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APPROXIMATION OF FUNCTIONS BY MATRIX-EULER SUMMABILITY MEANS OF FOURIER SERIES IN GENERALIZED HÖLDER METRIC

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Abstract. In this paper, a new estimate for the degree of trignometric approximation of a function $f \in H_r^{(w)}$, $(r \geq 1)$ class by Matrix-Euler means $(\Delta.E_1)$ of its Fourier Series has been determined.

1. Introduction

The degree of approximation of a function f belonging to $Lip\alpha$ class by Nörlund summability method (N, p_n) has been determined by several investigators like Khan[6], Qureshi[7, 8], Chandra[11], Leindler[9], Stepants[2] and Lal[16]. Working in quite different direction, Totik[17, 18], Mazhar[14], Totik and Mazhar[15] and Chandra[12] have studied the approximation of functions in Hölder space $H^{(w)}$. But till now no work seems to have done to obtain the degree of approximation of functions $f \in H_r^{(w)}$, $(r \geq 1)$, by Matrix-Euler $(\Delta.E_1)$ product summability means. In an attempt to make an advance study in this direction, in this paper, a new estimate for degree of trignometric approximation of a function $f \in H_r^{(w)}$, $(r \geq 1)$ space has been determined. It is important to note that $H_r^{(w)}$, $(r \geq 1)$, is a generalization of $H^{(w)}$, $H_{(\alpha),r}$ and $H_{(\alpha)}$ spaces. Some important applications of main theorem has been investigated.

2. Definition and Notations

Let $f(x)$ be a 2π periodic function, integrable in the Lebesgue sense over $[o, 2\pi]$ and belonging to $H_r^{(w)}$ class. Let the Fourier series of $f(x)$ is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1}$$

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with n^{th} partial sums $s_n(f;x)$.
Let $\sum_{n=0}^{\infty} u_n$ be an infinite series having n^{th} partial sum $s_n = \sum_{\nu=0}^{n} u_\nu$.
Let $T = (a_{n,k})$ be an infinite triangular matrix satisfying the condition of regularity (Silverman-Töeplitz [10]) i.e.

(i). $\sum_{k=0}^{n} a_{n,k} = 1$ as $n \to \infty$,
(ii). $a_{n,k} = 0$ for $k > n$,
(iii). $\sum_{k=0}^{n} |a_{n,k}| \leq M$, a finite constant.

The sequence-to-sequence transformation

$$t_n^{\Delta} = \sum_{k=0}^{n} a_{n,k} s_k = \sum_{k=0}^{n} a_{n,n-k} s_{n-k}$$

defines the sequence t_n^{Δ} of triangular matrix means of the sequence $\{s_n\}$, generated by the sequence of coefficients $(a_{n,k})$.

If $t_n^{\Delta} \to s$ as $n \to \infty$ then the series $\sum_{n=0}^{\infty} u_n$ is summable to s by triangular matrix Δ- method (Zygmund[1], p.74).

Let $E_n^{(1)} = \dfrac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} s_k$. If $E_n^{(1)} \to s$ as $n \to \infty$, then $\sum_{n=0}^{\infty} u_n$ is said to be summable to s by the Euler's method, E_1(Hardy[5]).
The triangular matrix Δ-transform of E_1 transform defines the $(\Delta.E_1)$ transform $t_n^{\Delta E}$ of the partial sums s_n of the series $\sum_{n=0}^{\infty} u_n$ by

$$t_n^{\Delta E} = \sum_{k=0}^{n} a_{n,k} E_k^1 = \sum_{k=0}^{n} a_{n,k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} s_\nu .$$

If $t_n^{\Delta E} \to s$ as $n \to \infty$, $\sum_{n=0}^{\infty} u_n$ is said to be summable $(\Delta.E_1)$ to s.

$$\begin{aligned} s_n \to s \quad &\Rightarrow \quad E_n^{(1)} = \frac{1}{2^n}\sum_{\nu=0}^{n}\binom{n}{\nu} s_\nu \to s \ as\ n \to \infty,\ E_1 \text{ method is regular,} \\ &\Rightarrow \quad t_n^{\Delta}(E_n^{(1)}) = t_n^{\Delta E} \to s \ as\ n \to \infty,\ \Delta \text{ method is regular,} \\ &\Rightarrow \quad (\Delta.E_1) \text{ method is regular.} \end{aligned}$$

Some important particular cases of triangular matrix-Euler means $(\Delta.E_1)$ are

(i). $(H, \frac{1}{n+1}).(E_1)$ means, when $a_{n,k} = \frac{1}{(n-k+1)\log n}$.
(ii). $(N, p_n).E_1$ means, when $a_{n,k} = \frac{p_{n-k}}{P_n}$, where $P_n = \sum_{k=0}^{n} p_k \neq 0$.
(iii). $(N, p_n, q_n).E_1$ means, when $a_{n,k} = \frac{p_{n-k} q_k}{R_n}$,
where $R_n = \sum_{k=0}^{n} p_k q_{n-k} \neq 0$.

