$$f = K^1_{m,n}\, {}^t\mathscr{R}_{m,n}\mathscr{R}_{m,n}(f),$$
$$g = \mathscr{R}_{m,n}K^1_{m,n}\, {}^t\mathscr{R}_{m,n}(g).$$

and

$$f = {}^t\mathscr{R}_{m,n}K^2_{m,n}\mathscr{R}_{m,n}(f),$$
$$g = K^2_{m,n}\mathscr{R}_{m,n}\, {}^t\mathscr{R}_{m,n}(g),$$

where the operators $K^1_{m,n}$ and $K^2_{m,n}$ are expressed in terms of the fractional powers of ℓ_α and Δ.

On the other hand, the uncertainty principles play an important role in harmonic analysis and have been studied by many authors and from many points of view [11, 12].These principles state that a function f and its Fourier transform $\widehat{f}$ cannot be simultaneously sharply localized.Theorems of Hardy, Morgan, Beurling,... are established for several Fourier transforms [6, 18, 21, 22].

In this context, we have studied and established some important uncertainty principles for the Fourier transform $\mathscr{F}_{m,n}$. More precisely we have proved the following Beurling-Hrmander type theorem

• Let $f \in L^2(d\nu_{m,n})$, and let d be a real number, $d \geqslant 0$. If

$$\int\int_{\Upsilon_+}\int_0^{+\infty}\int_{\mathbb{R}^n} \frac{|f(r,x)||\mathscr{F}_{m,n}(f)(\mu,\lambda)|}{(1+|(r,x)|+|\theta(\mu,\lambda)|)^d} e^{|(r,x)||\theta(\mu,\lambda)|}\, d\nu_{m,n}(r,x)\, d\gamma_{m,n}(\mu,\lambda) < +\infty.$$

Then there exist a positive constant a and a polynomial P on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable, such that

$$\forall (r,x) \in \mathbb{R}\times\mathbb{R}^n, \quad f(r,x) = P(r,x)e^{-a(r^2+|x|^2)},$$

with $deg(P) < \dfrac{d-(m+n+1)}{2}$.

Where

• $d\gamma_{m,n}$ is the spectral measure that will be defined in the second section.

• Υ_+ is the subset of Υ, given by

$$\Upsilon_+ = [0,+\infty[\times\mathbb{R}^n \cup \big\{(it,x) \mid (t,x) \in [0,+\infty[\times\mathbb{R}^n\ ,\ t \leqslant |x|\big\}.$$

• θ is the bijective function defined on the set Υ_+ by

$$\theta(\mu,\lambda) = (\sqrt{\mu^2+|\lambda|^2}, \lambda).$$

The precedent theorem allows us to establish the Gelfand-Shilov and Cowling-Price theorems.

• (Gelfand-Shilov) Let p, q be two conjugate exponents, $p, q \in]1,+\infty[$ and let ξ, η be non negative real numbers such that $\xi\eta \geqslant 1$. Let f be a measurable function on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable, such that $f \in L^2(d\nu_{m,n})$.

If

$$\int_0^{+\infty}\int_{\mathbb{R}^n} \frac{|f(r,x)|e^{\frac{\xi^p|(r,x)|^p}{p}}}{(1+|(r,x)|)^d}\, d\nu_{m,n}(r,x) < +\infty,$$

and

$$\iint_{\Upsilon_+} \frac{|\mathscr{F}_{m,n}(f)(\mu,\lambda)|e^{\frac{\eta^q|\theta(\mu,\lambda)|^q}{q}}}{(1+|\theta(\mu,\lambda)|)^d}\, d\gamma_{m,n}(\mu,\lambda) < +\infty\ ;\ d \geqslant 0.$$

Then

i) For $d \leqslant \dfrac{m+n+1}{2}$, $f = 0$.

ii) For $d > \dfrac{m+n+1}{2}$, we have

a) $f = 0$ for $\xi\eta > 1$.

b) $f = 0$ for $\xi\eta = 1$, and $p \neq 2$.

c) $f(r,x) = P(r,x)e^{-a(r^2+|x|^2)}$ for $\xi\eta = 1$ and $p = q = 2$,

where $a > 0$ and P is a polynomial on $\mathbb{R}\times\mathbb{R}^n$ even with respect to the first variable, with $deg(P) < d - \dfrac{m+n+1}{2}$.

• (Cowling-Price) Let $\xi, \eta, \omega_1, \omega_2$ be non negative real numbers such that $\xi\eta \geqslant \dfrac{1}{4}$. Let p, q be two exponents, $p, q \in [1, +\infty]$, and let f be a measurable function on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable such that $f \in L^2(d\nu_{m,n})$.
If

$$\left\| \frac{e^{\xi|(.,.)|^2}}{(1+|(.,.)|)^{\omega_1}} f \right\|_{p,\nu_{m,n}} < +\infty,$$

and

$$\left\| \frac{e^{\eta|\theta(.,.)|^2}}{(1+|\theta(.,.)|)^{\omega_2}} \mathscr{F}_{m,n}(f) \right\|_{q,\gamma_{m,n}} < +\infty,$$

then

i) For $\xi\eta > \dfrac{1}{4}$, $f = 0$.

ii) For $\xi\eta = \dfrac{1}{4}$, there exist a positive constant a and a polynomial P on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable, such that

$$f(r,x) = P(r,x)e^{-a(r^2+|x|^2)}.$$

2. The operators $\mathscr{R}_{m,n}$ and its dual ${}^t\mathscr{R}_{m,n}$

In this section, we define the operators $\mathscr{R}_{m,n}$ and its dual ${}^t\mathscr{R}_{m,n}$ and we give some properties.
Let m, n be two integers such that $m \geqslant n \geqslant 1$.

For every $(\mu,\lambda) \in \mathbb{C}\times\mathbb{C}^n$, the system

$$\begin{cases} \dfrac{\partial}{\partial x_j} u(r,x) = -i\lambda_j u(r,x), \text{if } 1 \leqslant j \leqslant n, \\ \Xi u(r,x) = -\mu^2 u(r,x), \\ u(0,0) = 1; \dfrac{\partial}{\partial r} u(0,x) = 0, \forall x \in \mathbb{R}^n, \end{cases}$$

admits a unique solution $\varphi_{\mu,\lambda}$, given by

$$\forall (r,x) \in \mathbb{R}\times\mathbb{R}^n; \quad \varphi_{\mu,\lambda}(r,x) = j_{\frac{m-1}{2}}(r\sqrt{\mu^2+\lambda^2})e^{-i\langle\lambda|x\rangle}, \tag{2.1}$$

where
- $\lambda^2 = \lambda_1^2 + ... + \lambda_n^2; \quad \lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{C}^n$,
- $\langle \lambda | x \rangle = \lambda_1 x_1 + ... + \lambda_n x_n; \quad x = (x_1, ..., x_n) \in \mathbb{R}^n$,
- Ξ is the operator given by

$$\Xi = \frac{\partial^2}{\partial r^2} + \frac{m}{r}\frac{\partial}{\partial r} - \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}. \tag{2.2}$$

- $j_{\frac{m-1}{2}}$ is the modified Bessel function defined by

$$j_{\frac{m-1}{2}}(z) = \frac{2^{\frac{m-1}{2}}\Gamma(\frac{m+1}{2})}{z^{\frac{m-1}{2}}} J_{\frac{m-1}{2}}(z) = \Gamma(\frac{m+1}{2}) \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(\frac{m+1}{2}+k)} (\frac{z}{2})^{2k}, \quad z \in \mathbb{C}, \tag{2.3}$$

and $J_{\frac{m-1}{2}}$ is the Bessel function of the first kind and index $\frac{m-1}{2}$ [8, 9, 17, 25]. The modified Bessel function $j_{\frac{m-1}{2}}$ has the following integral representation [1, 17], for every $z \in \mathbb{C}$, we have

$$j_{\frac{m-1}{2}}(z) = \frac{2\Gamma(\frac{m+1}{2})}{\sqrt{\pi}\Gamma(\frac{m}{2})} \int_0^1 (1-t^2)^{\frac{m}{2}-1} \cos(zt) dt. \tag{2.4}$$

From relation (2.4), we deduce that for every $z \in \mathbb{C}$, and for every $k \in \mathbb{N}$, we have

$$\left| j^{(k)}_{\frac{m-1}{2}}(z) \right| \leqslant e^{|Im(z)|}. \tag{2.5}$$

From the properties of the modified Bessel function $j_{\frac{m-1}{2}}$, we deduce that the eigenfunction $\varphi_{\mu,\lambda}$ is bounded on $\mathbb{R} \times \mathbb{R}^n$ if and only if (μ, λ) belongs to the set

$$\Upsilon = \mathbb{R} \times \mathbb{R}^n \cup \big\{ (it, x) \mid (t, x) \in \mathbb{R} \times \mathbb{R}^n \ , \ |t| \leqslant |x| \big\}, \tag{2.6}$$

and in this case

$$\sup_{(r,x) \in \mathbb{R} \times \mathbb{R}^n} |\varphi_{\mu,\lambda}(r, x)| = 1, \tag{2.7}$$

where $|x| = \sqrt{x_1^2 + ... + x_n^2}; \quad x = (x_1, ..., x_n) \in \mathbb{R}^n$.

For real numbers $a \geqslant b \geqslant -1/2$, we define the Sonine transform $S_{a,b}$ by

$$S_{a,b}(f)(r, x) = \begin{cases} \dfrac{2\Gamma(a+1)}{\Gamma(a-b)\Gamma(b+1)} \displaystyle\int_0^1 f(rt, x)(1-t^2)^{a-b-1} t^{2b+1} dt, & \text{if } a > b; \\ f(r, x), & \text{if } a = b. \end{cases} \tag{2.8}$$

It is well known, see for example [1, 17], that for every $(\mu, \lambda) \in \mathbb{C} \times \mathbb{C}^n$, we have

$$\forall (r, x) \in \mathbb{R} \times \mathbb{R}^n, \ S_{a,b}\big(j_b(\mu.)e^{-i\langle \lambda | . \rangle}\big)(r, x) = j_a(r\mu)e^{-i\langle \lambda | x \rangle}. \tag{2.9}$$

Proposition 2.1. *For every $(\mu, \lambda) \in \mathbb{C} \times \mathbb{C}^n$, the eigenfunction $\varphi_{\mu,\lambda}$ has the following integral representation*

$$\varphi_{\mu,\lambda}(r, x) = \begin{cases} \dfrac{2\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m-n}{2})\Gamma(\frac{n+1}{2})} \displaystyle\int_0^1 \int_{S^n} (1-t^2)^{\frac{m-n}{2}-1} \cos(rt\mu\eta) \\ \qquad \times e^{-i\langle \lambda | x + rt\xi \rangle} t^n dt d\sigma_n(\eta, \xi), & \textit{if } m > n, \\ \displaystyle\int_{S^n} \cos(r\mu\eta) e^{-i\langle \lambda | x + r\xi \rangle} d\sigma_n(\eta, \xi), & \textit{if } m = n, \end{cases} \tag{2.10}$$

where $S^n = \{(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n; \eta^2+|\xi|^2=1\}$ *is the unit sphere of* $\mathbb{R}\times\mathbb{R}^n$*, and* σ_n *is the surface measure on* S^n *normalized to have total measure one.*

Proof. • If $m>n$, it is well known that for every $(\mu,\lambda)\in\mathbb{C}\times\mathbb{C}^n$, we have

$$j_{\frac{n-1}{2}}(r\sqrt{\mu^2+\lambda^2}) = \int_{S^n}\cos(r\mu\eta)e^{-i\langle\lambda|r\xi\rangle}d\sigma_n(\eta,\xi). \tag{2.11}$$

On the other hand and according to relation (2.9), we have for every $(\mu,\lambda)\in\mathbb{C}\times\mathbb{C}^n$, and for every $(r,x)\in\mathbb{R}\times\mathbb{R}^n$

$$S_{\frac{m-1}{2},\frac{n-1}{2}}\big(j_{\frac{n-1}{2}}(.\sqrt{\mu^2+\lambda^2})e^{-i\langle\lambda|.\rangle}\big)(r,x) = j_{\frac{m-1}{2}}(r\sqrt{\mu^2+\lambda^2})e^{-i\langle\lambda|x\rangle}. \tag{2.12}$$

Hence by combining relations (2.8), (2.11) and (2.12), we get for every $(\mu,\lambda)\in\mathbb{C}\times\mathbb{C}^n$ and for every $(r,x)\in\mathbb{R}\times\mathbb{R}^n$,

$$\varphi_{\mu,\lambda}(r,x) = \frac{2\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m-n}{2})\Gamma(\frac{n+1}{2})}\int_0^1\int_{S^n}(1-t^2)^{\frac{m-n}{2}-1}\cos(rt\mu\eta)e^{-i\langle\lambda|x+rt\xi\rangle}t^n dt d\sigma_n(\eta,\xi).$$

• If $m=n$, then the result follows from relations (2.1) and (2.11). □

The precedent Mehler integral representation of the eigenfunction $\varphi_{\mu,\lambda}$ allows us to define the integral transform $\mathscr{R}_{m,n}$, connected with operators $\frac{\partial}{\partial x_j}; 1\leqslant j\leqslant n$ and Ξ. More precisely, we have

Definition 2.2. *We define the integral transform* $\mathscr{R}_{m,n}$ *associated with operators* $\frac{\partial}{\partial x_j}; 1\leqslant j\leqslant n$ *and* Ξ *to be*

$$\mathscr{R}_{m,n}(f)(r,x) = \begin{cases} \dfrac{2\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m-n}{2})\Gamma(\frac{n+1}{2})}\displaystyle\int_0^1\int_{S^n} f((0,x)+rt\omega) \\ \qquad\qquad\times(1-t^2)^{\frac{m-n}{2}-1}t^n dt d\sigma_n(\omega), & \text{if } m>n, \\ \displaystyle\int_{S^n} f((0,x)+r\omega)d\sigma_n(\omega), & \text{if } m=n, \end{cases} \tag{2.13}$$

where f *is a continuous function on* $\mathbb{R}\times\mathbb{R}^n$*, even with respect to the first variable.*

Remark 2.3. *i) From the Proposition 2.1 and Definition 2.2, we have*

$$\varphi_{\mu,\lambda}(r,x) = \mathscr{R}_{m,n}\Big(\cos(\mu.)\exp(-i\langle\lambda|.\rangle)\Big)(r,x). \tag{2.14}$$

ii) We can easily see, as in [5]*, that the integral transform* $\mathscr{R}_{m,n}$ *is continuous and injective from* $\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ *(the space of infinitely differentiable functions on* $\mathbb{R}\times\mathbb{R}^n$*, even with respect to the first variable) into itself.*

We denote by

• dm_{n+1} the measure defined on $[0,+\infty[\times\mathbb{R}^n$, by

$$dm_{n+1}(r,x) = \sqrt{\frac{2}{\pi}}\frac{1}{(2\pi)^{\frac{n}{2}}}dr\otimes dx, \tag{2.15}$$

where dx is the Lebesgue measure on $\mathbb{R}^n$.
• $L^p(dm_{n+1})$ the space of measurable functions f on $[0,+\infty[\times\mathbb{R}^n$, such that

$$\begin{aligned} \|f\|_{p,m_{n+1}} &= \Big(\int_0^{+\infty}\int_{\mathbb{R}^n} |f(r,x)|^p\, dm_{n+1}(r,x)\Big)^{\frac{1}{p}} < +\infty, && \text{if } p\in[1,+\infty[,\\ \|f\|_{\infty,m_{n+1}} &= \underset{(r,x)\in[0,+\infty[\times\mathbb{R}^n}{\text{ess sup}} |f(r,x)| < +\infty, && \text{if } p=+\infty. \end{aligned}$$

• $d\nu_{m,n}$, the measure defined on $[0,+\infty[\times\mathbb{R}^n$, by

$$d\nu_{m,n}(r,x) = \frac{r^m dr}{2^{\frac{m-1}{2}}\Gamma(\frac{m+1}{2})} \otimes \frac{dx}{(2\pi)^{\frac{n}{2}}}. \tag{2.16}$$

• $L^p(d\nu_{m,n})$, the Lebesgue space of measurable functions f on $[0,+\infty[\times\mathbb{R}^n$, such that $\|f\|_{p,\nu_{m,n}} < +\infty$.
• $S_e(\mathbb{R}\times\mathbb{R}^n)$ the Shwartz's space formed by the infinitely differentiable functions on $\mathbb{R}\times\mathbb{R}^n$, rapidly decreasing together with every their derivatives, and even with respect to the first variable.
• $C_e(\mathbb{R}\times\mathbb{R}^n)$ the space of continuous functions on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable.

