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ON A NEW APPLICATION OF ALMOST INCREASING SEQUENCES

(COMMUNICATED BY H. BOR)

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Abstract. A new result concerning absolute summability of infinite series using almost increasing sequence is presented. An application gives some generalization of Bor's result [1].

1. Introduction

Let $\sum a_n$ be an infinite series with sequence of partial sums (s_n). By u_n^α, t_n^α we denote the nth Cesaro mean of order $\alpha > -1$ of the sequences (s_n), (na_n) respectively, that is

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v, \tag{1.1}$$

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} v a_v. \tag{1.2}$$

The series $\sum a_n$ is summable $|C, \alpha|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left| u_n^\alpha - u_{n-1}^\alpha \right|^k \equiv \sum_{n=1}^{\infty} n^{-1} \left| t_n^\alpha \right|^k < \infty. \tag{1.3}$$

For $\alpha = 1$, $|C, \alpha|_k$ summability reduces to $|C, 1|_k$ summability.

Let (p_n) be a sequence of constants such that

$$P_n = p_0 + p_1 + ... + p_n \to \infty \text{ as } n \to \infty.$$

The sequence to sequence transformation

$$\delta_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v \tag{1.4}$$

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defines the sequence (δ_n) of the Nörlund mean of the sequence (s_n) generated by the sequence of coefficients (p_n). $\sum a_n$ is said to be summable $|N, p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left|\delta_n - \delta_{n-1}\right|^k < \infty \tag{1.5}$$

In the special case when

$$p_n = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}, \qquad \alpha \geq 0, \tag{1.6}$$

$|N, p_n|_k$ summability reduces to $|C, \alpha|_k$ summability.

A positive sequence (b_n) is said to be almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants M and N such that $Mc_n \leq b_n \leq Nc_n$. Every increasing sequence is almost increasing, but the converse need not to be true, see for example when $b_n = ne^{(-1)^n}$.

The following results are known:

Theorem 1.1. [3] *Let $p_0 > 0$, $p_n \geq 0$ and (p_n) be a non-increasing sequence. If $\sum a_n$ is summable $|C, 1|_k$, then the series $\sum a_n P_n (n+1)^{-1}$ is summable $|N, p_n|_k$, $k \geq 1$.*

Theorem 1.2. [1] *Let (p_n) be as in Theorem 1.1 and (X_n) be almost increasing sequence. If the conditions*

$$\sum_{n=1}^{\infty} n \left|\Delta^2 \lambda_n\right| X_n < \infty, \tag{1.7}$$

$$|\lambda_n| X_n = O(1) \ \textit{as} \quad n \to \infty, \tag{1.8}$$

$$\sum_{v=1}^{n} \frac{1}{v} |t_v|^k = O(X_n), \ \textit{as} \quad n \to \infty, \tag{1.9}$$

are satisfied, then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $|N, p_n|_k$, $k \geq 1$.

Lemma 1.3. [2] *Under the conditions (1.7) and (1.8), we have*

$$nX_n |\Delta\lambda_n| = O(1), \ \textit{as} \quad n \to \infty, \tag{1.10}$$

$$\sum_{n=1}^{\infty} X_n |\Delta\lambda_n| < \infty. \tag{1.11}$$

2. Results

We state and prove the following result

Theorem 2.1. *Let (p_n) be as in Theorem 1.1 and (X_n) be almost increasing sequence. If the conditions (1.7), (1.8) and*

$$\varphi_v = O(1), \quad \textit{as} \quad v \to \infty, \tag{2.1}$$

$$v\Delta\varphi_v = O(1), \quad \textit{as} \quad v \to \infty, \tag{2.2}$$

$$\sum_{v=1}^{n} \frac{1}{vX_v^{k-1}} |t_v|^k = O(X_n), \ \textit{as} \quad n \to \infty, \tag{2.3}$$

are satisfied, then the series $\sum a_n \lambda_n \varphi_n$ is summable $|C, 1|_k$, $k \geq 1$.