Let $C_{2\pi}$ denote the Banach space of all 2π-periodic and continuous functions defined on $[0, 2\pi]$ under the supremum norm.
For $0 < \alpha \leq 1$, let

$$H_{(\alpha)} = \{f \in C_{2\pi} : |f(x+t) - f(x)| = O(|t|^{\alpha})\}$$

The space $H_{(\alpha)}$ is a Banach space (Prössdorfff [13]) under the norm

$$\begin{aligned} \|f\|_{(\alpha)} &= \sup_{0\le x\le 2\pi} |f(x)| + \sup_{\substack{x,t\\ t\neq 0}} \frac{|f(x+t)-f(x)|}{|t|^{\alpha}}, \ 0<\alpha\le 1 \\ &= \|f\|_{\infty} + \sup_{\substack{x,t\\ t\neq 0}} \frac{|f(x+t)-f(x)|}{|t|^{\alpha}}, \ 0<\alpha\le 1. \end{aligned}$$

The metric induced by the norm $\|\|_{(\alpha)}$ on $H_{(\alpha)}$ is called the Hölder metric. Clearly $(H_{(\alpha)}, \|f\|_{(\alpha)})$ is a Banach space which decreases as α increases, i.e.,

$$H_{(\alpha)} \subseteq H_{(\beta)} \subseteq C_{2\pi}, \quad \text{for} \quad 0\le\beta<\alpha\le 1,$$

and

$$\|f\|_{(\beta)} \le (2\pi)^{\alpha-\beta} \|f\|_{(\alpha)}.$$

In general,

$$\sup_{0\le x\le 2\pi} |f(x)| \neq \sup_{\substack{x,t\\ t\neq 0}} \frac{|f(x+t)-f(x)|}{|t|^{\alpha}}, \quad 0<\alpha\le 1.$$

We define the norm $\|\|_r$ by

$$\|f\|_r = \begin{cases} \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^r\, dx\right\}^{\frac{1}{r}} \text{ for } 1\le r<\infty \\ ess \sup\limits_{0<x<2\pi} |f(x)| \text{ for } r=\infty. \end{cases}$$

Let $L^r[0,2\pi] = \left\{ f:[0,2\pi]\to\mathbb{R} : \int_0^{2\pi} |f(x)|^r\, dx < \infty \right\}, r\ge 1$, be the space of all 2π-periodic, integrable functions and for all t

$$H_{(\alpha),r} = \left\{ f\in L^r[0,2\pi] : \left(\int_0^{2\pi} |f(x+t)-f(x)|^r\, dx\right)^{\frac{1}{r}} = O\left(|t|^{\alpha}\right) \right\}.$$

The space $H_{(\alpha),r}$, $r\ge 1$, $0<\alpha\le 1$ is a Banach space under the norm $\|\|_{(\alpha),r}$:

$$\|f\|_{(\alpha),r} = \|f\|_r + \sup_{t\neq 0} \frac{\|f(.+t)-f(.)\|_r}{(|t|^{\alpha})}.$$

$$\|f\|_{(0),r} = \|f\|_r .$$

The metric induced by the norm $\|\|_{(\alpha),r}$ on $H_{(\alpha),r}$ is called Hölder continuous with degree r.
Easily, it can be obtained by

$$\|f\|_{(\beta),r} \le (2\pi)^{\alpha-\beta} \|f\|_{(\alpha),r}, \quad 0\le\beta<\alpha\le 1, \quad r\ge 1.$$

Since $f\in H_{(\alpha),r}$ if and only if $\|f\|_{(\alpha),r}<\infty$, we have

$$H_{(\alpha),r} \subseteq H_{(\beta),r} \subseteq L^r[0,2\pi], \quad 0\le\beta<\alpha\le 1, \quad r\ge 1.$$