Lemma 2.4. *i) For every function $f\in L^1(d\nu_{m,n})$, the function ${}^t\mathscr{R}_{m,n}(f)$ defined by*

$${}^t\mathscr{R}_{m,n}(f)(s,y) = \begin{cases} \dfrac{1}{2^{\frac{m}{2}}\Gamma(\frac{m-n}{2})\pi^{\frac{n}{2}}} \displaystyle\int\int_{s^2+|z|^2<r^2} f(r,z+y) \\ \qquad\qquad \times(r^2-s^2-|z|^2)^{\frac{m-n}{2}-1} r dr dz & \text{if } m>n,\\ \dfrac{1}{(2\pi)^{\frac{n}{2}}}\displaystyle\int_{\mathbb{R}^n} f(\sqrt{s^2+|x-y|^2},x)dx & \text{if } m=n. \end{cases}$$

belongs to $L^1(dm_{n+1})$ and we have

$$\|{}^t\mathscr{R}_{m,n}(f)\|_{1,m_{n+1}} \leqslant \|f\|_{1,\nu_{m,n}}. \tag{2.17}$$

ii) For every bounded function $f\in\mathcal{C}_e(\mathbb{R}\times\mathbb{R}^n)$, and for every function $g\in L^1(d\nu_{m,n})$, we have

$$\begin{aligned} &\int_0^{+\infty}\int_{\mathbb{R}^n} \mathscr{R}_{m,n}(f)(r,x)g(r,x)d\nu_{m,n}(r,x)\\ &\qquad = \int_0^{+\infty}\int_{\mathbb{R}^n} f(s,y)\,{}^t\mathscr{R}_{m,n}(g)(s,y)dm_{n+1}(s,y), \end{aligned} \tag{2.18}$$

Proof. $i)$ Let $f\in L^1(d\nu_{m,n})$.
• If $m>n$, we have

$$\begin{aligned} &\left|{}^t\mathscr{R}_{m,n}(f)(s,y)\right|\\ &\qquad \leqslant \frac{1}{2^{\frac{m}{2}}\Gamma(\frac{m-n}{2})\pi^{\frac{n}{2}}}\int\int_{s^2+|z|^2<r^2} |f(r,z+y)|(r^2-s^2-|z|^2)^{\frac{m-n}{2}-1} r dr dz, \end{aligned}$$

and using Fubini's theorem, we obtain

$$\int_0^{+\infty}\int_{\mathbb{R}^n}\Big|{}^t\mathscr{R}_{m,n}(f)(s,y)\Big|dm_{n+1}(s,y)$$
$$\leqslant \frac{1}{2^{\frac{m}{2}}\Gamma(\frac{m-n}{2})(2\pi)^{\frac{n}{2}}\pi^{\frac{n}{2}}}\sqrt{\frac{2}{\pi}}\int_0^{+\infty}\int_{\mathbb{R}^n}\Big(\int\int_{s^2+|z|^2<r^2}|f(r,z+y)|$$
$$\times(r^2-s^2-|z|^2)^{\frac{m-n}{2}-1}rdrdz\Big)dsdy$$
$$=\frac{1}{2^{\frac{m+n-1}{2}}\Gamma(\frac{m-n}{2})\pi^{n+\frac{1}{2}}}\int_0^{+\infty}\Big[\int\int_{s^2+|z|^2<r^2}\Big(\int_{\mathbb{R}^n}|f(r,z+y)|\,dy\Big)$$
$$\times(r^2-s^2-|z|^2)^{\frac{m-n}{2}-1}rdrdz\Big]ds$$
$$=\frac{1}{2^{\frac{m+n-1}{2}}\Gamma(\frac{m-n}{2})\pi^{n+\frac{1}{2}}}\int_0^{+\infty}\Big[\int\int_{s^2+|z|^2<r^2}\Big(\int_{\mathbb{R}^n}|f(r,y)|\,dy\Big)$$
$$\times(r^2-s^2-|z|^2)^{\frac{m-n}{2}-1}rdrdz\Big]ds$$
$$=\frac{1}{2^{\frac{m+n-1}{2}}\Gamma(\frac{m-n}{2})\pi^{n+\frac{1}{2}}}\int_0^{+\infty}\int_{\mathbb{R}^n}|f(r,y)|$$
$$\times\Big(\int\int_{s^2+|z|^2<r^2}(r^2-s^2-|z|^2)^{\frac{m-n}{2}-1}dsdz\Big)rdrdy$$
$$=\frac{1}{2^{\frac{m+n-3}{2}}\Gamma(\frac{m-n}{2})\Gamma(\frac{n+1}{2})\pi^{\frac{n}{2}}}\int_0^{+\infty}\int_{\mathbb{R}^n}|f(r,y)|\Big(\int_0^r(r^2-t^2)^{\frac{m-n}{2}-1}t^ndt\Big)rdrdy.$$

From the fact that

$$\int_0^r(r^2-t^2)^{\frac{m-n}{2}-1}t^ndt=\frac{\Gamma(\frac{m-n}{2})\Gamma(\frac{n+1}{2})}{2\Gamma(\frac{m+1}{2})}r^{m-1},$$

we deduce that

$$\int_0^{+\infty}\int_{\mathbb{R}^n}\Big|{}^t\mathscr{R}_{m,n}(f)(s,y)\Big|dm_{n+1}(s,y)$$
$$\leqslant\frac{1}{2^{\frac{m-1}{2}}\Gamma(\frac{m+1}{2})(2\pi)^{\frac{n}{2}}}\int_0^{+\infty}\int_{\mathbb{R}^n}|f(r,y)|r^mdrdy$$
$$=\|f\|_{1,\nu_{m,n}}.$$

• The case $m=n$ may be treated similarly .

$ii)$• If $m>n$, then by relation (2.13), we have

$$\mathscr{R}_{m,n}(f)(r,x)$$
$$=\frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m-n}{2})\pi^{\frac{n+1}{2}}}r^{1-m}\int\int_{s^2+|y|^2<r^2}f(s,x+y)(r^2-s^2-|y|^2)^{\frac{m-n}{2}-1}dsdy$$
$$=\frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m-n}{2})\pi^{\frac{n+1}{2}}}r^{1-m}\int\int_{s^2+|x-y|^2<r^2}f(s,y)(r^2-s^2-|x-y|^2)^{\frac{m-n}{2}-1}dsdy,$$

and by Fubini's theorem, we get

$$\int_0^{+\infty}\int_{\mathbb{R}^n}\mathscr{R}_{m,n}(f)(r,x)g(r,x)d\nu_{m,n}(r,x)$$

$$= \frac{1}{2^{\frac{m-1}{2}}\Gamma(\frac{m-n}{2})\pi^{\frac{n+1}{2}}(2\pi)^{\frac{n}{2}}}\int_0^{+\infty}\int_{\mathbb{R}^n} g(r,x)$$

$$\times\Big(\int\int_{s^2+|x-y|^2<r^2} f(s,y)(r^2-s^2-|x-y|^2)^{\frac{m-n}{2}-1}dsdy\Big)rdrdx$$

$$= \int_0^{+\infty}\int_{\mathbb{R}^n} f(s,y)\Big(\frac{1}{2^{\frac{m}{2}}\Gamma(\frac{m-n}{2})\pi^{\frac{n}{2}}}\int\int_{s^2+|x-y|^2<r^2} g(r,x)$$

$$\times (r^2-s^2-|x-y|^2)^{\frac{m-n}{2}-1}rdrdx\Big)dm_{n+1}(s,y).$$

• If $m=n$, we have

$$\int_0^{+\infty}\int_{\mathbb{R}^n}\mathscr{R}_{n,n}(f)(r,x)g(r,x)d\nu_{n,n}(r,x)$$

$$= \frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})(2\pi)^{\frac{n}{2}}}\int_0^{+\infty}\int_{\mathbb{R}^n}\Big(\int_{S^n} f(r\omega+(0,x))d\sigma_n(\omega)\Big)$$

$$\times g(r,x)r^n drdx,$$

and by Fubini's theorem,

$$\int_0^{+\infty}\int_{\mathbb{R}^n}\mathscr{R}_{n,n}(f)(r,x)g(r,x)d\nu_{n,n}(r,x)$$

$$= \frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\Big(\int_0^{+\infty}\int_{S^n} f(r\omega+(0,x))$$

$$\times g(r,x)r^n drd\sigma_n(\omega)\Big)dx$$

$$= \frac{\Gamma(\frac{n+1}{2})}{2\pi^{\frac{n+1}{2}}2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^{n+1}} f((s,y)+(0,x))$$

$$\times g(\sqrt{s^2+|y|^2},x)dsdy\Big)dx$$

$$= \frac{\Gamma(\frac{n+1}{2})}{2\pi^{\frac{n+1}{2}}2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^{n+1}} f(s,y)$$

$$\times g(\sqrt{s^2+|y-x|^2},x)dsdy\Big)dx$$

$$= \int_0^{+\infty}\int_{\mathbb{R}^n} f(s,y)\Big(\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} g(\sqrt{s^2+|y-x|^2},x)dx\Big)dm_{n+1}(s,y).$$

□

We denote by

• $\mathscr{B}_{\Upsilon_+}$ the σ-algebra defined on Υ_+ by

$$\mathscr{B}_{\Upsilon_+} = \{\theta^{-1}(B)\ ,\ B\in\mathscr{B}_{Bor}([0,+\infty[\times\mathbb{R}^n)\},$$

where the set Υ_+ and the function θ are defined in the introduction.

• $d\gamma_{m,n}$ the measure defined on $\mathscr{B}_{\Upsilon_+}$ by

$$\forall\ A\in\mathscr{B}_{\Upsilon_+};\ \gamma_{m,n}(A) = \nu_{m,n}(\theta(A)).$$

• $L^p(d\gamma_{m,n})$ the Lebesgue space of measurable functions f on Υ_+, such that $\|f\|_{p,\gamma_{m,n}} < +\infty$.

Then we have the following properties

Proposition 2.5. *i) For every nonnegative measurable function g on Υ_+, we have*

$$\int\int_{\Upsilon_+} g(\mu,\lambda)\, d\gamma_{m,n}(\mu,\lambda) = \frac{1}{2^{\frac{m-1}{2}}\Gamma(\frac{m+1}{2})(2\pi)^{\frac{n}{2}}}\Big(\int_0^{+\infty}\int_{\mathbb{R}^n} g(\mu,\lambda)(\mu^2+|\lambda|^2)^{\frac{m-1}{2}}\mu\, d\mu\, d\lambda + \int_{\mathbb{R}^n}\int_0^{|\lambda|} g(i\mu,\lambda)(|\lambda|^2-\mu^2)^{\frac{m-1}{2}}\mu\, d\mu\, d\lambda\Big).$$

ii) For every nonnegative measurable function f on $[0,+\infty[\times\mathbb{R}^n$ (respectively integrable on $[0,+\infty[\times\mathbb{R}^n$ with respect to the measure $d\nu_{m,n}$), $fo\theta$ is a measurable nonnegative function on Υ_+, (respectively integrable on Υ_+ with respect to the measure $d\gamma_{m,n}$) and we have

$$\int\int_{\Upsilon_+}(f\circ\theta)(\mu,\lambda)\, d\gamma_{m,n}(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)\, d\nu_{m,n}(r,x). \tag{2.19}$$

3. The Fourier transform associated with the operator $\mathscr{R}_{m,n}$

In the next, we shall define the translation operator and the convolution product associated with the integral transform $\mathscr{R}_{m,n}$. For this we need the following product formula satisfied by the function $\varphi_{\mu,\lambda}$, that is for every $(r,x),(s,y)\in[0,+\infty[\times\mathbb{R}^n$,

$$\varphi_{\mu,\lambda}(r,x)\varphi_{\mu,\lambda}(s,y) = \frac{\Gamma(\frac{m+1}{2})}{\sqrt{\pi}\Gamma(\frac{m}{2})}\int_0^{\pi}\varphi_{\mu,\lambda}(\sqrt{r^2+s^2+2rs\cos\theta}, x+y)\sin^{m-1}(\theta)d\theta. \tag{3.1}$$

Definition 3.1. *i) For every $(r,x)\in[0,+\infty[\times\mathbb{R}^n$, the translation operator $\tau_{(r,x)}$ associated with the integral transform $\mathscr{R}_{m,n}$ is defined on $L^p(d\nu_{m,n})$, $p\in[1,+\infty]$, by*

$$\tau_{(r,x)}f(s,y) = \frac{\Gamma(\frac{m+1}{2})}{\sqrt{\pi}\Gamma(\frac{m}{2})}\int_0^{\pi} f(\sqrt{r^2+s^2+2rs\cos\theta}, x+y)\sin^{m-1}(\theta)d\theta.$$

ii) The convolution product of $f,g\in L^1(d\nu_{m,n})$ is defined by

$$\forall (r,x)\in[0,+\infty[\times\mathbb{R}^n;\ f*g(r,x) = \int_0^{+\infty}\int_{\mathbb{R}^n}\tau_{(r,-x)}(\check{f})(s,y)g(s,y)d\nu_{m,n}(s,y),$$

where $\check{f}(s,y) = f(s,-y)$.

We have the following properties

• *relation (3.1) can be written: $\tau_{(r,x)}\varphi_{\mu,\lambda}(s,y) = \varphi_{\mu,\lambda}(r,x)\varphi_{\mu,\lambda}(s,y)$.*
• *If $f\in L^p(d\nu_{m,n})$, $1\leqslant p\leqslant+\infty$, then for every $(r,x)\in[0,+\infty[\times\mathbb{R}^n$, the function $\tau_{(r,x)}f$ belongs to $L^p(d\nu_{m,n})$ and we have*

$$||\tau_{(r,x)}f||_{p,\nu_{m,n}} \leqslant ||f||_{p,\nu_{m,n}}. \tag{3.2}$$

In particular, for every $f \in L^1(d\nu_{m,n})$ and $(s,y) \in [0,+\infty[\times\mathbb{R}^n$, the function $\tau_{(r,x)}f$ belongs to $L^1(d\nu_{m,n})$ and we have

$$\int_0^{+\infty}\int_{\mathbb{R}^n} \tau_{(s,y)}f(r,x)d\nu_{m,n}(r,x) = \int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)d\nu_{m,n}(r,x). \tag{3.3}$$

- *Let $p,q,r \in [1,+\infty]$ such that $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$. For every $f \in L^p(d\nu_{m,n})$, and $g \in L^q(d\nu_{m,n})$, the function $f*g$ belongs to $L^r(d\nu_{m,n})$ and we have*

$$||f*g||_{r,\nu_{m,n}} \leqslant ||f||_{p,\nu_{m,n}}||g||_{q,\nu_{m,n}}. \tag{3.4}$$

In the following, we will define the Fourier transform $\mathscr{F}_{m,n}$ connected with $\mathscr{R}_{m,n}$ and we give its connection with the translation operator and the convolution product defined above. Next, we shall give some properties that we need in the coming sections.

Definition 3.2. *The Fourier transform $\mathscr{F}_{m,n}$ associated with the integral transform $\mathscr{R}_{m,n}$ is defined on $L^1(d\nu_{m,n})$ by*

$$\forall(\mu,\lambda) \in \Upsilon\ ;\ \mathscr{F}_{m,n}(f)(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)\varphi_{\mu,\lambda}(r,x)\, d\nu_{m,n}(r,x),$$

where $\varphi_{\mu,\lambda}$ is the function given by (2.1) and Υ is the set defined by (2.6).

