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CONVERGENCE IN AN IMPULSIVE ADVANCED DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENT

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Abstract. In this work, we show the existence and uniqueness of the solution $x(t)$ of the initial value problem

$$\begin{cases} x'(t) = a(t)\left(x(t) - x([t+1])\right) + f(t), \quad t \neq n \in \mathbb{Z}^+ = \{1, 2, \ldots\},\ t \geq 0, \\ \Delta x(n) = d(n),\ n \in \mathbb{Z}^+, \\ x(0) = x_0. \end{cases}$$

Moreover, we prove that the limit of $x(t)$ is equal to a real constant as $t \to \infty$. Also, we formulate this limit value in terms of the initial condition, impulses, and the solution of an integral equation.

1. Introduction

The theory of differential equations with piecewise constant arguments ($DEPCA$) of the type

$$x'(t) = f\ \left(t, x(t), x(h(t))\right)$$

was initiated in ([13],[30]) where $h(t) = [t],\ [t-n],\ [t+n]$, *etc.* These types of equations have been intensively investigated for twenty five years. Systems described by $DEPCA$ exist in a large area such as biomedicine, chemistry, physics and mechanical engineering. Busenberg and Cooke [11] first established a mathematical model with a piecewise constant argument for analyzing vertically transmitted diseases. Examples in practice include machinery driven by servo units, charged particles moving in a piecewise constantly varying electric field, and elastic systems impelled by a Geneva wheel.

$DEPCA$ are also closely related to difference and differential equations. So, they describe hybrid dynamical systems and combine the properties of both differential and difference equations. The oscillation, periodicity and some asymptotic

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properties of various differential equations with piecewise constant arguments were methodically demonstrated in ([1]-[4],[21]-[23],[25]-[28],[31]). Also, Wiener's book [32] is a distinguished source in this area.

The theory of impulsive differential equations developed rapidly, in recent years. This improvement is particularly due to the fact that many phenomena and process in natural sciences such as physics, population dynamics, ecology, biology, *etc.*, can be simulated by these types of equations. There are many works about impulsive differential equations ([20], [26]). The monographs ([6], [29]) are good sources for impulsive differential equations.

But, there are only a few papers on impulsive differential equations with piecewise constant arguments (*IDEPCA*) ([24], [33]). In [24], Li and Shen considered the problem

$$y'(t) = f\ (t, y[t-k]),\ t \neq n,\ t \in J,$$
$$\Delta y(n^+) = I_n(y(n)),\ n = 1, 2, \ldots, p,\ y(0) = y(T).$$

Using the method of upper and lower solutions, they proved that it has at least one solution. In [33], Wiener and Lakshmikantham established the existence and uniqueness of solutions of the initial value problem

$$x'(t) = f\ (x(t), x(g(t))),\ x(0) = x_0,$$

and they also studied the cases of oscillation and stability, where f is a continuous function and $g : [0, \infty) \to [0, \infty),\ g(t) \leq t$, is a step function.

Lately, the problem of the asymptotic constancy of solutions was studied for some functional differential equations ([5],[7],[8],[10],[12],[14]-[19]) and as well the same problem has been considered for some impulsive delay differential equations ([9]). So, due to the practical reasons and the papers mentioned above one can be motivated to deal with the problem of asymptotic constancy of solutions of an impulsive differential equation with piecewise constant arguments. Let us note that in physical and engineering systems, the phenomena related to stepwise or piecewise constant variables or motions under piecewise constant forces can usually come out as first or second order differential equations with piecewise constant arguments.