For $f \in L^r[0,2\pi], r \ge 1$, the integral modulus of continuity is defined by $w_r(f,\delta) = \sup\limits_{0<t\le\delta} \left\{ \frac{1}{2\pi}\int_0^{2\pi} |f(x+t)-f(x)|^r\, dx \right\}^{\frac{1}{r}}$, for $f\in L^r[0,2\pi]$ where

$1 \leq r < \infty$ and if $r = \infty$, then
$w(f,\delta) = w_\infty(f,\delta) = \sup\limits_{o<t\leq\delta} \max\limits_{x} |f(x+t) - f((x)|$ for $f \in C_{2\pi}$.
It is known (Zygmund [1], p.45) that $w_r(f,\delta) \to 0$ as $\delta \to 0$.
Let $w : [0, 2\pi] \to \mathbb{R}$ be an arbitrary function with $w(t) > 0$ for $0 < t \leq 2\pi$ and $\lim\limits_{t\to 0^+} w(t) = w(0) = 0$.
The class of function $H^{(w)}$ has been defind by Leindler [9] as

$$H^{(w)} = \{f \in C_{2\pi} : |f(x+t) - f(x)| = O(w\,(t))\}$$

where w is a modulus of continuity, that is, w is a postive non-decreasing continuous function with the property: $w(0) = 0,\ w(t_1 + t_2) \leq w(t_1) + w(t_2)$.
We define $H_r^{(w)} = \left\{ f \in L^r[0, 2\pi] : 1 \leq r < \infty,\ \sup\limits_{t\neq 0} \dfrac{\|f(.+t) - f(.)\|_r}{w\,(t)} < \infty \right\}$
and $\|f\|_r^{(w)} = \|f\|_r + \sup\limits_{t\neq 0} \dfrac{\|f(.+t) - f(.)\|_r}{w\,(t)},\ r \geq 1$. Clearly $\|\|_r^{(w)}$ is a norm on $H_r^{(w)}$.
The completeness of the space $H_r^{(w)}$ can be discussed considering the completeness of $L^r\ (r \geq 1)$.
$\|f\|_r^{(v)} = \|f\|_r + \sup\limits_{t\neq 0} \dfrac{\|f(.+t) - f(.)\|_r}{v\,(|t|)},\ r \geq 1$. If $\frac{w(t)}{t}$ tends to zero as $t \to o^+$ then $f'(x)$ exists and is zero everywhere and f is constant.
Let $\left(\frac{w(t)}{v(t)}\right)$ be positive non decreasing.
Then $\|f\|_r^{(v)} \leq \max\left(1, \frac{w(2\pi)}{v(2\pi)}\right) \|f\|_r^{(w)} < \infty$. Thus,

$$H_r^{(w)} \subseteq H_r^{(v)} \subseteq L^r, \quad r \geq 1$$

Remarks.

(i). If we take $w(t) = t^\alpha$ then $H^{(w)}$ reduces to $H_{(\alpha)}$ class.
(ii). By taking $w(t) = t^\alpha$ in $H_r^{(w)}$, it reduces to $H_{(\alpha),r}$.
(iii). If we take $r \to \infty$ then $H_r^{(w)}$ class reduces to $H^{(w)}$.

The degree of approximation of a function $f : \mathbb{R} \to \mathbb{R}$ by a trignometric polynomial $t_n(x) = \frac{1}{2}a_0 + \sum_{\nu=1}^{\infty}(a_\nu \cos\nu x + b_\nu \sin\nu x)$ of order n is defined by (Zygmund [1], p.114-115)

$$E_n(f) = \min \|t_n - f\|_r\,.$$

We write,

$$\phi(x,t) = f(x+t) + f(x-t) - 2f(x), \Delta a_{n,k} = a_{n,k} - a_{n,k+1}, \quad 0 \leq k \leq n-1.$$

$$K_n^{\Delta E} = \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,k} \frac{\sin{(n-k+1)}(\frac{t}{2})\cos^{n-k}(\frac{t}{2})}{\sin{(\frac{t}{2})}}.$$

3. **Theorem**

In this paper, we prove the following theorem:

Theorem 3.1. *Let $A=(a_{n,k})$ be a regular lower triangular infinite matrix such that*

$$\sum_{k=0}^{n-1}|\Delta a_{n,k}|=O\left(\frac{1}{n+1}\right),(n+1)\,|a_{n,n}|=O\,(1)\,. \tag{2}$$