The Fourier transform $\mathscr{F}_{m,n}$ satisfies the properties

- For every $f \in L^1(d\nu_{m,n})$ and $(r,x) \in [0,+\infty[\times\mathbb{R}^n$, we have

$$\forall(\mu,\lambda) \in \Upsilon,\ \mathscr{F}_{m,n}(\tau_{(r,-x)}f)(\mu,\lambda) = \varphi_{\mu,\lambda}(r,x)\mathscr{F}_{m,n}(f)(\mu,\lambda). \tag{3.5}$$

- For every $f,g \in L^1(d\nu_{m,n})$, we have

$$\forall(\mu,\lambda) \in \Upsilon,\ \mathscr{F}_{m,n}(f*g)(\mu,\lambda) = \mathscr{F}_{m,n}(f)(\mu,\lambda)\mathscr{F}_{m,n}(g)(\mu,\lambda).$$

- For every $f \in L^1(d\nu_{m,n})$, and $(\mu,\lambda) \in \Upsilon$

$$\mathscr{F}_{m,n}(f)(\mu,\lambda) = \widetilde{\mathscr{F}}_{m,n}(f)\circ\theta(\mu,\lambda), \tag{3.6}$$

where for every $(\mu,\lambda) \in [0,+\infty[\times\mathbb{R}^n$,

$$\widetilde{\mathscr{F}}_{m,n}(f)(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)j_{\frac{m-1}{2}}(r\mu)e^{-i\langle\lambda|x\rangle}d\nu_{m,n}(r,x). \tag{3.7}$$

Moreover, relation (2.7) implies that the Fourier transform $\mathscr{F}_{m,n}$ is a bounded linear operator from $L^1(d\nu_{m,n})$ into $L^\infty(d\gamma_{m,n})$, and that for every $f \in L^1(d\nu_{m,n})$, we have

$$\|\mathscr{F}_{m,n}(f)\|_{\infty,\gamma_{m,n}} \leqslant \|f\|_{1,\nu_{m,n}}. \tag{3.8}$$

Theorem 3.3 (Inversion formula)**.** *Let $f \in L^1(d\nu_{m,n})$ such that $\mathscr{F}_{m,n}(f) \in L^1(d\gamma_{m,n})$, then for almost every $(r,x) \in [0,+\infty[\times\mathbb{R}^n$, we have*

$$\begin{aligned} f(r,x) &= \int\int_{\Upsilon_+} \mathscr{F}_{m,n}(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}\, d\gamma_{m,n}(\mu,\lambda) \\ &= \int_0^{+\infty}\int_{\mathbb{R}^n} \widetilde{\mathscr{F}}_{m,n}(f)(\mu,\lambda)j_{\frac{m-1}{2}}(r\mu)e^{i\langle\lambda|x\rangle}\, d\nu_{m,n}(\mu,\lambda). \end{aligned}$$

Theorem 3.4 (Plancherel theorem)**.** *The Fourier transform $\mathscr{F}_{m,n}$ can be extended to an isometric isomorphism from $L^2(d\nu_{m,n})$ onto $L^2(d\gamma_{m,n})$. In particular, we have the Parseval equality, for every $f, g \in L^2(d\nu_{m,n})$*

$$\int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)\overline{g(r,x)}d\nu_{m,n}(r,x) = \int\int_{\Upsilon_+} \mathscr{F}_{m,n}(f)(\mu,\lambda)\overline{\mathscr{F}_{m,n}(g)(\mu,\lambda)}d\gamma_{m,n}(\mu,\lambda).$$

Remark 3.5. *i) Let $f \in L^1(d\nu_{m,n})$ and $g \in L^2(d\nu_{m,n})$, by relation (3.3), the function $f * g$ belongs to $L^2(d\nu_{m,n})$; moreover*

$$\mathscr{F}_{m,n}(f * g) = \mathscr{F}_{m,n}(f)\mathscr{F}_{m,n}(g). \tag{3.9}$$

ii) For every $f, g \in L^2(d\nu_{m,n})$; the function $f * g$ belongs to the space $\mathcal{C}_{e,0}(\mathbb{R}\times\mathbb{R}^n)$ consisting of continuous functions h on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable and such that

$$\lim_{r^2+|x|^2\to+\infty} h(r,x) = 0.$$

Moreover,

$$f * g = \mathscr{F}^{-1}_{m,n}(\mathscr{F}_{m,n}(f)\mathscr{F}_{m,n}(g)), \tag{3.10}$$

where $\mathscr{F}^{-1}_{m,n}$ is the mapping defined on $L^1(d\gamma_{m,n})$ by

$$\mathscr{F}^{-1}_{m,n}(g)(r,x) = \int\int_{\Upsilon_+} g(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma_{m,n}(\mu,\lambda).$$

Remark 3.6. *From Lemma 2.4 and relation (2.14), we deduce that for every $f \in L^1(d\nu_{m,n})$, we have*

$$\forall(\mu,\lambda)\in\mathbb{R}\times\mathbb{R}^n, \quad \mathscr{F}_{m,n}(f)(\mu,\lambda) = \Lambda_{n+1}\circ {}^t\mathscr{R}_{m,n}(f)(\mu,\lambda),$$

where Λ_{n+1} is the usual Fourier transform defined on $[0,+\infty[\times\mathbb{R}^n$, by

$$\Lambda_{n+1}(f)(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)\cos(r\mu)e^{-i\langle\lambda|x\rangle}dm_{n+1}(r,x). \tag{3.11}$$

The following result is an immediate consequence of Fubini's theorem.

Lemma 3.7. *For a bounded function $g \in \mathcal{C}_e(\mathbb{R}\times\mathbb{R}^n)$, and a function $f \in S_e(\mathbb{R}\times\mathbb{R}^n)$, we have*

$$\int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)S_{\frac{m-1}{2},\frac{n-1}{2}}(g)(r,x)\,d\nu_{m,n}(r,x) = \int_0^{+\infty}\int_{\mathbb{R}^n} {}^tS_{\frac{m-1}{2},\frac{n-1}{2}}(f)(r,x)g(r,x)\,d\nu_{n,n}(r,x), \tag{3.12}$$

where ${}^tS_{a,b}$ is the dual of the Sonine transform defined for $f \in S_e(\mathbb{R}\times\mathbb{R}^n)$, by

$${}^tS_{a,b}(f)(r,x) = \begin{cases} \dfrac{1}{2^{a-b-1}\Gamma(a-b)}\displaystyle\int_r^{+\infty}(t^2-r^2)^{a-b-1}f(t,x)t\,dt, & \textit{if } a>b; \\ f(r,x), & \textit{if } a=b. \end{cases} \tag{3.13}$$

Proposition 3.8. *For every $f \in L^1(d\nu_{m,n})$, the function ${}^tS_{\frac{m-1}{2},\frac{n-1}{2}}(f)$ belongs to $L^1(d\nu_{n,n})$, and we have*

$$\|{}^tS_{\frac{m-1}{2},\frac{n-1}{2}}(f)\|_{1,\nu_{n,n}} \leqslant \|f\|_{1,\nu_{m,n}}. \tag{3.14}$$

Moreover, for every $f \in S_e(\mathbb{R}\times\mathbb{R}^n)$ we have

$$\forall(\mu,\lambda) \in [0,+\infty[\times\mathbb{R}^n,\ \widetilde{\mathscr{F}}_{m,n}(f)(\mu,\lambda) = \widetilde{\mathscr{F}}_{n,n} \circ {}^tS_{\frac{m-1}{2},\frac{n-1}{2}}(f)(\mu,\lambda). \tag{3.15}$$

The relation (3.15) follows from (2.12) and Lemma (3.7).

Remark 3.9. *Since for every $m \geqslant n$, the Fourier transform $\widetilde{\mathscr{F}}_{m,n}$ is a topological isomorphism from $S_e(\mathbb{R}\times\mathbb{R}^n)$ onto itself, then by relation (3.15) we deduce that the dual transform ${}^tS_{\frac{m-1}{2},\frac{n-1}{2}}$ is also a topological isomorphism from $S_e(\mathbb{R}\times\mathbb{R}^n)$ onto itself.*

4. Fractional powers of Bessel and the Laplacian operators

In the next section, we will establish inversion formulas for the operators $\mathscr{R}_{m,n}$ and its dual ${}^t\mathscr{R}_{m,n}$. More precisely, we define some functions spaces where the operators $\mathscr{R}_{m,n}$ and ${}^t\mathscr{R}_{m,n}$ are topological isomorphisms, and we exhibit the inverse operators in terms of integro-differential operators. For this we define and study in this section, the fractional powers of Bessel and Laplacian operators.

We denote by

• $\mathscr{E}_e(\mathbb{R})$ the space of even infinitely differentiable functions on $\mathbb{R}$.
• $S_e(\mathbb{R})$ the subspace of $\mathscr{E}_e(\mathbb{R})$, consisting of functions rapidly decreasing together with every their derivatives.
• $S'_e(\mathbb{R})$ the space of even tempered distributions on $\mathbb{R}$.
• $S'_e(\mathbb{R}\times\mathbb{R}^n)$ the space of tempered distributions on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable.
Each of these spaces is equipped with its usual topology.
• For $\alpha \in \mathbb{R}$, $\alpha \geqslant \frac{-1}{2}$, $d\omega_\alpha$ the measure defined on $[0,+\infty[$ by

$$d\omega_\alpha(r) = \frac{1}{2^\alpha\Gamma(\alpha+1)} r^{2\alpha+1}dr. \tag{4.1}$$

• ℓ_α the Bessel operator defined on $]0,+\infty[$ by

$$\ell_\alpha = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r}, \quad \alpha \geqslant -\frac{1}{2}. \tag{4.2}$$

• For an even measurable function f on $\mathbb{R}$, $T_f^{\omega_\alpha}$ denotes the even tempered distribution defined by

$$\forall\varphi \in S_e(\mathbb{R}),\ \langle T_f^{\omega_\alpha},\varphi\rangle = \int_0^{+\infty} f(r)\varphi(r)d\omega_\alpha(r). \tag{4.3}$$

• For a measurable function g on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable, $T_g^{m_{n+1}}$ (resp.$T_g^{\nu_{m,n}}$) denotes the even tempered distribution, defined by

$$\forall\varphi\in S_e^{'}(\mathbb{R}\times\mathbb{R}^n),\ \langle T_g^{m_{n+1}},\varphi\rangle=\int_0^{+\infty}\int_{\mathbb{R}^n}g(r,x)\varphi(r,x)dm_{n+1}(r,x),$$

$$\Big(resp.\langle T_g^{\nu_{m,n}},\varphi\rangle=\int_0^{+\infty}\int_{\mathbb{R}^n}g(r,x)\varphi(r,x)d\nu_{m,n}(r,x)\Big),\tag{4.4}$$

where dm_{n+1} and $d\nu_{m,n}$ are the measures given by relations (2.15) and (2.16).

Let $a\in\mathbb{C}$, such that $\mathrm{Re}(a)>-2(\alpha+1)$, then the function $r\mapsto|r|^a$ defines an even tempered distribution $T_{|r|^a}^{\omega_\alpha}$ on $\mathbb{R}$. Indeed, let $m\in\mathbb{N}$ satisfying

$$\int_0^{+\infty}\frac{r^{Re(a)+2\alpha+1}}{(1+r^2)^m}dr<+\infty,$$

then for every $\varphi\in S_e(\mathbb{R})$;

$$|\langle T_{|r|^a}^{\omega_\alpha},\varphi\rangle|\leqslant C_{m,\alpha,a}P_m(\varphi),$$

where

$$C_{m,\alpha,a}=\frac{1}{2^\alpha\Gamma(\alpha+1)}\int_0^{+\infty}\frac{r^{Re(a)+2\alpha+1}}{(1+r^2)^m}dr,$$

and

$$P_m(\varphi)=\sup_{\substack{k_1,k_2\leqslant m\\ x\in\mathbb{R}}}(1+x^2)^{k_1}|\varphi^{(k_2)}(x)|.$$

Now let $a\in\mathbb{C}\backslash\{-2(\alpha+1)-k,k\in\mathbb{N}\}$ and $m\in\mathbb{N}^*$ such that $\mathrm{Re}(a)>-m-2(\alpha+1)$, then the value of the following expression

$$\int_0^1\Big(\varphi(r)-\sum_{j=0}^{m-1}\frac{\varphi^{(j)}(0)}{j!}r^j\Big)r^a d\omega_\alpha(r)+\frac{1}{2^\alpha\Gamma(\alpha+1)}\sum_{j=0}^{m-1}\frac{\varphi^{(j)}(0)}{j!(j+a+2\alpha+2)},$$

is independent of the choice of the parameter m. Hence the mapping

$$a\mapsto T_{|r|^a}^{\omega_\alpha}$$

may be extended on $\mathbb{C}\backslash\{-2(\alpha+1)-k,k\in\mathbb{N}\}$, by setting

$$\langle T_{|r|^a}^{\omega_\alpha},\varphi\rangle=\int_0^1\Big(\varphi(r)-\sum_{j=0}^{m-1}\frac{\varphi^{(j)}(0)}{j!}r^j\Big)r^a d\omega_\alpha(r)+\int_1^{+\infty}r^a\varphi(r)d\omega_\alpha(r)$$
$$+\frac{1}{2^\alpha\Gamma(\alpha+1)}\sum_{j=0}^{m-1}\frac{\varphi^{(j)}(0)}{j!(j+a+2\alpha+2)}$$

where m is an any integer satisfying $\mathrm{Re}(a)>-m-2(\alpha+1)$, and therefore $T_{|r|^a}^{\omega_\alpha}$ is an even tempered distribution on $\mathbb{R}$. Thus the mapping

$$a\mapsto T_{|r|^a}^{\omega_\alpha}$$

can be extended to a valued function in $S_e^{'}(\mathbb{R})$, analytic on $\mathbb{C}\backslash\{-2(\alpha+1)-k,k\in\mathbb{N}\}$. On the other hand, the points $-2(\alpha+1)-k,k\in\mathbb{N}$ are simples poles for $T_{|r|^a}^{\omega_\alpha}$ and we have

$$\mathrm{Res}(T_{|r|^a}^{\omega_\alpha},-2(\alpha+1)-k)=\frac{(-1)^k}{2^\alpha\Gamma(\alpha+1)}\frac{\delta^{(k)}}{k!},$$

in particular

$$\mathrm{Res}(T^{\omega_\alpha}_{|r|^a}, -2(\alpha+1)-2k-1) = \frac{-1}{2^\alpha\Gamma(\alpha+1)}\frac{\delta^{(2k+1)}}{(2k+1)!} = 0, \quad \text{over } S_e(\mathbb{R}).$$

This means that the mapping

$$a \mapsto T^{\omega_\alpha}_{|r|^a}$$

is analytic on $\mathbb{C}\backslash\{-2(\alpha+k+1), k\in\mathbb{N}\}$ and

$$\mathrm{Res}(T^{\omega_\alpha}_{|r|^a}, -2(\alpha+k+1)) = \frac{1}{2^\alpha\Gamma(\alpha+1)}\frac{\delta^{(2k)}}{(2k)!}.$$

Definition 4.1. *i) The Bessel translation operator τ^α_r is defined on $S_e(\mathbb{R})$ by*

$$\tau^\alpha_r(f)(s) = \begin{cases} \dfrac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\displaystyle\int_0^\pi f(\sqrt{r^2+s^2+2rs\cos\theta})\sin^{2\alpha}(\theta)d\theta, & \text{if } \alpha > -\frac{1}{2};\\ \dfrac{1}{2}[f(r+s)+f(r-s)], & \text{if } \alpha = -\frac{1}{2}.\end{cases}$$

ii) The Bessel convolution product of $f\in S_e(\mathbb{R})$ and $T\in S^{'}_e(\mathbb{R})$ is the function defined by

$$\forall\, r\in\mathbb{R},\quad T *_\alpha f(r) = \langle T, \tau^\alpha_r(f)\rangle. \tag{4.5}$$

iii) The Fourier-Bessel transform is defined on $S_e(\mathbb{R})$ by

$$\forall\, \mu\in\mathbb{R},\quad F_\alpha(f)(\mu) = \int_0^{+\infty} f(r)j_\alpha(r\mu)d\omega_\alpha(r), \tag{4.6}$$

and on $S^{'}_e(\mathbb{R})$ by

$$\forall\, \varphi\in S_e(\mathbb{R}),\quad \langle F_\alpha(T),\varphi\rangle = \langle T, F_\alpha(\varphi)\rangle. \tag{4.7}$$