In this paper, we consider the first order nonhomogeneous linear impulsive advanced differential equation with piecewise constant argument

$$x'(t) = a(t)(x(t) - x([t+1])) + f(t), \quad t \neq n \in \mathbb{Z}^+ = \{1, 2, \ldots\},\ t \geq 0, \quad (1.1)$$

$$\Delta x(n) = d(n),\ n \in \mathbb{Z}^+, \quad (1.2)$$

and the initial condition

$$x(0) = x_0, \quad (1.3)$$

where $a(t)$ and $f(t)$ are continuous real valued functions on $[0, \infty),\ d : \mathbb{Z}^+ \longrightarrow \mathbb{R},\ x_0 \in \mathbb{R},\ \Delta x(n) = x(n^+) - x(n^-),\ x(n^+) = \lim_{t \to n^+} x(t),\ x(n^-) = \lim_{t \to n^-} x(t)$ and $[.]$ denotes the greatest integer function.

Here, we purpose to give sufficient conditions for asymptotic constancy of the solution $x(t)$ of $(1.1) - (1.3)$, that is, $\lim_{t \to \infty} x(t) = \ell \in \mathbb{R}$. We also aim to calculate this limit value in terms of initial condition and the solution of an integral equation. As we know, this problem has not been studied, yet.

Definition 1.1. *A function $x(t)$ defined on $[0,\infty)$ is said to be a solution of* $(1.1)-(1.3)$ *if it satisfies the following conditions:*
(i) $x:[0,\infty)\to\mathbb{R}$ *is continuous with the possible exception of the points* $t\in\mathbb{Z}^+$*,*
(ii) $x(t)$ *is right continuous and has left-hand limits at the points* $t\in\mathbb{Z}^+$*,*
(iii) $x'(t)$ *exists for every* $t\in[0,\infty)$ *with the possible exception of the points* $t\in\mathbb{Z}^+$ *where one-sided derivatives exist,*
(iv) $x(t)$ *satisfies* (1.1) *for any* $t\in(0,\infty)$ *with the possible exception of the points* $t\in\mathbb{Z}^+$*,*
(v) $x(t)$ *satisfies* (1.2) *for every* $t=n\in\mathbb{Z}^+$*,*
(vi) $x(0)=x_0$*.*
Before given the main results we can prove the existence and uniqueness of solutions of $(1.1)-(1.3)$ *:*

Theorem 1.1. *The initial value problem* $(1.1)-(1.3)$ *has a unique solution on* $[0,\infty)$*.*

Proof. For $t\in[0,1)$, (1.1) can be written as

$$x'(t)=a(t)(x(t)-x(1))+f(t).$$

Integrating both sides from 0 to t,

$$x(t)=\exp\left(\int_0^t a(u)\,du\right)x_0+\left(1-\exp\left(\int_0^t a(u)\,du\right)\right)x(1)$$
$$+\int_0^t\exp\left(\int_s^t a(u)\,du\right)f(s)\,ds. \qquad (1.4)$$

By using impulse condition (1.2) for $n=1$, we obtain

$$x(1)=x\left(1^-\right)+d(1).$$

From (1.4),

$$x(1)=x_0+\int_0^1\exp\left(-\int_0^s a(u)\,du\right)f(s)\,ds+\exp\left(-\int_0^1 a(u)\,du\right)d(1). \qquad (1.5)$$

Substituting (1.5) into (1.4), we have

$$\begin{aligned}x(t)\;=\;&\exp\left(\int_0^t a(u)\,du\right)x_0\\&+\left(1-\exp\left(\int_0^t a(u)\,du\right)\right)\left(x_0+\int_0^1\exp\left(-\int_0^s a(u)\,du\right)f(s)\,ds\right.\\&\left.+\exp\left(-\int_0^1 a(u)\,du\right)d(1)\right)+\int_0^t\exp\left(\int_s^t a(u)\,du\right)f(s)\,ds. \qquad (1.6)\end{aligned}$$

For $t\in[1,2)$, (1.1) is reduced to

$$x'(t)=a(t)(x(t)-x(2))+f(t).$$

Integrating both sides from 1 to t,

$$x(t) = \exp\left(\int_1^t a(u)\,du\right) x(1) + \left(1-\exp\left(\int_1^t a(u)\,du\right)\right) x(2) + \int_1^t \exp\left(\int_s^t a(u)\,du\right) f(s)\,ds. \quad (1.7)$$