If $f:[0,2\pi]\to\mathbb{R}$ be a 2π-periodic, Lebesgue integrable and belonging to the generalized class $H_r^{(w)}$,$r\geq 1$; w,v be modulus of continuity and $\frac{w(t)}{v(t)}$ be positive, non-decreasing then the degree of approximation of f by triangular matrix-Euler means $t_n^{\Delta E}=\sum_{k=0}^{n}a_{n,k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}s_\nu$ of its Fourier series (1) is given by

$$\|t_n^{\Delta E}-f\|_r^{(v)}=O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{t^2v(t)}dt\right). \tag{3}$$

4. **Lemmas**

Following Lemmas are required to prove the theorems:

Lemma 4.1. *For $0<t\leq(n+1)^{-1}$, $K_n^{\Delta E}(t)=O(n+1)$.*

Proof. For $0<t\leq(n+1)^{-1}$, $\sin\frac{t}{2}\geq\frac{t}{\pi}$, $\sin nt\leq nt$, $|\cos t|\leq 1$. We have

$$\begin{aligned}\left|K_n^{\Delta E}(t)\right| &= \left|\frac{1}{2\pi}\sum_{k=0}^{n}a_{n,k}\frac{\sin{(n-k+1)(\frac{t}{2})}\cos^{n-k}(\frac{t}{2})}{\sin{(\frac{t}{2})}}\right|\\ &\leq \frac{1}{2\pi}\sum_{k=0}^{n}|a_{n,k}|\,\frac{(n-k+1)(\frac{t}{2})\left|\cos^{n-k}(\frac{t}{2})\right|}{(\frac{t}{\pi})}\\ &\leq \frac{1}{4}(n+1)\sum_{k=0}^{n}|a_{n,k}|\\ &\leq \frac{M}{4}(n+1)\\ &= O(n+1).\end{aligned}$$

Lemma 4.2. *For $(n+1)^{-1}<t<\pi$, $K_n^{\Delta E}(t)=O\left(\frac{1}{(n+1)t^2}\right)$.*

Proof. For $(n+1)^{-1} < t < \pi,\ \sin\frac{t}{2} \geq \frac{t}{\pi}$, using Abel's lemma, we get

$$\begin{aligned}
\left|K_n^{\Delta E}(t)\right| &= \left|\frac{1}{2\pi}\sum_{k=0}^{n} a_{n,k}\frac{\sin(n-k+1)(\frac{t}{2})\cos^{n-k}(\frac{t}{2})}{\sin(\frac{t}{2})}\right| \\
&\leq \frac{1}{2t}\left|\sum_{k=0}^{n-1}(a_{n,k}-a_{n,k+1})\sum_{\nu=0}^{k}\sin(n-\nu+1)\left(\frac{t}{2}\right)\cos^{n-\nu}\left(\frac{t}{2}\right)\right. \\
&\qquad \left. +a_{n,n}\sum_{k=0}^{n}\sin(n-k+1)\left(\frac{t}{2}\right)\cos^{n-k}\left(\frac{t}{2}\right)\right| \\
&\leq \frac{1}{2t}\left[\sum_{k=0}^{n-1}|\Delta a_{n,k}|\left|\frac{\sin(2n-k+2)(\frac{t}{4})\sin(n+1)(\frac{t}{4})}{\sin(\frac{t}{4})}\right| + |a_{n,n}|\left|\frac{\sin(n+2)(\frac{t}{4})\sin(n+1)(\frac{t}{4})}{\sin(\frac{t}{4})}\right|\right] \\
&\leq \frac{\pi}{t^2}\left[\sum_{k=o}^{n-1}|\Delta a_{n,k}| + |a_{n,n}|\right]\max_{0\leq k\leq n}\left|\sin(2n-k+2)\left(\frac{t}{2}\right)sin(n+1)\left(\frac{t}{2}\right)\right| \\
&= \frac{\pi}{t^2}\left[\sum_{k=o}^{n-1}|\Delta a_{n,k}| + |a_{n,n}|\right] \\
&= \frac{\pi}{t^2}\left[O\left(\frac{1}{n+1}\right)+O\left(\frac{1}{n+1}\right)\right]\ by\ (2) \\
&= O\left(\frac{1}{(n+1)t^2}\right).
\end{aligned}$$