We have the following properties

• F_α is an isomorphism from $S_e(\mathbb{R})$ (resp. $S^{'}_e(\mathbb{R})$) onto itself, and we have

$$F^{-1}_\alpha = F_\alpha. \tag{4.8}$$

• For $f\in S_e(\mathbb{R})$, and $r\in\mathbb{R}$, the function $\tau^\alpha_r(f)$ belongs to $S_e(\mathbb{R})$ and we have

$$F_\alpha(\tau^\alpha_r f)(\mu) = j_\alpha(r\mu)F_\alpha(f)(\mu). \tag{4.9}$$

• For $f\in S_e(\mathbb{R})$ and $T\in S^{'}_e(\mathbb{R})$, the function $T *_\alpha f$ belongs to $\mathscr{E}_e(\mathbb{R})$, and is slowly increasing, moreover

$$F_\alpha(T^{\omega_\alpha}_{T*_\alpha f}) = F_\alpha(f)F_\alpha(T). \tag{4.10}$$

Proposition 4.2. *The mappings*

$$a\mapsto T^{\omega_\alpha}_{|r|^a},\;\; a\mapsto T^{\omega_\alpha}_{\frac{\Gamma(\frac{a}{2}+\alpha+1)}{\Gamma(\frac{-a}{2})}2^{a+\alpha+1}|r|^{-a-2\alpha-2}} \tag{4.11}$$

defined initially for $-2(\alpha+1) < Re(a) < 0$, can be extended to a valued functions in $S^{'}_e(\mathbb{R})$, analytic on $\mathbb{C}\backslash\{-2(\alpha+k+1), k\in\mathbb{N}\}$, and we have

$$F_\alpha(T^{\omega_\alpha}_{|r|^a}) = T^{\omega_\alpha}_{\frac{\Gamma(\frac{a}{2}+\alpha+1)}{\Gamma(\frac{-a}{2})}2^{a+\alpha+1}|r|^{-a-2\alpha-2}}. \tag{4.12}$$

Proof. Let $a \in \mathbb{C},\ -2(\alpha+1) < \mathrm{Re}(a) < 0$ and let ψ_t be the function defined by

$$\psi_t(r) = e^{\frac{-tr^2}{2}},\ t > 0.$$

We have

$$F_\alpha(\psi_t)(\lambda) = t^{-\alpha-1} e^{\frac{-\lambda^2}{2t}}. \tag{4.13}$$

On the other hand, for every $\varphi \in S_e(\mathbb{R})$, we have

$$\int_0^{+\infty} F_\alpha(\psi_t)(r)\varphi(r)d\omega_\alpha(r) = \int_0^{+\infty} F_\alpha(\varphi)(r)\psi_t(r)d\omega_\alpha(r), \tag{4.14}$$

from relations (4.13) and (4.14), we obtain

$$\int_0^{+\infty} t^{-\alpha-1} e^{\frac{-r^2}{2t}}\varphi(r)d\omega_\alpha(r) = \int_0^{+\infty} F_\alpha(\varphi)(r) e^{\frac{-tr^2}{2}} d\omega_\alpha(r).$$

Multiplying both sides by $t^{-\frac{a+2}{2}}$ and integrating over $]0,+\infty[$, we obtain

$$\int_0^{+\infty}\int_0^{+\infty} t^{-\alpha-1} t^{-\frac{a+2}{2}} e^{\frac{-\lambda^2}{2t}}\varphi(r)d\omega_\alpha(r)dt \tag{4.15}$$

$$= \int_0^{+\infty}\int_0^{+\infty} F_\alpha(\varphi)(r) e^{\frac{-tr^2}{2}} t^{-\frac{a+2}{2}} d\omega_\alpha(r)dt, \tag{4.16}$$

and by using Fubini's theorem, we deduce that

$$\int_0^{+\infty} F_\alpha(\varphi)(r) r^a d\omega_\alpha(r) = 2^{a+\alpha+1}\frac{\Gamma(\frac{a}{2}+\alpha+1)}{\Gamma(\frac{-a}{2})}\int_0^{+\infty}\varphi(r) r^{-a-2\alpha-2} d\omega_\alpha(r).$$

This shows that for every $a \in \mathbb{C}$, such that $-2(\alpha+1) < \mathrm{Re}(a) < 0$, we have

$$F_\alpha(T^{\omega_\alpha}_{|r|^a}) = T^{\omega_\alpha}_{\frac{\Gamma(\frac{a}{2}+\alpha+1)}{\Gamma(\frac{-a}{2})}2^{a+\alpha+1}|r|^{-a-2\alpha-2}}.$$

The result is then obtained by analytic continuation. □

Definition 4.3. *For $a \in \mathbb{C}\backslash\{-(\alpha+k+1), k \in \mathbb{N}\}$, the fractional power of Bessel operator ℓ_α is defined on $S_e(\mathbb{R})$ by*

$$(-\ell_\alpha)^a f(r) = \Big(T^{\omega_\alpha}_{\frac{\Gamma(a+\alpha+1)}{\Gamma(-a)}2^{2a+\alpha+1}|s|^{-2a-2\alpha-2}} *_\alpha f\Big)(r). \tag{4.17}$$

It is well known that for $f \in S_e(\mathbb{R})$, the function $(-\ell_\alpha)^a f$ belongs to $\mathscr{E}_e(\mathbb{R})$ and is slowly increasing, moreover by relations (4.10) and (4.12), we deduce that for $f \in S_e(\mathbb{R})$ and $a \in \mathbb{C}\backslash\{-(\alpha+k+1), k \in \mathbb{N}\}$, we have

$$F_\alpha(T^{\omega_\alpha}_{(-\ell_\alpha)^a f}) = F_\alpha(f) T^{\omega_\alpha}_{|r|^{2a}}. \tag{4.18}$$

Let $\mathscr{A}$ be the transform defined on $S_e(\mathbb{R}\times\mathbb{R}^n)$ by

$$\mathscr{A}(\varphi)(\rho) = \int_{S^n}\varphi(\rho\omega)d\sigma_n(\omega),$$

then $\mathscr{A}$ is a continuous mapping from $S_e(\mathbb{R}\times\mathbb{R}^n)$ into $S_e(\mathbb{R})$.

Let $a \in \mathbb{C}$, such that Re(a) $> -\frac{n+1}{2}$, then the function $(r,x) \mapsto (r^2+|x|^2)^a$ defines an even tempered distribution $T^{m_{n+1}}_{(r^2+|x|^2)^a}$ on $\mathbb{R}\times\mathbb{R}^n$ and we have

$$\langle T^{m_{n+1}}_{(r^2+|x|^2)^a}, \varphi\rangle = \int_0^{+\infty}\int_{\mathbb{R}^n}(r^2+|x|^2)^a\varphi(r,x)dm_{n+1}(r,x)$$
$$= \langle T^{\omega_{\frac{n-1}{2}}}_{|r|^{2a}}, \mathscr{A}(\varphi)\rangle. \tag{4.19}$$

From relation (4.19) and Proposition 4.2, we deduce that the valued function in $S^{'}_{e}(\mathbb{R}\times\mathbb{R}^n)$ defined by

$$a \mapsto T^{m_{n+1}}_{(r^2+|x|^2)^a}$$

is analytic on $\mathbb{C}\backslash\{-(\frac{n+1}{2}+k), k\in\mathbb{N}\}$. Moreover, we have

Proposition 4.4. *the mappings*

$$a \mapsto T^{m_{n+1}}_{(r^2+|x|^2)^a},\ a \mapsto T^{m_{n+1}}_{\frac{\Gamma(\frac{n+1}{2}+a)}{\Gamma(-a)}2^{\frac{n+1}{2}+2a}(r^2+|x|^2)^{-(\frac{n+1}{2}+a)}} \tag{4.20}$$

defined firstly for $-\frac{n+1}{2} < Re(a) < 0$, *can be extended to valued functions in* $S^{'}_{e}(\mathbb{R}\times\mathbb{R}^n)$, *analytic on* $\mathbb{C}\backslash\{-(\frac{n+1}{2}+k), k\in\mathbb{N}\}$, *and we have*

$$\Lambda_{n+1}(T^{m_{n+1}}_{(r^2+|x|^2)^a}) = T^{m_{n+1}}_{\frac{\Gamma(\frac{n+1}{2}+a)}{\Gamma(-a)}2^{\frac{n+1}{2}+2a}(r^2+|x|^2)^{-(\frac{n+1}{2}+a)}}, \tag{4.21}$$

where Λ_{n+1} *is defined on* $S^{'}_{e}(\mathbb{R}\times\mathbb{R}^n)$ *by*

$$\langle\Lambda_{n+1}(T),\varphi\rangle = \langle T, \Lambda_{n+1}(\varphi)\rangle, \quad \varphi\in S_e(\mathbb{R}\times\mathbb{R}^n), \tag{4.22}$$

and $\Lambda_{n+1}(\varphi)$ *is given by relation (3.11).*

Proof. Let $a\in\mathbb{C}$, such that $-(\frac{n+1}{2}) <$ Re(a) < 0, and let $\psi_t,\ t>0$, be the function defined on $\mathbb{R}\times\mathbb{R}^n$, by

$$\psi_t(r,x) = e^{-\frac{t}{2}(r^2+|x|^2)}.$$

We have

$$\Lambda_{n+1}(\psi_t)(\mu,\lambda) = t^{-\frac{n+1}{2}}e^{-\frac{1}{2t}(\mu^2+|\lambda|^2)}.$$

On the other hand, for every $\varphi\in S_e(\mathbb{R}\times\mathbb{R}^n)$, we have

$$\int_0^{+\infty}\int_{\mathbb{R}^n}\Lambda_{n+1}(\psi_t)(r,x)\varphi(r,x)dm_{n+1}(r,x)$$
$$= \int_0^{+\infty}\int_{\mathbb{R}^n}\psi_t(r,x)\Lambda_{n+1}(\varphi)(r,x)dm_{n+1}(r,x),$$

hence,

$$\int_0^{+\infty}\int_{\mathbb{R}^n}t^{-\frac{n+1}{2}}e^{-\frac{1}{2t}(r^2+|x|^2)}\varphi(r,x)dm_{n+1}(r,x)$$
$$= \int_0^{+\infty}\int_{\mathbb{R}^n}e^{-\frac{t}{2}(r^2+|x|^2)}\Lambda_{n+1}(\varphi)(r,x)dm_{n+1}(r,x).$$

Multiplying both sides by $t^{-(a+1)}$ and integrating over $]0,+\infty[$, we obtain

$$\int_0^{+\infty}\Big(\int_0^{+\infty}\int_{\mathbb{R}^n} e^{-\frac{1}{2t}(r^2+|x|^2)}\varphi(r,x)dm_{n+1}(r,x)\Big)t^{-(\frac{n+1}{2}+a+1)}dt$$
$$=\int_0^{+\infty}\Big(\int_0^{+\infty}\int_{\mathbb{R}^n} e^{-\frac{t}{2}(r^2+|x|^2)}\Lambda_{n+1}(\varphi)(r,x)dm_{n+1}(r,x)\Big)t^{-(a+1)}dt,$$

using Fubini's theorem, we deduce that

$$\int_0^{+\infty}\int_{\mathbb{R}^n}\Lambda_{n+1}(\varphi)(r,x)(r^2+|x|^2)^a dm_{n+1}(r,x)$$
$$=\frac{\Gamma(\frac{n+1}{2}+a)}{\Gamma(-a)}2^{\frac{n+1}{2}+2a}\int_0^{+\infty}\int_{\mathbb{R}^n}\varphi(r,x)(r^2+|x|^2)^{-(\frac{n+1}{2}+a)}dm_{n+1}(r,x).$$

This shows that for every $a\in\mathbb{C}$, such that $-\frac{n+1}{2}<\mathrm{Re(a)}<0$, we have

$$\Lambda_{n+1}(T^{m_{n+1}}_{(r^2+|x|^2)^a})=T^{m_{n+1}}_{\frac{\Gamma(\frac{n+1}{2}+a)}{\Gamma(-a)}2^{\frac{n+1}{2}+2a}(r^2+|x|^2)^{-(\frac{n+1}{2}+a)}}.$$

Then the proof is complete by analytic continuation . □

Definition 4.5. *For $a\in\mathbb{C}\backslash\{-(\frac{n+1}{2}+k),k\in\mathbb{N}\}$, the fractional power of the Laplacian operator $\Delta=\frac{\partial^2}{\partial r^2}+\sum_{j=1}^n(\frac{\partial}{\partial x_j})^2$ is defined on $S_e(\mathbb{R}\times\mathbb{R}^n)$ by*

$$(-\Delta)^a f(r,x)=\Big(T^{m_{n+1}}_{\frac{\Gamma(\frac{n+1}{2}+a)}{\Gamma(-a)}2^{\frac{n+1}{2}+2a}(s^2+|y|^2)^{-(\frac{n+1}{2}+a)}}\star f\Big)(r,x),\tag{4.23}$$

where
i) $\star$ is the usual convolution product defined by

$$T\star f(r,x)=\langle T,\sigma_{(r,-x)}(\check{f})\rangle,\ \ T\in S_e'(\mathbb{R}\times\mathbb{R}^n),\ \ f\in S_e(\mathbb{R}\times\mathbb{R}^n);\tag{4.24}$$

ii) $\sigma_{(r,x)}$ is the translation operator associated with Λ_{n+1} and given by

$$\sigma_{(r,x)}(f)(s,y)=\frac{1}{2}\Big[f(r+s,x+y)+f(r-s,x+y)\Big],\ \ f\in S_e(\mathbb{R}\times\mathbb{R}^n).\tag{4.25}$$

It is well known that for $f\in S_e(\mathbb{R}\times\mathbb{R}^n)$ and $T\in S_e'(\mathbb{R}\times\mathbb{R}^n)$, the function $T\star f$ belongs to $\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$ and is slowly increasing. Moreover

$$\Lambda_{n+1}(T^{m_{n+1}}_{T\star f})=\Lambda_{n+1}(f)\Lambda_{n+1}(T),\tag{4.26}$$

thus from relations (4.21) and (4.26), we deduce that for $f\in S_e(\mathbb{R}\times\mathbb{R}^n)$ and $a\in\mathbb{C}\backslash\{-(\frac{n+1}{2}+k),k\in\mathbb{N}\}$,

$$\Lambda_{n+1}(T^{m_{n+1}}_{(-\Delta)^a f})=\Lambda_{n+1}(f)T^{m_{n+1}}_{(r^2+|x|^2)^a}.\tag{4.27}$$

5. Inversion formulas for $\mathscr{R}_{m,n}$ and ${}^t\mathscr{R}_{m,n}$

In this section, we will define some subspaces of $S_e(\mathbb{R}\times\mathbb{R}^n)$ where the operator $\mathscr{R}_{m,n}$ and its dual ${}^t\mathscr{R}_{m,n}$ are topological isomorphisms. Using the fractional powers defined in the precedent section we give nice expression of the inverse operators.