From the impulse condition (1.2), for $n=2$, we have

$$x(2) = x\left(2^-\right) + d(2).$$

By using (1.7),

$$x(2) = x(1) + \int_1^2 \exp\left(-\int_1^s a(u)\,du\right) f(s)\,ds + \exp\left(-\int_1^2 a(u)\,du\right) d(2). \quad (1.8)$$

Substituting (1.5) and (1.8) into (1.7)

$$\begin{aligned} x(t) = {} & \exp\left(\int_1^t a(u)\,du\right)\left(x_0 + \int_0^1 \exp\left(-\int_0^s a(u)\,du\right) f(s)\,ds \right. \\ & \left. + \exp\left(-\int_0^1 a(u)\,du\right) d(1)\right) \\ & + \left(1-\exp\left(\int_1^t a(u)\,du\right)\right)\left(x_0 + \int_0^1 \exp\left(-\int_0^s a(u)\,du\right) f(s)\,ds \right. \\ & + \exp\left(-\int_0^1 a(u)\,du\right) d(1) + \int_1^2 \exp\left(-\int_1^s a(u)\,du\right) f(s)\,ds \\ & \left. + \exp\left(-\int_1^2 a(u)\,du\right) d(2)\right) + \int_1^t \exp\left(\int_s^t a(u)\,du\right) f(s)\,ds. \quad (1.9) \end{aligned}$$

Following this way and using mathematical induction method, we obtain that

$$x(t) = \exp\left(\int_n^t a(u)\,du\right) x(n) + \left(1-\exp\left(\int_n^t a(u)\,du\right)\right) x(n+1) + \int_n^t \exp\left(\int_s^t a(u)\,du\right) f(s)\,ds$$

for $n \le t < n+1$, where

$$x(n) = x_0 + \sum_{r=0}^{n-1}\left(\int_r^{r+1} \exp\left(-\int_r^s a(u)\,du\right) f(s)\,ds + \exp\left(-\int_r^{r+1} a(u)\,du\right) d(r+1)\right). \quad (1.10)$$

Therefore, the problem $(1.1)-(1.3)$ has the unique solution defined on $[0,\infty)$

$$\begin{aligned} x(t) &= \exp\left(\int_{[t]}^{t} a(u)\,du\right)\left\{x_0+\sum_{r=0}^{[t-1]}\left(\int_{r}^{r+1}\exp\left(-\int_{r}^{s} a(u)\,du\right)f(s)\,ds\right.\right. \\ &\quad \left.\left.+\exp\left(-\int_{r}^{r+1} a(u)\,du\right)d(r+1)\right)\right\} \\ &\quad +\left(1-\exp\left(\int_{[t]}^{t} a(u)\,du\right)\right)\left\{x_0+\sum_{r=0}^{[t]}\left(\int_{r}^{r+1}\exp\left(-\int_{r}^{s} a(u)\,du\right)f(s)\,ds\right.\right. \\ &\quad \left.\left.+\exp\left(-\int_{r}^{r+1} a(u)\,du\right)d(r+1)\right)\right\}+\int_{[t]}^{t}\exp\left(\int_{s}^{t} a(u)\,du\right)f(s)\,ds. \end{aligned} \tag{1.11}$$

□

In this while, we note that a straightforward verification shows that the solution of the initial value problem $(1.1)-(1.3)$ *satisfies the following integral equation*

$$x(t)=x_0+\int_0^t a(s)\,x(s)\,ds-\int_0^t a(s)\,x([s+1])\,ds+\int_0^t f(s)\,ds+\sum_{i=1}^{[t]} d(i) \tag{1.12}$$

which we use to prove our main results.
This paper is organized as follows. In Section 2, the main results Theorem 2.1 and Theorem 2.2 are presented. Section 3 and Section 4 contain the proofs of main results, respectively and at the end of the Section 4 we give an example. Section 5 is devoted to conclusion.

2. Main Results

Our main results are given as follows.