Proof. Let T_n be the nth $(C,1)$ mean of the sequence $(na_n\lambda_n\varphi_n)$. Therefore

$$T_n = \frac{1}{n+1}\sum_{v=1}^{n} va_v\lambda_v\varphi_v.$$

Abel's transformation gives

$$\begin{aligned}
T_n &= \frac{1}{n+1}\left(\sum_{v=1}^{n-1}\Delta\left(\lambda_v\varphi_v\right)\sum_{r=1}^{v} ra_r + \lambda_n\varphi_n\sum_{v=1}^{n} va_v\right)\\
&= \frac{1}{n+1}\left(\sum_{v=1}^{n-1}(v+1)t_v\Delta\varphi_v\lambda_v + \sum_{v=1}^{n-1}(v+1)t_v\varphi_{v+1}\Delta\lambda_v\right) + t_n\varphi_n\lambda_n\\
&= T_{n1}+T_{n2}+T_{n3}.
\end{aligned}$$

In order to complete the proof, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{1}{n}\left|T_{nj}\right|^k < \infty, \quad j=1,2,3.$$

Applying Hölder's inequality, we have

$$\begin{aligned}
\sum_{n=2}^{m+1}\frac{1}{n}\left|T_{n1}\right|^k &= \sum_{n=2}^{m+1}\frac{1}{n}\left|\frac{1}{n+1}\sum_{v=1}^{n-1}(v+1)t_v\Delta\varphi_v\lambda_v\right|^k\\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{k+1}}\sum_{v=1}^{n-1}v^k\left|t_v\right|^k\left|\Delta\varphi_v\right|^k\left|\lambda_v\right|^k\left(\sum_{v=1}^{n-1}1\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^2}\sum_{v=1}^{n-1}v^k\left|t_v\right|^k\left|\Delta\varphi_v\right|^k\left|\lambda_v\right|^k\\
&= O(1)\sum_{v=1}^{m}v^k\left|t_v\right|^k\left|\Delta\varphi_v\right|^k\left|\lambda_v\right|^k\sum_{n=v+1}^{m+1}\frac{1}{n^2}\\
&= O(1)\sum_{v=1}^{m}v^{-1}\left|t_v\right|^k\left|\lambda_v\right|^k\\
&= O(1)\sum_{v=1}^{m}\frac{\left|t_v\right|^k}{vX_v^{k-1}}\left|\lambda_v\right|\\
&= O(1)\sum_{v=1}^{m-1}\left|\Delta\lambda_v\right|\sum_{r=1}^{v}\frac{\left|t_r\right|^k}{rX_r^{k-1}} + O(1)\left|\lambda_m\right|\sum_{v=1}^{m}\frac{\left|t_v\right|^k}{vX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}X_v\left|\Delta\lambda_v\right| + O(1)X_m\left|\lambda_m\right| = O(1).
\end{aligned}$$

$$\begin{aligned}
\sum_{n=2}^{m+1}\frac{1}{n}\left|T_{n2}\right|^k &= \sum_{n=2}^{m+1}\frac{1}{n}\left|\frac{1}{n+1}\sum_{v=1}^{n-1}(v+1)t_v\varphi_{v+1}\Delta\lambda_v\right|^k \\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{k+1}}\sum_{v=1}^{n-1}v^k\frac{\left|t_v\right|^k\left|\varphi_{v+1}\right|^k\left|\Delta\lambda_v\right|}{X_v^{k-1}}\left(\sum_{v=1}^{n-1}X_v\left|\Delta\lambda_v\right|\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}v^k\frac{\left|t_v\right|^k\left|\Delta\lambda_v\right|}{X_v^{k-1}}\sum_{n=v+1}^{m+1}\frac{1}{n^{k+1}} \\
&= O(1)\sum_{v=1}^{m}\frac{\left|t_v\right|^k}{vX_v^{k-1}}v\left|\Delta\lambda_v\right| \\
&= O(1)\sum_{v=1}^{m-1}\left|\Delta\left(v\left|\Delta\lambda_v\right|\right)\right|\sum_{r=1}^{v}\frac{\left|t_r\right|^k}{rX_r^{k-1}}+O(1)m\left|\Delta\lambda_m\right|\sum_{v=1}^{m}\frac{\left|t_v\right|^k\left|\Delta\lambda_v\right|}{vX_v^{k-1}} \\
&= O(1)\sum_{v=1}^{m-1}v\left|\Delta^2\lambda_v\right|X_v+O(1)\sum_{v=1}^{m-1}X_v\left|\Delta\lambda_v\right|+O(1)m\left|\Delta\lambda_m\right|X_m \\
&= O(1).
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{m}\frac{1}{n}\left|T_{n3}\right|^k &= \sum_{n=1}^{m}\frac{1}{n}\left|t_n\varphi_n\lambda_n\right|^k \\
&= O(1)\sum_{n=1}^{m}\frac{\left|t_n\right|^k}{nX_n^{k-1}}\left|\lambda_n\right| \\
&= O(1),\ \text{as in the case of } T_{n1}.
\end{aligned}$$