5. **Proof of the Theorem3.1**

Following Titchmarsh [4], $s_k(f;x)$ of Fourier series (1) is given by

$$s_k(f;x) - f(x) = \frac{1}{2\pi}\int_0^\pi \phi(x,t)\frac{sin\left(k+\frac{1}{2}\right)t}{sin\left(\frac{t}{2}\right)}dt, \quad k = 0,1,2\ldots .$$

Then

$$\begin{aligned}
\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(s_k(f;x)-f(x)) &= \frac{1}{2\pi}\int_0^\pi \phi(x,t)\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\frac{sin\left(k+\frac{1}{2}\right)t}{sin\left(\frac{t}{2}\right)}dt \\
or \qquad E_n^1(x) - f(x) &= \frac{1}{2\pi}\int_0^\pi \phi(x,t)\frac{1}{2^n\sin\left(\frac{t}{2}\right)}\left\{I_m\sum_{k=0}^{n}\binom{n}{k}e^{i(k+\frac{1}{2})t}\right\}dt \\
&= \frac{1}{2\pi}\int_0^\pi \phi(x,t)\frac{1}{2^n\sin\left(\frac{t}{2}\right)}\left\{I_m\sum_{k=0}^{n}\binom{n}{k}e^{ikt}e^{i\frac{t}{2}}\right\}dt \\
&= \frac{1}{2\pi}\int_0^\pi \phi(x,t)\frac{1}{2^n\sin\left(\frac{t}{2}\right)}\left\{I_m\left(1+e^{it}\right)^n.e^{i\frac{t}{2}}\right\}dt \\
&= \frac{1}{2\pi}\int_0^\pi \phi(x,t)\frac{\sin\left\{(n+1)\left(\frac{t}{2}\right)\right\}\cos^n\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)}dt.
\end{aligned}$$

Now

$$\begin{aligned} t_n^{\Delta E}(x) - f(x) &= \frac{1}{P_n} \sum_{k=0}^{n} a_{n,k} \left\{ E_{n-k}^1(x) - f(x) \right\} \\ &= \frac{1}{2\pi} \int_0^{\pi} \phi(x,t) \sum_{k=0}^{n} a_{n,k} \frac{\sin\left\{(n-k+1)(\frac{t}{2})\right\} \cos^{n-k}(\frac{t}{2})}{\sin(\frac{t}{2})} dt. \end{aligned}$$

Let

$$l_n(x) = t_n^{\Delta E}(x) - f(x) = \int_0^{\pi} \phi(x,t) K_n^{\Delta E}(t) dt.$$

Then

$$l_n(x+y) - l_n(x) = \int_0^{\pi} (\phi(x+y,t) - \phi(x,t)) K_n^{\Delta E}(t) dt.$$

By generalized Minkowski's inequality (Chui[3], p.37), we get

$$\|l_n(.+y) - l_n(.)\|_r \le \int_0^{\pi} \|\phi(.+y,t) - \phi(.,t)\|_r \left|K_n^{\Delta E}(t)\right| dt$$

$$\begin{aligned} &= \int_0^{\frac{1}{n+1}} \left(\|\phi(.+y,t) - \phi(.,t)\|_r \left|K_n^{\Delta E}(t)\right| \right) dt + \int_{\frac{1}{n+1}}^{\pi} \left(\|\phi(.+y,t) - \phi(.,t)\|_r \left|K_n^{\Delta E}(t)\right| \right) dt \\ &= I_1 + I_2. \end{aligned} \tag{4}$$

Clearly

$$\begin{aligned} |\phi(x+y,t) - \phi(x,t)| \le\ & |f(x+y+t) - f(x+y)| + |f(x+y-t) - f(x+y)| \\ & + |f(x+t) - f(x)| + |f(x-t) - f(x)|. \end{aligned}$$

Applying Minkowski's inequality, we have

$$\begin{aligned} \|\phi(.+y,t) - \phi(.,t)\|_r &\le \|f(.+y+t) - f(.+y)\|_r + \|f(.+y-t) - f(.+y)\|_r \\ &\quad + \|f(.+t) - f(.)\|_r + \|f(.-t) - f(.)\|_r \\ &= O\left(w(t)\right). \end{aligned} \tag{5}$$

Also

$$\begin{aligned} \|\phi(.+y,t) - \phi(.,t)\|_r &\le \|f(.+y+t) - f(.+t)\|_r + \|f(.+y-t) - f(.-t)\|_r \\ &\quad + 2\|f(.+y) - f(.)\|_r \\ &= O\left(w(|y|)\right). \end{aligned} \tag{6}$$