Theorem 2.1. *Let* $a(t)$ *and* $f(t)$ *be continuous functions on the interval* $[0,\infty)$*. If*

(i) $\int_0^\infty |a(s)|\,ds \le K_1 < \infty$,

(ii) $\int_0^\infty |f(s)|\,ds \le K_2 < \infty$,

(iii) $\sum_{i=1}^\infty |d(i)| \le L_1 < \infty$,

then, the solution $x(t)$ *of* $(1.1)-(1.3)$ *tends to a constant as* $t\to\infty$.

Theorem 2.2. *Suppose that all assumptions of Theorem 2.1 are satisfied. Let* $x(t)$ *be the solution of* $(1.1)-(1.3)$ *and* $\lim_{t\to\infty} x(t)=\ell(x_0)$.

If

$$\int_{[t]}^{t} |a(s)|\, ds \le \rho < 1, \tag{2.1}$$

then

$$\ell(x_0) = x_0 + \int_0^{\infty} y(s) f(s)\, ds + \sum_{i=1}^{\infty} y\left(i^-\right) d(i) \tag{2.2}$$

where y is a solution of the integral equation

$$y(t) = 1 - \int_{[t]}^{t} y(s) a(s)\, ds,\ \ t \ge 0. \tag{2.3}$$

and $y(i^-) = \lim_{t \to i^-} y(t)$.

3. Proof of Theorem 2.1

For the proof of Theorem 2.1 we need the following lemma:

Lemma 3.1. *Assume that all hypotheses of Theorem 2.1 are satisfied. Then for a solution $x(n)$ of the corresponding difference equation*

$$\begin{aligned} x(n+1) &= x(n) + \int_n^{n+1} \exp\left(-\int_n^s a(u)\, du\right) f(s)\, ds \\ &\quad + \exp\left(-\int_n^{n+1} a(u)\, du\right) d(n+1),\ \ n \ge 0, \end{aligned} \tag{3.1}$$

there is a positive constant L_2 such that

$$|x(n)| \le L_2,\ \ n = 0, 1, 2, \ldots. \tag{3.2}$$

Proof. The solution $x(n)$ of Eq.(3.1) is

$$\begin{aligned} x(n) &= x_0 + \sum_{r=0}^{n-1} \left(\int_r^{r+1} \exp\left(-\int_r^s a(u)\, du\right) f(s)\, ds \right. \\ &\quad \left. + \exp\left(-\int_r^{r+1} a(u)\, du\right) d(r+1),\ \ n \ge 0. \right) \end{aligned} \tag{3.3}$$

By the hypotheses of (i), (ii), (iii), it is easy to see that

$$\sum_{r=0}^{\infty} \left(\int_r^{r+1} \exp\left(\int_s^{r+1} a(u)\, du \right) f(s)\, ds + \exp\left(-\int_r^{r+1} a(u)\, du \right) d(r+1) \right) < \infty.$$

This means that $\lim_{n \to \infty} x(n) \in \mathbb{R}$ and thus the solution $x(n)$ is bounded; that is, there is a $L_2 > 0$ such that (3.2) is satisfied. □

Now, we can give the proof of Theorem 2.1.

Proof of Theorem 2.1. *Let $x(t)$ be the solution of $(1.1)-(1.3)$. Then, from (1.12), we have*

$$\begin{aligned}|x(t)| &\leq |x_0| + \int_0^t |a(s)|\,|x(s)|\,ds + \int_0^t |a(s)|\,|x([s+1])|\,ds + \int_0^t |f(s)|\,ds + \sum_{i=1}^{[t]} |d(i)| \\ &\leq |x_0| + \int_0^t |a(s)|\,|x(s)|\,ds + L_2 \int_0^\infty |a(s)|\,ds + \int_0^\infty |f(s)|\,ds + \sum_{i=1}^{\infty} |d(i)|.\end{aligned}$$