□

3. Remarks

Remark 3.1. (a) It may be mentioned that condition (2.3)is weaker than (1.9). In fact is (1.9) is satisfied, then

$$\sum_{n=1}^{m}\frac{\left|t_n\right|^k}{nX_n^{k-1}}=O\left(\frac{1}{X_1^{k-1}}\right)\sum_{n=1}^{m}\frac{1}{n}\left|t_n\right|^k=O(X_m),$$

while if (2.3) is satisfied then,

$$\begin{aligned}
\sum_{n=1}^{m}\frac{1}{n}\left|t_n\right|^k &= \sum_{n=1}^{m}\frac{1}{nX_n^{k-1}}\left|t_n\right|^k X_n^{k-1} \\
&= \sum_{n=1}^{m-1}\left(\sum_{v=1}^{n}\frac{\left|t_v\right|^k}{vX_v^{k-1}}\right)\Delta X_n^{k-1}+\left(\sum_{n=1}^{m}\frac{\left|t_n\right|^k}{nX_n^{k-1}}\right)X_m^{k-1} \\
&= O(1)\sum_{n=1}^{m-1}X_n\left|\Delta X_n^{k-1}\right|+O(X_m)X_m^{k-1} \\
&= O(X_{m-1})\sum_{n=1}^{m-1}\left(X_{n+1}^{k-1}-X_n^{k-1}\right)+O(X_m^k) \\
&= O(X_{m-1})\left(X_m^{k-1}-X_1^{k-1}\right)+O(X_m^k) \\
&= O(X_m^k).
\end{aligned}$$

or we can deal with this case as follows

$$\sum_{n=1}^{m}\frac{1}{n}\left|t_n\right|^k=\sum_{n=1}^{m}\frac{1}{nX_n^{k-1}}\left|t_n\right|^k X_n^{k-1}=O(X_m^{k-1})\sum_{n=1}^{m}\frac{1}{nX_n^{k-1}}\left|t_n\right|^k=O(X_m^k).$$

Therefore (1.9) implies (2.3) but not conversely.

(b) The other advantage of condition (2.3) is that this condition leave no losing through estimation concerning powers of $|\lambda_n|$. As an example through the proof of Theorem 1.2, ıt has been substituted $|\lambda_n|^k=|\lambda_n|^{k-1}|\lambda_n|=O(|\lambda_n|)$, which implies that $|\lambda_n|^{k-1}$ has been lost.

Remark 3.2. By putting $\varphi_n=P_n/(n+1)$ in Theorem 2.1, we obtain Theorem 1.2 via Theorem 1.1, as follows:

As (p_n) is non-increasing, then $P_n\leq(n+1)p_0$ which implies $\varphi_n=O(1)$. Also

$$\begin{aligned}
n\Delta\varphi_n &= n\left(\frac{P_n}{n+1}-\frac{P_{n+1}}{n+2}\right)=n\left(\frac{P_n}{n+1}-\frac{P_n}{n+2}-\frac{P_{n+1}}{n+2}\right)=n\frac{P_n}{(n+1)(n+2)}-n\frac{P_{n+1}}{n+2} \\
&= O\left(\frac{P_n}{n+1}\right)+O(p_{n+1})=O(1).
\end{aligned}$$

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References

[1] H. Bor, *On a new application of almost increasing sequences,*,to be published in Mathem and Computer Modelling, 35 (2011),230-233.

[2] S. M. Mazhar, *Absolute summability factors of infinite series*, Kyungpook Math. J., 39 (1999), 67-73.

[3] R. S. Varma, *On the absolute Nörlund summability factors*, Riv. Mat. Univ. Parma (4), 3(1977)27-33.

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