For v is positive, non decreasing, $t \le |y|$, we obtained

$$\begin{aligned} \|\phi(.+y,t) - \phi(.,t)\|_r &= O\left(w(t)\right) \\ &= O\left(v(t)\left(\frac{w(t)}{v(t)}\right)\right) \\ &= O\left(v(|y|)\left(\frac{w(t)}{v(t)}\right)\right). \end{aligned}$$

Since $\frac{w(t)}{v(t)}$ is positive, non-decreasing, if $t \geq |y|$, then $\frac{w(t)}{v(t)} \geq \frac{w(|y|)}{v(|y|)}$, so that

$$\begin{aligned} \|\phi(.+y,t)-\phi(.,t)\|_r &= O\left(w(|y|)\right) \\ &= O\left(v(|y|)\left(\frac{w(t)}{v(t)}\right)\right). \end{aligned} \tag{7}$$

Using lemma (4.1) and (7) we obtain

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{n+1}} \|\phi(.+y,t)-\phi(.,t)\|_r \left|K_n^{\Delta E}(t)\right| dt \\ &= O\left(\int_0^{\frac{1}{n+1}} v(|y|)\frac{w(t)}{v(t)}(n+1)dt\right) \\ &= O\left((n+1)v(|y|)\int_0^{\frac{1}{n+1}} \frac{w(t)}{v(t)}dt\right) \\ &= O\left((n+1)v(|y|)\frac{w\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\int_0^{\frac{1}{n+1}} dt\right) \\ &= O\left(v(|y|)\frac{w\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right). \end{aligned} \tag{8}$$

Also,using Lemma (4.2) and (7) we get

$$\begin{aligned} I_2 &= \int_{\frac{1}{n+1}}^{\pi} \|\phi(.+y,t)-\phi(.,t)\|_r \left|K_n^{\Delta E}(t)\right| dt \\ &= O\left(\int_{\frac{1}{n+1}}^{\pi} v(|y|)\frac{w(t)}{v(t)}\frac{1}{(n+1)t^2}dt\right) \\ &= O\left(\frac{1}{n+1}\int_{\frac{1}{n+1}}^{\pi} v(|y|)\frac{w(t)}{t^2v(t)}dt\right). \end{aligned} \tag{9}$$

By (4), (8) and (9), we have

$$\begin{aligned} \|l_n(.+y)-l_n(.)\|_r &= O\left(v(|y|)\frac{w\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right) \\ &\quad +O\left(\frac{1}{n+1}\int_{\frac{1}{n+1}}^{\pi} v(|y|)\frac{w(t)}{t^2v(t)}dt\right). \end{aligned}$$

Thus,

$$\begin{aligned} \sup_{y\neq 0}\frac{\|l_n(.+y)-l_n(.)\|_r}{v(|y|)} &= O\left(\frac{w\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right) \\ &\quad +O\left(\frac{1}{n+1}\int_{\frac{1}{n+1}}^{\pi} \frac{w(t)}{t^2v(t)}dt\right). \end{aligned} \tag{10}$$

Clearly

$$\begin{aligned}|\phi(x,t)| &= |f(x+t)+f(x-t)-2f(x)| \\ &\leq |f(x+t)-f(x)|+|f(x-t)-f(x)|\end{aligned}$$

Applying Minkowski's inequality, we have

$$\begin{aligned}\|\phi(.,t)\|_r &\leq \|f(.+t)-f(.)\|_r+\|f(.-t)-f(.)\|_r \\ &= O\left(w(t)\right). \qquad (11)\end{aligned}$$

Using(11), Lemma (4.1),Lemma (4.2) we obtain

$$\begin{aligned}\|l_n(.)\|_r &= \left\|t_n^{\Delta E}-f\right\|_r \leq \left(\int_0^{\frac{1}{n+1}}+\int_{\frac{1}{n+1}}^{\pi}\right)\|\phi(.,t)\|_r\left|K_n^{\Delta E}(t)\right|dt \\ &= \int_0^{\frac{1}{n+1}}\|\phi(.,t)\|_r\left|(K_n^{\Delta E}(t)\right|dt+\int_{\frac{1}{n+1}}^{\pi}\|\phi(.,t)\|_r\left|K_n^{\Delta E}(t)\right|dt \\ &= O\left((n+1)\int_0^{\frac{1}{n+1}}w(t)dt\right)+O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{t^2}dt\right) \\ &= O\left(w\left(\frac{1}{(n+1)}\right)\right)+O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{t^2}dt\right). \qquad (12)\end{aligned}$$