We denote by

• $\mathscr{N}$ the subspace of $S_e(\mathbb{R}\times\mathbb{R}^n)$, consisting of functions f satisfying

$$\forall\, k\in\mathbb{N},\forall\, x\in\mathbb{R}^n,\ \ (\frac{\partial}{\partial r^2})^k f(0,x)=0, \tag{5.1}$$

where $\dfrac{\partial}{\partial r^2}$ is the singular partial differential operator defined by

$$\frac{\partial}{\partial r^2}=\frac{1}{r}\frac{\partial}{\partial r}. \tag{5.2}$$

• $S_{e,0}(\mathbb{R}\times\mathbb{R}^n)$ the subspace of $S_e(\mathbb{R}\times\mathbb{R}^n)$, constituted by the functions f satisfying

$$\forall\, k\in\mathbb{N},\forall\, x\in\mathbb{R}^n,\ \ \int_0^{+\infty} f(r,x)r^{2k}dr=0. \tag{5.3}$$

• $S_e^0(\mathbb{R}\times\mathbb{R}^n)$ the subspace of $S_e(\mathbb{R}\times\mathbb{R}^n)$, constituted by the functions f satisfying

$$\text{supp}(\widetilde{\mathscr{F}}_{m,n}(f))\subset\Big\{(\mu,\lambda)\in\mathbb{R}\times\mathbb{R}^n;\ |\mu|\geqslant|\lambda|\Big\}. \tag{5.4}$$

Lemma 5.1. *i) The usual Fourier transform Λ_{n+1} defined by relation (3.11) is an isomorphism from $S_{e,0}(\mathbb{R}\times\mathbb{R}^n)$ onto $\mathscr{N}$.*
ii) The subspace $\mathscr{N}$ can be written as

$$\mathscr{N}=\Big\{f\in S_e(\mathbb{R}\times\mathbb{R}^n);\forall\, k\in\mathbb{N},\forall\, x\in\mathbb{R}^n,(\frac{\partial}{\partial r})^{2k}f(0,x)=0\Big\}. \tag{5.5}$$

Proof. Let $f\in S_{e,0}(\mathbb{R}\times\mathbb{R}^n)$.
i) For $m\geqslant 0$, we have

$$(\frac{\partial}{\partial\mu^2})^k(j_{\frac{m-1}{2}})(r\mu)=\frac{(-1)^k\Gamma(\frac{m+1}{2})}{2^k\Gamma(\frac{m+1}{2}+k)}r^{2k}j_{\frac{m-1}{2}+k}(r\mu), \tag{5.6}$$

thus, from the expression of Λ_{n+1}, given in Remark 3.6, and the fact that $j_{\frac{-1}{2}}(s)=\cos s$, we obtain

$$(\frac{\partial}{\partial\mu^2})^k(\Lambda_{n+1}(f))(0,\lambda)=\frac{(-1)^k}{2^{k-\frac{1}{2}}\Gamma(k+\frac{1}{2})(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\int_0^{+\infty}f(r,x)r^{2k}e^{-i\langle\lambda|x\rangle}drdx, \tag{5.7}$$

which gives the result.
ii) The proof of ii) is immediate. □

Theorem 5.2. *i) For every real number a, the transforms $\mathscr{A}_a(f)$ and $\mathscr{B}_a(f)$ defined respectively on $\mathscr{N}$ by*

$$\mathscr{A}_a(f)(r,x)=(r^2+|x|^2)^a f(r,x),$$

and

$$\mathscr{B}_a(f)(r,x)=|r|^a f(r,x),$$

are isomorphisms from the space $\mathscr{N}$ onto itself.
ii) For $f\in\mathscr{N}$, the function $B^{-1}(f)$ defined by

$$B^{-1}(f)(r,x)=\begin{cases} f(\sqrt{r^2-|x|^2},x), & \text{if } |r|\geqslant|x|,\\ 0, & \text{otherwise},\end{cases}$$

belongs to the space $S_e(\mathbb{R}\times\mathbb{R}^n)$.

Proof. i) Let a be a real number, by induction for every $k \in \mathbb{N}$, there is a polynomial P_k of $n+1$ variables such that

$$(\frac{\partial}{\partial r})^k (r^2 + |x|^2)^a = P_k(r,x)(r^2 + |x|^2)^{a-k}.$$

Hence, for every $(k,\alpha) \in \mathbb{N} \times \mathbb{N}^n$, there is a polynomial $P_{k,\alpha}$ satisfying for every $(r,x) \in \mathbb{R} \times \mathbb{R}^n \backslash \{(0,0)\}$,

$$(\frac{\partial}{\partial r})^k D_x^\alpha (r^2 + |x|^2)^a = P_{k,\alpha}(r,x)(r^2 + |x|^2)^{a-k-|\alpha|}. \tag{5.8}$$

Let f be a function of the space $\mathscr{N}$, and let $(k,\alpha) \in \mathbb{N} \times \mathbb{N}^n$, then by Leibniz's formula, we deduce that for every $(r,x) \in \mathbb{R} \times \mathbb{R}^n \backslash \{(0,0)\}$, we have

$$\begin{aligned}(\frac{\partial}{\partial r})^k D_x^\alpha ((r^2 + |x|^2)^a f(r,x)) = &\sum_{(k_1,\beta) \leqslant (k,\alpha)} \frac{(k,\alpha)!}{(k_1,\beta)!(k-k_1,\alpha-\beta)!} \\ &\times (\frac{\partial}{\partial r})^{k_1} D_x^\beta (r^2 + |x|^2)^a (\frac{\partial}{\partial r})^{k-k_1} D_x^{\alpha-\beta}(f)(r,x),\end{aligned}$$

and from relation (5.8), we get

$$\begin{aligned}(\frac{\partial}{\partial r})^k D_x^\alpha ((r^2 + |x|^2)^a f(r,x)) = &\sum_{(k_1,\beta) \leqslant (k,\alpha)} \frac{(k,\alpha)!}{(k_1,\beta)!(k-k_1,\alpha-\beta)!} P_{k_1,\beta}(r,x) \\ &\times (r^2 + |x|^2)^{a-k_1-|\beta|} (\frac{\partial}{\partial r})^{k-k_1} D_x^{\alpha-\beta}(f)(r,x).\end{aligned}$$

Let $m, \ell \in \mathbb{N}$ satisfying $a - m + \ell > 0$, and let $k, k^{'} \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$ such that $k^{'} \leqslant m$ and $k + |\alpha| \leqslant m$. Since f belongs to the space $\mathscr{N}$, then by using Taylor's formula we have

$$\begin{aligned}&(\frac{\partial}{\partial r})^k D_x^\alpha ((r^2 + |x|^2)^a f(r,x)) \\ &\quad = \sum_{(k_1,\beta) \leqslant (k,\alpha)} \frac{(k,\alpha)!}{(k_1,\beta)!(k-k_1,\alpha-\beta)!} P_{k_1,\beta}(r,x)(r^2 + |x|^2)^{a-k_1-|\beta|} \\ &\quad \times \frac{r^{2\ell}}{(2\ell-1)!} \int_0^1 (1-t)^{2\ell-1} (\frac{\partial}{\partial r})^{2\ell+k-k_1} D_x^{\alpha-\beta}(f)(rt,x)dt; \ if \ |r| \leqslant 1,\end{aligned}$$

and

$$\begin{aligned}&(\frac{\partial}{\partial r})^k D_x^\alpha ((r^2 + |x|^2)^a f(r,x)) \\ &\quad = -\sum_{(k_1,\beta) \leqslant (k,\alpha)} \frac{(k,\alpha)!}{(k_1,\beta)!(k-k_1,\alpha-\beta)!} P_{k_1,\beta}(r,x)(r^2 + |x|^2)^{a-k_1-|\beta|} \\ &\quad \times \frac{r^{2\ell}}{(2\ell-1)!} \int_1^{+\infty} (1-t)^{2\ell-1} (\frac{\partial}{\partial r})^{2\ell+k-k_1} D_x^{\alpha-\beta}(f)(rt,x)dt; \ if \ |r| > 1.\end{aligned}$$

This shows that the function $\mathscr{A}_a(f)$ is infinitely differentiable on $\mathbb{R} \times \mathbb{R}^n$, even with respect the first variable and for every $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

$$(\frac{\partial}{\partial r})^k \Big[\mathscr{A}_a(f)(r,x))\Big]_{r=0} = 0.$$

Furthermore, there exist $m_0 \in \mathbb{N}$ and a constant $C > 0$ such that for every $(k_1,\beta) \leqslant (k,\alpha)$, $k + |\alpha| \leqslant m$, we have

$$|P_{k_1,\beta}(r,x)| \leqslant C(1 + r^2 + |x|^2)^{m_0}.$$

• For $(r,x)\in\mathbb{R}\times\mathbb{R}^n$; $|r|\leqslant 1$, we have

$$\begin{aligned}(1+r^2+&|x|^2)^{k'}\Big|(\frac{\partial}{\partial r})^k D_x^\alpha((r^2+|x|^2)^a f(r,x))\Big|\\ &\leqslant C\sum_{(k_1,\beta)\leqslant(k,\alpha)}\frac{(k,\alpha)!}{(k_1,\beta)!(k-k_1,\alpha-\beta)!}(1+r^2+|x|^2)^{k'+[a]+1-m+\ell+m_0}\\ &\times\int_0^1\Big|(\frac{\partial}{\partial r})^{2\ell+k-k_1}D_x^{\alpha-\beta}(f)(rt,x)\Big|dt\\ &\leqslant C2^{[a]+1+\ell}\sum_{(k_1,\beta)\leqslant(k,\alpha)}\frac{(k,\alpha)!}{(k_1,\beta)!(k-k_1,\alpha-\beta)!}\\ &\times\int_0^1(1+(rt)^2+|x|^2)^{[a]+1+\ell+m_0}\Big|(\frac{\partial}{\partial r})^{2\ell+k-k_1}D_x^{\alpha-\beta}(f)(rt,x)\Big|dt\\ &\leqslant C2^{[a]+1+\ell+m}\mathscr{P}_{2\ell+m+[a]+m_0+1}(f).\end{aligned}\tag{5.9}$$

• For $(r,x)\in\mathbb{R}\times\mathbb{R}^n$; $|r|>1$, we have

$$\begin{aligned}(1+r^2+&|x|^2)^{k'}\Big|(\frac{\partial}{\partial r})^k D_x^\alpha((r^2+|x|^2)^a f(r,x))\Big|\\ &\leqslant C\sum_{(k_1,\beta)\leqslant(k,\alpha)}\frac{(k,\alpha)!}{(k_1,\beta)!(k-k_1,\alpha-\beta)!}\\ &\times\int_1^{+\infty}(1+(rt)^2+|x|^2)^{[a]+1+3\ell+m_0}\Big|(\frac{\partial}{\partial r})^{2\ell+k-k_1}D_x^{\alpha-\beta}(f)(rt,x)\Big|\frac{dt}{1+t^2}\\ &\leqslant C\pi 2^m\mathscr{P}_{3\ell+m+m_0+[a]+1}(f).\end{aligned}\tag{5.10}$$

Combining relations (5.9) and (5.10), we deduce that $\mathscr{A}_a(f)$ belongs to the space $\mathscr{N}$, and for every $m\in\mathbb{N}$,

$$\mathscr{P}_m(\mathscr{A}_a(f))\leqslant 2^{m+\ell+[a]+2}\mathscr{P}_{3\ell+m+m_0+[a]+1}(f).\tag{5.11}$$

where $\mathscr{P}_m(\varphi)=\sup\limits_{\substack{(r,x)\in\mathbb{R}\times\mathbb{R}^n\\ k_1\leqslant m\\ k_2+|\alpha|\leqslant m}}(1+r^2+|x|^2)^{k_1}\Big|(\frac{\partial}{\partial r})^{k_2}D_x^\alpha\varphi(r,x)\Big|$,
$\varphi\in S_e(\mathbb{R}\times\mathbb{R}^n)$.
Hence, for every $a\in\mathbb{R}$, the transform $\mathscr{A}_a$ is continuous from the space $\mathscr{N}$ into itself, and consequently $\mathscr{A}_a$ is an isomorphism from $\mathscr{N}$ onto itself, and $\mathscr{A}_a^{-1}=\mathscr{A}_{-a}$.
Similarly, one can prove that for every $a\in\mathbb{R}$ the transform $\mathscr{B}_a$ is an isomorphism from $\mathscr{N}$ onto itself, and $\mathscr{B}_a^{-1}=\mathscr{B}_{-a}$.
ii) Let $f\in S_e(\mathbb{R}\times\mathbb{R}^n)$ and let

$$g(r,x)=B^{-1}(f)(r,x)=\begin{cases} f(\sqrt{r^2-|x|^2},x), & \text{if } |r|\geqslant|x|,\\ 0, & \text{otherwise.}\end{cases}$$

Then by induction for every $k\in\mathbb{N}$, there exist real polynomials P_k, $0\leqslant k\leqslant n$, such that

$$(\frac{\partial}{\partial r})^k(g)(r,x)=\sum_{\ell=0}^k P_\ell(r)(\frac{\partial}{\partial r^2})^\ell(f)(\sqrt{r^2-|x|^2},x).$$

On the other hand, for every $j \in \{1,...,n\}$ and again by induction on $\alpha_j \in \mathbb{N}$, we get

$$(\frac{\partial}{\partial x_j})^{\alpha_j}(\frac{\partial}{\partial r})^k(g)(r,x) = \sum_{\ell=0}^{k} P_\ell(r) C_j^{\alpha_j}((\frac{\partial}{\partial r^2})^\ell(f))(\sqrt{r^2-|x|^2},x),$$

where

$$C_j = -x_j\frac{\partial}{\partial r^2} + \frac{\partial}{\partial x_j}.$$

Hence, for every $(k,\alpha) \in \mathbb{N}\times\mathbb{N}^n$, we have

$$(\frac{\partial}{\partial r})^k D_x^\alpha(g)(r,x) = \sum_{\ell=0}^{k} P_\ell(r) C_1^{\alpha_1}...C_n^{\alpha_n}((\frac{\partial}{\partial r^2})^\ell(f))(\sqrt{r^2-|x|^2},x),$$

this shows that $B^{-1}(f)$ is a C^∞ function on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable.

Let $m \in \mathbb{N}$, $k, k^{'} \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$ satisfying $k+|\alpha| \leqslant m$ and $k^{'} \leqslant m$, then for every $(r,x) \in \mathbb{R}\times\mathbb{R}^n$, we get

$$\begin{aligned}&(1+r^2+|x|^2)^{k^{'}}(\frac{\partial}{\partial r})^k D_x^\alpha(B^{-1}(f))(r,x)\\ &\quad = (1+r^2+|x|^2)^{k^{'}}\sum_{\ell=0}^{k} P_\ell(r) C_1^{\alpha_1}...C_n^{\alpha_n}((\frac{\partial}{\partial r^2})^\ell(f))(\sqrt{r^2-|x|^2},x).\end{aligned}$$

Let $m_0 \in \mathbb{N}$ and $M > 0$ such that for every $\ell \leqslant k \leqslant m$

$$|P_\ell(r)| \leqslant \mathrm{M}(1+r^2)^{m_0} \leqslant M(1+r^2+|x|^2)^{m_0},$$

then for every $(r,x) \in \mathbb{R}\times\mathbb{R}^n$, we have

$$\begin{aligned}&(1+r^2+|x|^2)^{k^{'}}\Big|(\frac{\partial}{\partial r})^k D_x^\alpha(B^{-1}(f))(r,x)\Big|\\ &\quad\leqslant M\sum_{\ell=0}^{k}(1+r^2+|x|^2)^{k^{'}+m_0}\Big|C_1^{\alpha_1}...C_n^{\alpha_n}((\frac{\partial}{\partial r^2})^\ell(f))(\sqrt{r^2-|x|^2},x)\Big|\\ &\quad\leqslant M\sum_{\ell=0}^{k}(1+r^2+2|x|^2)^{k^{'}+m_0}\Big|C_1^{\alpha_1}...C_n^{\alpha_n}((\frac{\partial}{\partial r^2})^\ell(f))(\sqrt{r^2-|x|^2},x)\Big|\\ &\quad\leqslant M\sum_{\ell=0}^{k}\mathscr{P}_{m+m_0}\Big(C_1^{\alpha_1}...C_n^{\alpha_n}((\frac{\partial}{\partial r^2})^\ell(f))\Big).\end{aligned} \tag{5.12}$$

Therefore the function $B^{-1}(f)$ belongs to the space $S_e(\mathbb{R}\times\mathbb{R}^n)$. □

Theorem 5.3. *The Fourier transform $\mathscr{F}_{m,n}$ associated with the integral transform $\mathscr{R}_{m,n}$ is an isomorphism from $S_e^0(\mathbb{R}\times\mathbb{R}^n)$ onto $\mathscr{N}$.*

Proof. Let $f \in S_e^0(\mathbb{R}\times\mathbb{R}^n)$. From relation (3.6), we get

$$\begin{aligned}(\frac{\partial}{\partial \mu^2})^k(\mathscr{F}_{m,n}(f))(0,\lambda) &= (\frac{\partial}{\partial \mu^2})^k\big(\widetilde{\mathscr{F}}_{m,n}(f)\circ\theta\big)(0,\lambda)\\ &= \Big((\frac{\partial}{\partial \mu^2})^k\widetilde{\mathscr{F}}_{m,n}(f)\Big)\circ\theta(0,\lambda)\\ &= (\frac{\partial}{\partial \mu^2})^k(\widetilde{\mathscr{F}}_{m,n}(f))(|\lambda|,\lambda) = 0,\end{aligned} \tag{5.13}$$

because $\operatorname{supp}(\widetilde{\mathscr{F}}_{m,n}(f)) \subset \Big\{(\mu,\lambda)\in\mathbb{R}\times\mathbb{R}^n;\ |\mu|\geqslant|\lambda|\Big\}$. This shows that $\mathscr{F}_{m,n}$ maps injectively $S_e^0(\mathbb{R}\times\mathbb{R}^n)$ onto $\mathscr{N}$. On the other hand, let $h\in\mathscr{N}$ and

$$g(r,x)=\begin{cases} h(\sqrt{r^2-|x|^2},x), & \text{if } |r|\geqslant|x|,\\ 0, & \text{otherwise.}\end{cases}$$

From Theorem 5.2 ii), it follows that g belongs to $S_e(\mathbb{R}\times\mathbb{R}^n)$, then there exists $f\in S_e(\mathbb{R}\times\mathbb{R}^n)$ such that $\widetilde{\mathscr{F}}_{m,n}(f)=g$. Consequently, $f\in S_e^0(\mathbb{R}\times\mathbb{R}^n)$ and $\mathscr{F}_{m,n}(f)=h$. □

From Lemma 5.1, and Theorem 5.3, we deduce the following result

Corollary 5.4. *The dual transform ${}^t\mathscr{R}_{m,n}$ is an isomorphism from $S_e^0(\mathbb{R}\times\mathbb{R}^n)$ onto $S_{e,0}(\mathbb{R}\times\mathbb{R}^n)$.*

Theorem 5.5. *The operator $K^1_{m,n}$ defined by*

$$K^1_{m,n}(f)=\frac{\sqrt{\pi}}{2^{\frac{m}{2}}\Gamma(\frac{m+1}{2})}\Big(-\frac{\partial^2}{\partial r^2}\Big)^{\frac12}(-\Delta)^{\frac{m-1}{2}}f, \tag{5.14}$$

is an isomorphism from $S_{e,0}(\mathbb{R}\times\mathbb{R}^n)$ onto itself, where

$$\Big(-\frac{\partial^2}{\partial r^2}\Big)^{\frac12}f(r,x)=(-\ell_{\frac{-1}{2}})^{\frac12}\,(f(.,x))\,(r). \tag{5.15}$$

Proof. Let $f\in S_{e,0}(\mathbb{R}\times\mathbb{R}^n)$ and $\varphi\in S_e(\mathbb{R}\times\mathbb{R}^n)$. Using Fubini's theorem, we get

$$\begin{aligned}&\langle\Lambda_{n+1}(T^{m_{n+1}}_{(-\frac{\partial^2}{\partial r^2})^{\frac12}f}),\varphi\rangle\\ &=\int_0^{+\infty}\int_{\mathbb{R}^n}\Big(-\frac{\partial^2}{\partial r^2}\Big)^{\frac12}f(r,x)\Lambda_{n+1}(\varphi)(r,x)dm_{n+1}(r,x)\\ &=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big\langle T^{\omega_{\frac{-1}{2}}}_{(-\ell_{\frac{-1}{2}})^{\frac12}(f(.,x))},F_{\frac{-1}{2}}(\varphi(.,y))\Big\rangle e^{-i\langle x|y\rangle}dxdy,\end{aligned}$$

and by relation (4.18), we obtain

$$\begin{aligned}&\langle\Lambda_{n+1}(T^{m_{n+1}}_{(-\frac{\partial^2}{\partial r^2})^{\frac12}f}),\varphi\rangle\\ &=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big\langle F_{\frac{-1}{2}}(f(.,x))T^{\omega_{\frac{-1}{2}}}_{|r|},\varphi(.,y)\Big\rangle e^{-i\langle x|y\rangle}dxdy\\ &=\int_0^{+\infty}\int_{\mathbb{R}^n}s\varphi(s,y)\Big(\int_0^{+\infty}\int_{\mathbb{R}^n}f(r,x)\cos(rs)e^{-i\langle x|y\rangle}dm_{n+1}(r,x)\Big)dm_{n+1}(s,y),\end{aligned}$$

this shows that

$$\Lambda_{n+1}(T^{m_{n+1}}_{(-\frac{\partial^2}{\partial r^2})^{\frac12}f})=T^{m_{n+1}}_{|r|\Lambda_{n+1}(f)}. \tag{5.16}$$

Now, from Lemma 5.1, we deduce that the function

$$(\mu,\lambda)\to|\mu|\Lambda_{n+1}(f)(\mu,\lambda) \tag{5.17}$$

belongs to the subspace $\mathscr{N}$, then from relation (5.16), it follows that the function $(-\dfrac{\partial^2}{\partial r^2})^{\frac{1}{2}}f$ belongs to the subspace $S_{e,0}(\mathbb{R}\times\mathbb{R}^n)$, and we have

$$\forall(\mu,\lambda)\in\mathbb{R}\times\mathbb{R}^n,\quad \Lambda_{n+1}((-\frac{\partial^2}{\partial r^2})^{\frac{1}{2}}f)(\mu,\lambda)=|\mu|\Lambda_{n+1}(f)(\mu,\lambda). \tag{5.18}$$

By the same way, and using relation (4.27), we deduce that for every $f\in S_{e,0}(\mathbb{R}\times\mathbb{R}^n)$, the function $(-\Delta)^{\frac{m-1}{2}}f$ belongs to the subspace $S_{e,0}(\mathbb{R}\times\mathbb{R}^n)$, and for every $(\mu,\lambda)\in\mathbb{R}\times\mathbb{R}^n$,

$$\Lambda_{n+1}((-\Delta)^{\frac{m-1}{2}}f)(\mu,\lambda)=(\mu^2+|\lambda|^2)^{\frac{m-1}{2}}\Lambda_{n+1}(f)(\mu,\lambda). \tag{5.19}$$

From relations (5.18) and (5.19), we deduce that

$$K^1_{m,n}(f)(r,x)=\frac{\sqrt{\pi}}{2^{\frac{m}{2}}\Gamma(\frac{m+1}{2})}\Lambda^{-1}_{n+1}\Big((\mu^2+|\lambda|^2)^{\frac{m-1}{2}}|\mu|\Lambda_{n+1}(f)\Big)(r,x). \tag{5.20}$$

Hence, the Theorem follows from Lemma 5.1 and Theorem 5.2. □

We denote by

- For $T\in S'_e(\mathbb{R}\times\mathbb{R}^n)$, $\varphi\in S_e(\mathbb{R}\times\mathbb{R}^n)$,

$$\langle S_{a,b}(T),\varphi\rangle=\langle T,\ {}^tS_{a,b}(\varphi)\rangle, \tag{5.21}$$

where $S_{a,b}$; $a\geqslant b\geqslant -\dfrac{1}{2}$, is the Sonine transform defined by relation (2.8).

- For $T\in S'_e(\mathbb{R}\times\mathbb{R}^n)$, $\varphi\in S_e(\mathbb{R}\times\mathbb{R}^n)$,

$$T*\varphi(r,x)=\langle T,\tau_{(r,-x)}\check{\varphi}\rangle, \tag{5.22}$$

where $\tau_{(r,x)}$ is the translation operator given by Definition 3.1.

- $\widetilde{\mathscr{F}}_{m,n}$ is the mapping defined on $S'_e(\mathbb{R}\times\mathbb{R}^n)$ by

$$\langle\widetilde{\mathscr{F}}_{m,n}(T),\varphi\rangle=\langle T,\widetilde{\mathscr{F}}_{m,n}(\varphi)\rangle;\quad \varphi\in S_e(\mathbb{R}\times\mathbb{R}^n). \tag{5.23}$$

- $L_{\frac{m-1}{2}}$ is the operator defined on $S_e(\mathbb{R}\times\mathbb{R}^n)$ by

$$L_{\frac{m-1}{2}}(f)(r,x)=(-\ell_{\frac{m-1}{2}})^{m-1}\left(f(.,x)\right)(r), \tag{5.24}$$

where $(-\ell_{\frac{m-1}{2}})^a$ is the fractional power of Bessel operator given by Definition 4.3.

Theorem 5.6. *The operator $K^2_{m,n}$ defined by*

$$K^2_{m,n}(f)(r,x)=S_{\frac{m-1}{2},\frac{n-1}{2}}(T)*((-\Xi)L_{\frac{m-1}{2}}(\check{f}))(r,-x) \tag{5.25}$$

is an isomorphism from $S^0_e(\mathbb{R}\times\mathbb{R}^n)$ onto itself, where

- *T is the distribution defined by*

$$\langle T,\varphi\rangle=\frac{\pi}{2^m\Gamma^2(\frac{m+1}{2})(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\varphi(|y|,y)dy, \tag{5.26}$$

- *Ξ is the operator given by relation (2.2).*

Proof. For $f\in S^0_e(\mathbb{R}\times\mathbb{R}^n)$, we have

$$\begin{aligned}K^2_{m,n}(f)(r,x)&=\langle S_{\frac{m-1}{2},\frac{n-1}{2}}(T),\tau_{(r,x)}(-\Xi)L_{\frac{m-1}{2}}f\rangle\\&=\frac{\pi}{2^m\Gamma^2(\frac{m+1}{2})(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}{}^tS_{\frac{m-1}{2},\frac{n-1}{2}}\big(\tau_{(r,x)}(-\Xi)L_{\frac{m-1}{2}}f\big)(|y|,y)dy.\end{aligned}$$

Using inversion formula for the Fourier Bessel transform $F_{\frac{n-1}{2}}$ and applying the Fubini's theorem, we deduce that

$$K^2_{m,n}(f)(r,x) = \frac{\pi}{2^m\Gamma^2(\frac{m+1}{2})(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\Big\{\int_0^{+\infty}\int_0^{+\infty} {}^tS_{\frac{m-1}{2},\frac{n-1}{2}}\big(\tau_{(r,x)}(-\Xi)L_{\frac{m-1}{2}}f\big)(t,y) \times j_{\frac{n-1}{2}}(ts)j_{\frac{n-1}{2}}(s|y|)d\omega_n(t)d\omega_n(s)\Big\}dy. \tag{5.27}$$

From the fact that

$$\frac{\sqrt{\pi}}{2^{\frac{n}{2}}\Gamma(\frac{n+1}{2})}s^{n-1}j_{\frac{n-1}{2}}(s|y|) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{|\lambda|<s}\frac{e^{-i\langle\lambda|y\rangle}}{\sqrt{s^2-|\lambda|^2}}d\lambda, \tag{5.28}$$

and again by Fubini's theorem, we obtain

$$K^2_{m,n}(f)(r,x) = \frac{\sqrt{\pi}}{2^{m-\frac{1}{2}}\Gamma^2(\frac{m+1}{2})(2\pi)^{\frac{n}{2}}}\int_0^{+\infty}\int_{|\lambda|<s}\Big[\int_0^{+\infty}\int_{\mathbb{R}^n} {}^tS_{\frac{m-1}{2},\frac{n-1}{2}}\Big(\tau_{(r,x)}(-\Xi)L_{\frac{m-1}{2}}f\Big)(t,y) \times j_{\frac{n-1}{2}}(ts)e^{-i\langle\lambda|y\rangle}d\nu_{n,n}(t,y)\Big]\frac{sdsd\lambda}{\sqrt{s^2-|\lambda|^2}}. \tag{5.29}$$

Using relation (3.15), we get

$$K^2_{m,n}(f)(r,x) = \frac{\sqrt{\pi}}{2^{m-\frac{1}{2}}\Gamma^2(\frac{m+1}{2})(2\pi)^{\frac{n}{2}}}\int_0^{+\infty}\int_{|\lambda|<s}\widetilde{\mathscr{F}}_{m,n}\Big(\tau_{(r,x)}(-\Xi)L_{\frac{m-1}{2}}f\Big)(s,\lambda)\frac{sdsd\lambda}{\sqrt{s^2-|\lambda|^2}}. \tag{5.30}$$

Since for every $f\in S_e(\mathbb{R}\times\mathbb{R}^n)$, we have

$$\forall (r,x),(s,\lambda)\in\mathbb{R}\times\mathbb{R}^n,\quad \widetilde{\mathscr{F}}_{m,n}(\tau_{(r,x)}f)(s,\lambda) = j_{\frac{m-1}{2}}(rs)e^{i\langle\lambda|x\rangle}\widetilde{\mathscr{F}}_{m,n}(f)(s,\lambda), \tag{5.31}$$

we get

$$K^2_{m,n}(f)(r,x) = \frac{\sqrt{\pi}}{2^{m-\frac{1}{2}}\Gamma^2(\frac{m+1}{2})(2\pi)^{\frac{n}{2}}}\int_0^{+\infty}\int_{|\lambda|<s}\widetilde{\mathscr{F}}_{m,n}((-\Xi)L_{\frac{m-1}{2}}f)(s,\lambda) \times j_{\frac{m-1}{2}}(rs)e^{i\langle\lambda|x\rangle}\frac{sdsd\lambda}{\sqrt{s^2-|\lambda|^2}}. \tag{5.32}$$

On the other hand, for $f\in S^0_e(\mathbb{R}\times\mathbb{R}^n)$, the function $L_{\frac{m-1}{2}}f$ belongs to $\mathscr{E}_e(\mathbb{R}\times\mathbb{R}^n)$, and is slowly increasing. Moreover, we have

$$\widetilde{\mathscr{F}}_{m,n}(T^{\nu_{m,n}}_{(-\ell_{\frac{m-1}{2}})^{m-1}f(.,x)(r)}) = T^{\nu_{m,n}}_{|r|^{2m-2}\widetilde{\mathscr{F}}_{m,n}(f)}. \tag{5.33}$$

But, for $f\in S^0_e(\mathbb{R}\times\mathbb{R}^n)$, the function $\widetilde{\mathscr{F}}_{m,n}(f)$ belongs to the subspace $\mathscr{N}$, hence according to Theorem 5.2, we deduce that the function $L_{\frac{m-1}{2}}f$ belongs to $S_e(\mathbb{R}\times\mathbb{R}^n)$, and we have

$$\forall(\mu,\lambda)\in\mathbb{R}\times\mathbb{R}^n,\quad \widetilde{\mathscr{F}}_{m,n}(L_{\frac{m-1}{2}}f)(\mu,\lambda) = |\mu|^{2m-2}\widetilde{\mathscr{F}}_{m,n}(f)(\mu,\lambda). \tag{5.34}$$

This implies that

$$K^2_{m,n}(f)(r,x) = \frac{\sqrt{\pi}}{2^{m-\frac{1}{2}}\Gamma^2(\frac{m+1}{2})(2\pi)^{\frac{n}{2}}}\int_0^{+\infty}\int_{|\lambda|<s} s^{2m-2}(s^2-|\lambda|^2) \times \widetilde{\mathscr{F}}_{m,n}(f)(s,\lambda)j_{\frac{m-1}{2}}(rs)e^{i\langle\lambda|x\rangle}\frac{sdsd\lambda}{\sqrt{s^2-|\lambda|^2}}. \quad (5.35)$$