Now, By (10) and (12)

$$\begin{aligned}\|l_n(.)\|_r^{(v)} &= \|l_n(.)\|_r+\sup_{y\neq 0}\frac{\|l_n(.+y)-l_n(.)\|_r}{v(|y|)} \\ &= O\left(w\left(\frac{1}{n+1}\right)\right)+O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{t^2}dt\right) \\ &\quad +O\left(\frac{w\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right)+O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{v(t)t^2}dt\right).\end{aligned}$$

Using the fact that $w(t)=\frac{w(t)}{v(t)}.v(t) \ \leq \ v(\pi)\frac{w(t)}{v(t)}$, $0<t\leq\pi$, we get

$$\|l_n(.)\|_r^{(v)} = O\left(\frac{w\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right)+O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{v(t)t^2}dt\right). \qquad (13)$$

Since w and v are modulus of continuity such that $\frac{w(t)}{v(t)}$ is positive, non decreasing, therefore

$$\frac{1}{n+1}\int_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{v(t)t^2}dt \geq \frac{w\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\left(\frac{1}{n+1}\right)\int_{\frac{1}{n+1}}^{\pi}\frac{dt}{t^2} \geq \frac{w\left(\frac{1}{n+1}\right)}{2v\left(\frac{1}{n+1}\right)}.$$

Then

$$\frac{w\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)} = O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{t^2v(t)}dt\right). \qquad (14)$$

By (13) and (14), we have

$$\left\|t_n^{\Delta E} - f\right\|_r^{(v)} = O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{t^2 v(t)}dt\right).$$

This completes the proof of theorem 3.1.

6. Applications

We obtain the following corollaries from the main Theorems.

Corollary 6.1. *Let* $f \in H_{(\alpha),r}$, $r \geq 1$, $0 < \alpha \leq 1$ *then*

$$\left\|t_n^{\Delta E} - f\right\|_{(\beta),r} = \begin{cases} O\left(\frac{1}{(n+1)^{\alpha-\beta}}\right), & 0 \leq \beta < \alpha < 1, \\ O\left(\frac{log(n+1)\pi}{n+1}\right), & \beta = 0, \alpha = 1. \end{cases}$$

Proof. If we take $w(t) = t^\alpha$, $v(t) = t^\beta$ in theorem 3.1.

Corollary 6.2. *If we take* $a_{n,k} = \frac{1}{(n-k+1)\log n}$, *in theorem 3.1, then degree of approximation of a function* $f \in H_r^{(w)}$ *by* $(H, \frac{1}{n+1}).E_1$ *means*

$$t_n^{HE} = \frac{1}{\log n}\sum_{k=0}^{n}\frac{1}{n-k+1}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}s_\nu$$

of the fourier series (1) is given by

$$\left\|t_n^{HE} - f\right\|_r^{(v)} = O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{t^2 v(t)}dt\right).$$

Corollary 6.3. *If we take* $a_{n,k} = \frac{p_{n-k}}{P_n}$, *where* $P_n = \sum_{k=0}^{n} p_k \neq 0$ *in theorem 3.1, then degree of approximation of a function* $f \in H_r^{(w)}$ *by* $(N, p_n).E_1$ *means*

$$t_n^{NE} = \frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}s_\nu$$

of the fourier series (1) is given by

$$\left\|t_n^{NE} - f\right\|_r^{(v)} = O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{t^2 v(t)}dt\right).$$

Corollary 6.4. *If we take* $a_{n,k} = \frac{p_{n-k}q_k}{R_n}$, *where* $R_n = \sum_{k=0}^{n} p_k q_{n-k} \neq 0$ *in theorem 3.1, then degree of approximation of a function* $f \in H_r^{(w)}$ *by* $(N, p, q).E_1$ *means*

$$t_n^{NE} = \frac{1}{R_n}\sum_{k=0}^{n}p_{n-k}q_k\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}s_\nu$$

of the fourier series (1) is given by

$$\left\|t_n^{NE} - f\right\|_r^{(v)} = O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{t^2 v(t)}dt\right).$$

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