By a change of variables, and using Fubini's theorem, we get

$$K^2_{m,n}(f)(r,x) = \frac{\sqrt{\pi}}{2^{m-\frac{1}{2}}\Gamma^2(\frac{m+1}{2})(2\pi)^{\frac{n}{2}}}\int_0^{+\infty}\int_{\mathbb{R}^n} \mu^2(\mu^2+|\lambda|^2)^{m-1}\widetilde{\mathscr{F}}_{m,n}(f)(\sqrt{\mu^2+|\lambda|^2},\lambda) \times j_{\frac{m-1}{2}}(r\sqrt{\mu^2+|\lambda|^2})e^{i\langle\lambda|x\rangle}d\mu d\lambda. \quad (5.36)$$

From relations (2.19) and (3.6), we deduce that

$$K^2_{m,n}(f)(r,x) = \frac{\sqrt{\pi}}{2^{\frac{m}{2}}\Gamma(\frac{m+1}{2})}\mathscr{F}^{-1}_{m,n}\Big((\mu^2+|\lambda|^2)^{\frac{m-1}{2}}|\mu|\mathscr{F}_{m,n}(f)\Big)(r,x). \quad (5.37)$$

Then the result follows from relation (5.37), Lemma 5.1, Theorems 5.2 and 5.3. □

Theorem 5.7. *i) For $f \in S_{e,0}(\mathbb{R}\times\mathbb{R}^n)$ and $g \in S^0_e(\mathbb{R}\times\mathbb{R}^n)$, there exists the inversion formulas for $\mathscr{R}_{m,n}$*

$$f = K^1_{m,n}\,{}^t\mathscr{R}_{m,n}\mathscr{R}_{m,n}(f), \qquad g = \mathscr{R}_{m,n}K^1_{m,n}\,{}^t\mathscr{R}_{m,n}(g). \quad (5.38)$$

ii) For $f \in S_{e,0}(\mathbb{R}\times\mathbb{R}^n)$ and $g \in S^0_e(\mathbb{R}\times\mathbb{R}^n)$, there exist the inversion formulas for ${}^t\mathscr{R}_{m,n}$

$$f = {}^t\mathscr{R}_{m,n}K^2_{m,n}\mathscr{R}_{m,n}(f), \qquad g = K^2_{m,n}\mathscr{R}_{m,n}\,{}^t\mathscr{R}_{m,n}(g). \quad (5.39)$$

Proof. i) Let $g \in S^0_e(\mathbb{R}\times\mathbb{R}^n)$. From relations (2.14), (5.14), Theorem 3.3, Remark 3.6, and Theorem 5.3, we have

$$\begin{aligned}
g(r,x) &= \frac{1}{2^{\frac{m-1}{2}}\Gamma(\frac{m+1}{2})(2\pi)^{\frac{n}{2}}}\int_0^{+\infty}\int_{\mathbb{R}^n}\mu(\mu^2+|\lambda|^2)^{\frac{m-1}{2}} \\
&\times \mathscr{F}_{m,n}(g)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}d\mu d\lambda \\
&= \frac{\sqrt{\pi}}{2^{\frac{m}{2}}\Gamma(\frac{m+1}{2})}\int_0^{+\infty}\int_{\mathbb{R}^n}\mu(\mu^2+|\lambda|^2)^{\frac{m-1}{2}}\mathscr{F}_{m,n}(g)(\mu,\lambda) \\
&\times \mathscr{R}_{m,n}(\cos(\mu.)e^{i\langle\lambda|.\rangle})dm_{n+1}(\mu,\lambda) \\
&= \mathscr{R}_{m,n}\Big(\frac{\sqrt{\pi}}{2^{\frac{m}{2}}\Gamma(\frac{m+1}{2})}\int_0^{+\infty}\int_{\mathbb{R}^n}\mu(\mu^2+|\lambda|^2)^{\frac{m-1}{2}}\Lambda_{n+1}\circ{}^t\mathscr{R}_{m,n}(g)(\mu,\lambda) \\
&\times \cos(\mu.)e^{i\langle\lambda|.\rangle}dm_{n+1}(\mu,\lambda)\Big)(r,x) \\
&= \mathscr{R}_{m,n}\Big(\Lambda^{-1}_{n+1}\big(\frac{\sqrt{\pi}}{2^{\frac{m}{2}}\Gamma(\frac{m+1}{2})}\mu(\mu^2+|\lambda|^2)^{\frac{m-1}{2}}\Lambda_{n+1}\circ{}^t\mathscr{R}_{m,n}(g)(\mu,\lambda)\big)\Big)(r,x) \\
&= \mathscr{R}_{m,n}K^1_{m,n}\,{}^t\mathscr{R}_{m,n}(g). \qquad (5.40)
\end{aligned}$$

This relation, together with Corollary 5.4, relation (5.20) and Theorem 5.5, imply that the integral transform $\mathscr{R}_{m,n}$ is an isomorphism from $S_{e,0}(\mathbb{R}\times\mathbb{R}^n)$ onto $S^0_e(\mathbb{R}\times$

$\mathbb{R}^n$), and that $K^1_{m,n}\,{}^t\mathscr{R}_{m,n}$ is its inverse, in particular for $S_{e,0}(\mathbb{R}\times\mathbb{R}^n)$, we have

$$f = K^1_{m,n}\,{}^t\mathscr{R}_{m,n}\mathscr{R}_{m,n}(f). \tag{5.41}$$

ii) Let $f\in S_{e,0}(\mathbb{R}\times\mathbb{R}^n)$. From i), we have

$$f = K^1_{m,n}\,{}^t\mathscr{R}_{m,n}\mathscr{R}_{m,n}(f). \tag{5.42}$$

Let $g=\mathscr{R}_{m,n}(f)$, then $g\in S^0_e(\mathbb{R}\times\mathbb{R}^n)$, and we have

$$\mathscr{R}^{-1}_{m,n}(g) = K^1_{m,n}\,{}^t\mathscr{R}_{m,n}(g), \tag{5.43}$$

and from Remark 3.6, it follows that

$$\mathscr{R}^{-1}_{m,n}(g) = \Lambda^{-1}_{n+1}\big(\frac{\sqrt{\pi}}{2^{\frac{m}{2}}\Gamma(\frac{m+1}{2})}\mu(\mu^2+|\lambda|^2)^{\frac{m-1}{2}}\mathscr{F}_{m,n}(g)\big), \tag{5.44}$$

$$\mathscr{R}^{-1}_{m,n}\mathscr{R}^{-1}_{m,n}(g) = \mathscr{F}^{-1}_{m,n}\big(\frac{\sqrt{\pi}}{2^{\frac{m}{2}}\Gamma(\frac{m+1}{2})}\mu(\mu^2+|\lambda|^2)^{\frac{m-1}{2}}\mathscr{F}_{m,n}(g)\big) = K^2_{m,n}(g),$$

which gives

$$f = {}^t\mathscr{R}_{m,n}K^2_{m,n}\mathscr{R}_{m,n}(f). \tag{5.45}$$

□

6. Uncertainty principles for the Fourier transform $\mathscr{F}_{m,n}$

In this section, we shall use the well known generalized Beurling-Hrmander theorem established by Bonami, Demange and Jaming in [4], to prove the same result for the Fourier transform $\mathscr{F}_{m,n}$. Next, we use this result to establish two other uncertainty principles for this transform.

Theorem 6.1. [4] *Let f be a measurable function on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable such that $f\in L^2(dm_{n+1})$, and let d be a real number, $d\geqslant 0$. If*

$$\int_0^{+\infty}\int_{\mathbb{R}^n}\int_0^{+\infty}\int_{\mathbb{R}^n}\frac{|f(r,x)||\Lambda_{n+1}(f)(s,y)|}{(1+|(r,x)|+|(s,y)|)^d}e^{|(r,x)||(s,y)|}dm_{n+1}(r,x)dm_{n+1}(s,y) < +\infty,$$

then there exist a positive constant a and a polynomial P on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable, such that

$$f(r,x) = P(r,x)e^{-a(r^2+|x|^2)},$$

with $deg(P) < \dfrac{d-(n+1)}{2}$.

Theorem 6.2 (Hörmander-Beurling for $\mathscr{F}_{m,n}$)**.** *Let $f\in L^2(d\nu_{m,n})$, and let d be a real number, $d\geqslant 0$. If*

$$\iint_{\Upsilon_+}\int_0^{+\infty}\int_{\mathbb{R}^n}\frac{|f(r,x)||\mathscr{F}_{m,n}(f)(\mu,\lambda)|}{(1+|(r,x)|+|\theta(\mu,\lambda)|)^d}e^{|(r,x)||\theta(\mu,\lambda)|}\,d\nu_{m,n}(r,x)\,d\gamma_{m,n}(\mu,\lambda) < +\infty.$$

Then there exist a positive constant a and a polynomial P on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable, such that

$$\forall (r,x)\in\mathbb{R}\times\mathbb{R}^n,\quad f(r,x) = P(r,x)e^{-a(r^2+|x|^2)},$$

with $deg(P) < \dfrac{d-(m+n+1)}{2}$.

Proof. Let $d\lambda_{m+n+1}$ be the normalized Lebesgue measure defined on $\mathbb{R}^{m+1}\times\mathbb{R}^n$ by

$$d\lambda_{m+n+1}(y,x)=\frac{dy}{(2\pi)^{\frac{m+1}{2}}}\otimes\frac{dx}{(2\pi)^{\frac{n}{2}}},$$

and let $L^2(d\lambda_{m+n+1})$ be the space of square integrable functions on $\mathbb{R}^{m+1}\times\mathbb{R}^n$ with respect to the measure $d\lambda_{m+n+1}$.
For $f\in L^2(d\nu_{m,n})$, we denote by g the function defined on $\mathbb{R}^{m+1}\times\mathbb{R}^n$ by

$$\forall (y,x)\in\mathbb{R}^{m+1}\times\mathbb{R}^n,\quad g(y,x)=f(|y|,x),$$

then the function g belongs to $L^2(d\lambda_{m+n+1})$, and we have

$$\|g\|_{2,\lambda_{m+n+1}}=\|f\|_{2,\nu_{m,n}}.$$

Furthermore,

$$\forall (\mu,\lambda)\in\mathbb{R}^{m+1}\times\mathbb{R}^n,\quad \Lambda_{m+n+1}(g)(\mu,\lambda)=\widetilde{\mathscr{F}}_{m,n}(f)(|\mu|,\lambda), \tag{6.1}$$

where Λ_{m+n+1} is the usual Fourier transform defined on $\mathbb{R}^{m+1}\times\mathbb{R}^n$ by

$$\Lambda_{m+n+1}(g)(\mu,\lambda)=\int_{\mathbb{R}^{m+1}}\int_{\mathbb{R}^n} g(y,x)e^{-i\langle\mu|y\rangle}e^{-i\langle\lambda|x\rangle}d\lambda_{m+n+1}(y,x).$$

If

$$\int\int_{\Upsilon_+}\int_0^{+\infty}\int_{\mathbb{R}^n}\frac{|f(r,x)||\mathscr{F}_{m,n}(f)(\mu,\lambda)|}{(1+|(r,x)|+|\theta(\mu,\lambda)|)^d}e^{|(r,x)||\theta(\mu,\lambda)|}\,d\nu_{m,n}(r,x)\,d\gamma_{m,n}(\mu,\lambda)<+\infty.$$

Then by relations (2.19) and (6.1), we have

$$\begin{aligned}
&\int_{\mathbb{R}^{m+1}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^{m+1}}\int_{\mathbb{R}^n}\frac{|g(y,x)||\Lambda_{m+n+1}(g)(\zeta,\lambda)|}{(1+|(y,x)|+|(\zeta,\lambda)|)^d}e^{|(r,x)||(\zeta,\lambda)|}\,d\lambda_{m+n+1}(y,x)\,d\lambda_{m+n+1}(\zeta,\lambda)\\
&\quad=\int_0^{+\infty}\int_{\mathbb{R}^n}\int_0^{+\infty}\int_{\mathbb{R}^n}\frac{|f(r,x)||\widetilde{\mathscr{F}}_{m,n}(f)(\mu,\lambda)|}{(1+|(r,x)|+|(\mu,\lambda)|)^d}e^{|(r,x)||\mu,\lambda|}\,d\nu_{m,n}(r,x)\,d\nu_{m,n}(\mu,\lambda)\\
&\quad=\int\int_{\Upsilon_+}\int_0^{+\infty}\int_{\mathbb{R}^n}\frac{|f(r,x)||\mathscr{F}_{m,n}(f)(\mu,\lambda)|}{(1+|(r,x)|+|\theta(\mu,\lambda)|)^d}e^{|(r,x)||\theta(\mu,\lambda)|}\,d\nu_{m,n}(r,x)\,d\gamma_{m,n}(\mu,\lambda)\\
&\quad<+\infty,
\end{aligned}$$

and therefore by applying Theorem 6.1, we deduce that there exists a positive constant a and a polynomial $\widetilde{P}$ on $\mathbb{R}^{m+1}\times\mathbb{R}^n$, such that

$$g(y,x)=\widetilde{P}(y,x)e^{-a(|y|^2+|x|^2)},$$

with $deg(\widetilde{P})<\dfrac{d-(m+n+1)}{2}$.
Now, the polynomial P defined on $\mathbb{R}\times\mathbb{R}^n$ by

$$P(r,x)=\widetilde{P}((r,0,\dots,0),x),$$

is even with respect to the first variable with $deg(P)<\dfrac{d-(m+n+1)}{2}$ and

$$\forall (r,x)\in\mathbb{R}\times\mathbb{R}^n,\quad f(r,x)=P(r,x)e^{-a(r^2+|x|^2)}.$$

□

Lemma 6.3. *Let P be a polynomial on $\mathbb{R}\times\mathbb{R}^n$, $P\neq 0$, with $deg(P)=k$. Then there exist two positive constants A and C such that*

$$\forall t\geqslant A,\quad p(t)=\frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}\int_{S^n}\big|P(t\omega)\big|d\sigma_n(\omega)\geqslant Ct^k.$$

Proof. Let P be a polynomial on $\mathbb{R}\times\mathbb{R}^n$, $P\neq 0$, with $deg(P)=k$. Then, we have

$$p(t)=\frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}\int_{S^n}\big|\sum_{j=0}^{k}a_j(\omega)t^j\big|d\sigma_n(\omega),$$

where the functions a_j, $0\leqslant j\leqslant k$, are continuous on S^n. It's clear that the function p is continuous on $[0,+\infty[$, and by dominate convergence theorem's, we have

$$p(t)\sim C_k t^k\quad (t\longrightarrow+\infty), \tag{6.2}$$

where $C_k=\dfrac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}\displaystyle\int_{S^n}|a_k(\omega)|d\sigma_n(\omega)>0.$

Now relation (6.2) implies that there is a positive constant A such that

$$\forall t\geqslant A,\ p(t)\geqslant\frac{C_k}{2}t^k.$$

□

Theorem 6.4 (Gelfand-Shilov for $\mathscr{R}_{m,n}$)**.** *Let p,q be two conjugate exponents, $p,q\in]1,+\infty[$ and let ξ,η be non negative real numbers such that $\xi\eta\geqslant 1$. Let f be a measurable function on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable, such that $f\in L^2(d\nu_{m,n})$.*
If

$$\int_0^{+\infty}\int_{\mathbb{R}^n}\frac{|f(r,x)|e^{\frac{\xi^p|(r,x)|^p}{p}}}{(1+|(r,x)|)^d}\,d\nu_{m,n}(r,x)<+\infty,$$

and

$$\iint_{\Upsilon_+}\frac{|\mathscr{F}_{m,n}(f)(\mu,\lambda)|e^{\frac{\eta^q|\theta(\mu,\lambda)|^q}{q}}}{(1+|\theta(\mu,\lambda)|)^d}\,d\gamma_{m,n}(\mu,\lambda)<+\infty\ ;\ d\geqslant 0.$$

Then
i) For $d\leqslant\dfrac{m+n+1}{2}$, $f=0$.
ii) For $d>\dfrac{m+n+1}{2}$, we have
  a) $f=0$ for $\xi\eta>1$.
  b) $f=0$ for $\xi\eta=1$, and $p\neq 2$.
  c) $f(r,x)=P(r,x)e^{-a(r^2+|x|^2)}$ for $\xi\eta=1$ and $p=q=2$,
where $a>0$ and P is a polynomial on $\mathbb{R}\times\mathbb{R}^n$ even with respect to the first variable, with $deg(P)<d-\dfrac{m+n+1}{2}$.

Proof. Let f be a function satisfying the hypothesis. Since $\xi\eta \geqslant 1$, and by a convexity argument, we have

$$\begin{aligned}
&\iint_{\Upsilon_+}\int_0^{+\infty}\int_{\mathbb{R}^n}\frac{|f(r,x)||\mathscr{F}_{m,n}(f)(\mu,\lambda)|}{(1+|(r,x)|+|\theta(\mu,\lambda)|)^{2d}}e^{|(r,x)||\theta(\mu,\lambda)|}\,d\nu_{m,n}(r,x)\,d\gamma_{m,n}(\mu,\lambda)\\
&\leqslant \iint_{\Upsilon_+}\int_0^{+\infty}\int_{\mathbb{R}^n}\frac{|f(r,x)||\mathscr{F}_{m,n}(f)(\mu,\lambda)|}{(1+|(r,x)|)^{d}(1+|\theta(\mu,\lambda)|)^{d}}e^{\xi\eta|(r,x)||\theta(\mu,\lambda)|}\,d\nu_{m,n}(r,x)\,d\gamma_{m,n}(\mu,\lambda)\\
&\leqslant \Big(\int_0^{+\infty}\int_{\mathbb{R}^n}\frac{|f(r,x)|}{(1+|(r,x)|)^{d}}e^{\frac{\xi^p|(r,x)|^p}{p}}\,d\nu_{m,n}(r,x)\Big)\\
&\times\Big(\iint_{\Upsilon_+}\frac{|\mathscr{F}_{m,n}(f)(\mu,\lambda)|}{(1+|\theta(\mu,\lambda)|)^{d}}e^{\frac{\eta^q|\theta(\mu,\lambda)|^q}{q}}\,d\gamma_{m,n}(\mu,\lambda)\Big)\\
&< +\infty. \qquad (6.3)
\end{aligned}$$

Then from the Beurling-Hörmander theorem, we deduce that there exist a positive constant a and a polynomial P such that

$$f(r,x) = P(r,x)e^{-a(r^2+|x|^2)}, \qquad (6.4)$$

with $deg(P) < d - \dfrac{m+n+1}{2}$. In particular if $d \leqslant \dfrac{m+n+1}{2}$, then f vanishes almost everywhere.

Suppose now that $d > \dfrac{m+n+1}{2}$. By a standard computation, we obtain

$$\widetilde{\mathscr{F}}_{m,n}(f)(\mu,\lambda) = Q(\mu,\lambda)e^{-\frac{1}{4a}(\mu^2+|\lambda|^2)}, \qquad (6.5)$$

where Q is a polynomial on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable, with

$$deg(P) = deg(Q).$$

On the other hand, from relations (2.19), (3.6), (6.3), (6.4) and (6.5), we get

$$\begin{aligned}
\int_0^{+\infty}\int_{\mathbb{R}^n}\int_0^{+\infty}\int_{\mathbb{R}^n}&\frac{|P(r,x)||Q(\mu,\lambda)|}{(1+|(r,x)|)^{d}(1+|(\mu,\lambda)|)^{d}}e^{\xi\eta|(r,x)||(\mu,\lambda)|-a(r^2+|x|^2)}\\
&\times e^{-\frac{1}{4a}(\mu^2+|\lambda|^2)}\,d\nu_{m,n}(r,x)\,d\nu_{m,n}(\mu,\lambda) < +\infty,
\end{aligned}$$

hence,

$$\int_0^{+\infty}\int_0^{+\infty}\frac{\varphi(s)}{(1+s)^d}\frac{\psi(\rho)}{(1+\rho)^d}e^{\xi\eta s\rho}e^{-as^2}e^{-\frac{1}{4a}\rho^2}s^{m+n}\rho^{m+n}ds d\rho < +\infty, \qquad (6.6)$$

where

$$\varphi(s) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}\int_{S^n}\big|P(s\omega)\big||\omega_1|^m d\sigma_n(\omega), \quad \omega = (\omega_1,\dots,\omega_n)$$

and

$$\psi(\rho) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}\int_{S^n}\big|Q(\rho\omega)\big||\omega_1|^m d\sigma_n(\omega).$$

• Suppose that $\xi\eta > 1$. If $f \neq 0$, then each of polynomials P and Q is not identically zero, let $k = deg(P) = deg(Q)$.

From lemma 6.3, there exist two positive constants A and C such that

$$\forall t \geqslant A, \quad \varphi(s) \geqslant Cs^k,$$

and

$$\forall \rho \geqslant A, \quad \psi(\rho) \geqslant C\rho^k.$$

Then, the inequality (6.6) leads to

$$\int_A^{+\infty}\int_A^{+\infty}\frac{e^{\xi\eta s\rho}}{(1+s)^d(1+\rho)^d}e^{-as^2}e^{-\frac{1}{4a}\rho^2}dsd\rho<+\infty. \tag{6.7}$$

Let $\varepsilon>0$, such that $\xi\eta-\varepsilon=\sigma>1$. relation (6.7) implies that

$$\int_A^{+\infty}\int_A^{+\infty}\frac{e^{\varepsilon s\rho}}{(1+s)^d(1+\rho)^d}e^{\sigma s\rho}e^{-as^2}e^{-\frac{1}{4a}\rho^2}dsd\rho<+\infty. \tag{6.8}$$

However, for every $s\geqslant A\geqslant\dfrac{d}{\varepsilon}$ and $\rho\geqslant A$, we have

$$\frac{e^{\varepsilon\rho s}}{(1+s)^d(1+\rho)^d}\geqslant\frac{e^{\varepsilon A^2}}{(1+A)^{2d}},$$

and by relation (6.8) it follows that

$$\int_A^{+\infty}\int_A^{+\infty}e^{\sigma s\rho}e^{-as^2}e^{-\frac{1}{4a}\rho^2}dsd\rho<+\infty. \tag{6.9}$$

Let $F(s)=\displaystyle\int_A^{+\infty}e^{\sigma\rho s-\frac{1}{4a}\rho^2}d\rho$, then F can be written

$$F(s)=e^{a\sigma^2s^2}\Big(\int_A^{+\infty}e^{-\frac{1}{4a}\rho^2}d\rho+2a\sigma e^{-\frac{A^2}{4a}}\int_0^s e^{A\sigma w-a\sigma^2w^2}dw\Big),$$

in particular

$$F(s)\geqslant e^{a\sigma^2s^2}\int_A^{+\infty}e^{-\frac{1}{4a}\rho^2}d\rho.$$

Since $\sigma>1$, then

$$\begin{aligned}\int_A^{+\infty}\int_A^{+\infty}e^{\sigma s\rho}e^{-as^2}e^{-\frac{1}{4a}\rho^2}dsd\rho&=\int_A^{+\infty}e^{-as^2}F(s)ds\\&\geqslant\int_A^{+\infty}e^{-\frac{1}{4a}\rho^2}d\rho\int_A^{+\infty}e^{a(\sigma^2-1)s^2}ds=+\infty.\end{aligned}$$

This contradicts relation (6.9) and shows that $f=0$.

• Suppose that $\xi\eta=1$ and $p\neq2$. In this case we have $p>2$ or $q>2$. Suppose that $q>2$, then from the second hypothesis and relation (6.5), we have

$$\int_0^{+\infty}\frac{\psi(\rho)e^{-\frac{\rho^2}{4a}}e^{\frac{\eta^q\rho^q}{q}}}{(1+\rho)^d}\rho d\rho<+\infty. \tag{6.10}$$

If $f\neq0$, then the polynomial Q is not identically zero, and by Lemma 6.3 and by relation (6.10), it follows that

$$\int_0^{+\infty}\frac{e^{-\frac{\rho^2}{4a}}e^{\frac{\eta^q\rho^q}{q}}}{(1+\rho)^d}d\rho<+\infty,$$

which is impossible since $q>2$ and the proof of Theorem 6.4 is complete. □

Theorem 6.5 (Cowling-Price for $\mathscr{R}_{m,n}$). *Let $\xi,\eta,\omega_1,\omega_2$ be non negative real numbers such that $\xi\eta\geqslant\dfrac{1}{4}$. Let p,q be two exponents, $p,q\in[1,+\infty]$, and let f be a measurable function on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable such that*

$f \in L^2(d\nu_{m,n})$.
If

$$\left\| \frac{e^{\xi|(.,.)|^2}}{(1+|(.,.)|)^{\omega_1}} f \right\|_{p,\nu_{m,n}} < +\infty, \tag{6.11}$$

and

$$\left\| \frac{e^{\eta|\theta(.,.)|^2}}{(1+|\theta(.,.)|)^{\omega_2}} \mathscr{F}_{m,n}(f) \right\|_{q,\gamma_{m,n}} < +\infty, \tag{6.12}$$

then
i) For $\xi\eta > \frac{1}{4}$, $f = 0$.
ii) For $\xi\eta = \frac{1}{4}$, *there exist a positive constant* a *and a polynomial* P *on* $\mathbb{R} \times \mathbb{R}^n$, *even with respect to the first variable, such that*

$$f(r,x) = P(r,x)e^{-a(r^2+|x|^2)}.$$

Proof. Let p' and q' be the conjugate exponents of p respectively q. Let us pick $d_1, d_2 \in \mathbb{R}$, such that $d_1 > m+n+1$ and $d_2 > m+n+1$. Finally, let d be a positive real number such that $d > max\big(\omega_1 + \frac{d_1}{p'}, \omega_2 + \frac{d_2}{q'}, \frac{m+n+1}{2}\big)$.
From Hölder's inequality and relations (6.11) and (6.12), we deduce that

$$\int_0^{+\infty}\int_{\mathbb{R}^n} \frac{|f(r,x)|e^{\xi|(r,x)|^2}}{(1+|(r,x)|)^{\omega_1+\frac{d_1}{p'}}}\, d\nu_{m,n}(r,x) < +\infty,$$

and

$$\iint_{\Upsilon_+} \frac{|\mathscr{F}_{m,n}(f)(\mu,\lambda)|e^{\eta|\theta(\mu,\lambda)|^2}}{(1+|\theta(\mu,\lambda)|)^{\omega_2+\frac{d_2}{q'}}}\, d\gamma_{m,n}(\mu,\lambda) < +\infty.$$

Consequently, we have

$$\int_0^{+\infty}\int_{\mathbb{R}^n} \frac{|f(r,x)|e^{\xi|(r,x)|^2}}{(1+|(r,x)|)^{d}}\, d\nu_{m,n}(r,x) < +\infty,$$

and

$$\iint_{\Upsilon_+} \frac{|\mathscr{F}_{m,n}(f)(\mu,\lambda)|e^{\eta|\theta(\mu,\lambda)|^2}}{(1+|\theta(\mu,\lambda)|)^{d}}\, d\gamma_{m,n}(\mu,\lambda) < +\infty.$$

Then, the desired result follows from Theorem 6.4. □

Remark 6.6. *Hardy's Theorem is a special case of Theorem 6.5 when* $p = q = +\infty$.

References

[1] G. Andrews, R. Askey, and R. Roy, *Special functions,* Cambridge University Press, New-York, 1999.
[2] C. Baccar, S. Omri, and L. T. Rachdi, *Fock spaces connected with spherical mean operator and associated operators,* Mediterr. J. Math., **6 1** (2009) 1-25.
[3] C. Baccar, N.B. Hamadi, L.T. Rachdi, *Inversion formulas for the Riemann-Liouville transform and its dual associated with singular partial differential operators*, Int. J. Math. Math. Sci., Article ID 86238, **2006** (2006) 1-26.
[4] A. Bonami, B. Demange, and P. Jaming, *Hermite functions and uncertainty priciples for the Fourier and the widowed Fourier transforms,* Rev. Mat. Iberoamericana., **19** (2003) 23-55.
[5] R. Courant and D. Hilbert, *Methods of mathematical physics,* Vol. II: Partial differential equations, Wiley Interscience, New-York, 1962.

[6] M. G. Cowling and J. F. Price, *Generalization of Heisenberg's inequality in harmonic analysis,* (Cortona, 1982), Lecture Notes in math., **992** (1983) 443-449.
[7] M. Dziri, M. Jelassi, and L. T. Rachdi, *Spaces of DLp type and a convolution product associated with the spherical mean operator,* Int. J. Math. Math. Sci., **3** (2005) 357-381.
[8] A. Erdely and all, *Tables of integral transforms,* Mc Graw-Hill Book Compagny., New-York, 1954.
[9] A. Erdely and all, *Asymptotic expansions,* Dover publications., New-York, 1956.
[10] J. A. Fawcett, *Inversion of N-dimensional spherical means,* SIAM. J. Appl. Math., **45** (1985) 336-341.
[11] G. B. Folland and A. Sitaram, *The uncertainty principle: A mathematical survey,* J. Fourier Anal. Appl., **3** (1997) 207-238.
[12] V. Havin and B. Jöricke, *An uncertainty principle in harmonic analysis,* Springer Verlag., Berlin, 1994.
[13] H. Helesten, L. E. Andersson, *An inverse method for the processing of synthetic aperture radar data,* Inverse Problems, **4** (1987) 111-124.
[14] M. Herberthson, *A numerical implementation of an inverse formula for CARABAS Raw Data,* Internal Report D 30430-3.2, National Defense Research Institute, FOA, Box 1165, S-581 11, Linköping, sweden, 1986.
[15] K. Hleili, S. Omri, L. T. Rachdi, *Uncertainty principle for the Riemann-Liouville operator,* Cubo, **13 03** (2011) 119126.
[16] M. Jelassi, L. T. Rachdi, *On the range of the Fourier transform associated with the spherical mean operator,* Fract. Calc. Appl. Anal., **7 4** (2004) 379-402.
[17] N.N. Lebedev, *Special functions and their applications,* Dover publications, Inc., New-York, 1972.
[18] G. W. Morgan, *A note on Fourier transforms,* J. London. Math. Soc., **9** (1934) 178-192.
[19] N. Msehli, L.T. Rachdi, *Beurling-Hörmander Uncertainty Principle for the spherical mean operator,* J. Inequal. Pure Appl. Math., **10 2** (2009) Art. 38.
[20] M. Nessibi, L. T. Rachdi, K. Trimèche, *Ranges and inversion formulas for spherical mean operator and its Dual,* J. Math. Anal. Appl., **196** (1995) 861-884.
[21] S. Omri, *Local uncertainty principle for the Hankel transform,* Integral Transforms Spec. Funct.,**21 9** (2010) 703-712.
[22] S. Omri, L. T. Rachdi, *Heisenberg-Pauli-Weyl uncertainty principle for the Riemann- Liouvill operator,* J. Inequal. Pure Appl. Math., **9 3** (2008) Art. 88.
[23] L. Peng, J. Zhao, *Wavelet and Weyl transforms associated with the spherical mean operator,* Integr. equ. oper. theory., **50** (2004) 279-290.
[24] L. T. Rachdi, K. Trimèche, *Weyl transforms associated with the spherical mean operator,* Anal. Appl., **1 2** (2003) 141-164.
[25] G.N. Watson, *A treatise on the Theory of Bessel Functions,* 2nd ed. Cambridge Univ. Press., London/New-York, 1966.
[26] J. Zhao, *Localization operators associated with the sphercial mean operator,* Anal. Theory Appl., **21 4** (2005) 317-325.

Khaled Hleili
Department of Mathematics and Informatics, National Institute of applied science and technology of Tunis, Centre Urbain Nord BP 676 - 1080 Tunis cedex, Tunisia
E-mail address: khaled.hleili@gmail.com

Slim Omri
Preparatory institute of Engineers study, Campus Universitaire, Merazka, 8000 Nabeul, Tunisia
E-mail address: slim.omri@ipein.rnu.tn

Lakhdar Tannech Rachdi
Department of Mathematics, Faculty of Sciences of Tunis, 2092 El Manar 2 Tunis, Tunisia
E-mail address: lakhdartannech.rachdi@fst.rnu.tn