By using (i), (ii), (iii) and Lemma 3.1, we obtain

$$|x(t)| \leq c + \int_0^t |a(s)|\,|x(s)|\,ds$$

where $c = |x_0| + L_2K_1 + K_2 + L_1$. Applying Gronwall inequality,

$$|x(t)| \leq c\exp\left(\int_0^t |a(s)|\,ds\right) \leq c\exp\left(\int_0^\infty |a(s)|\,ds\right).$$

Hence, $x(t)$ is bounded on the interval $[0,\infty)$; that is,

$$|x(t)| \leq ce^{K_1} \leq M,\ \ t \geq 0, \tag{3.4}$$

where $M = \max\left\{L_2,\ ce^{K_1}\right\}$.
On the other hand, by (1.12),

$$\begin{aligned}|x(t)-x(s)| \leq\ & \int_s^t |a(u)|\,|x(u)|\,du + \int_s^t |a(u)|\,|x([u+1])|\,du \\ & + \int_s^t |f(u)|\,du + \sum_{i=[s]+1}^{[t]} |d(i)|\end{aligned} \tag{3.5}$$

for $0 \leq s \leq t < \infty$.
Using (3.2) and (3.4), we have

$$|x(t)-x(s)| \leq 2M\int_s^\infty |a(u)|\,du + \int_s^\infty |f(u)|\,du + \sum_{i=[s]+1}^{\infty} |d(i)|.$$

Because of (i), (ii) and (iii),

$$\lim_{s\to\infty} |x(t)-x(s)| = 0.$$

So, by the Cauchy convergence criterion, $\lim\limits_{t\to\infty} x(t) \in \mathbb{R}$.

4. Proof of Theorem 2.2

The proof of Theorem 2.2 is based on the technique presented in [8]*. Therefore, it is necessary to prove the following theorem and lemmas. But, we first demonstrate the notation of the set of piecewise right continuous functions by* $PRC\left([0,\infty),\mathbb{R}\right):$ $\varphi \in PRC\left([0,\infty),\mathbb{R}\right)$ *implies that* $\varphi:[0,\infty)\to\mathbb{R}$ *is continuous for* $t\in[0,\infty),\ t\neq n\in\mathbb{Z}^{+}$, *and is right continuous for* $t=n\in\mathbb{Z}^{+}$.

Theorem 4.1. *Suppose* $a(t)$ *is continuous on* $[0,\infty)$ *and* (2.1) *is satisfied. Then, there is a unique bounded function* $y\in PRC\left([0,\infty),\mathbb{R}\right)$ *such that Eq.*(2.3) *holds.*

Proof. Let us consider the Banach space

$$B=\left\{y\in PRC\left([0,\infty),\mathbb{R}\right):|y|_B\leq\lambda,\ \lambda\geq\frac{1}{1-\rho}\right\}$$

where $\rho\in(0,1)$ is the same as in (2.1) and

$$|y|_B=\sup_{t\geq 0}|y(t)|,\ y\in B.$$

For $y\in B$ and $t\geq 0$, let us define the following operator

$$Ty(t)=1-\int_{[t]}^{t}y(s)\,a(s)\,ds.$$

It can be easily shown that for $n\in\mathbb{Z}^{+}$

$$Ty\left(n^{+}\right)=\lim_{t\to n^{+}}Ty(t)=1=Ty(n),$$

$$Ty\left(n^{-}\right)=\lim_{t\to n^{-}}Ty(t)=1-\int_{n-1}^{n}y(s)\,a(s)\,ds$$

and for $t_*\in(n,n+1)$

$$Ty\left(t_*^{+}\right)=Ty\left(t_*^{-}\right)=Ty(t_*).$$

So, $Ty\in PRC\left([0,\infty),\mathbb{R}\right)$.
Moreover, from (2.1),

$$|Ty|_B\leq 1+\rho\,|y|_B\leq\lambda,$$

that is, T takes the bounded functions of B into B. Hence T maps B into itself.
On the other hand, for y and $z\in B$

$$|Ty-Tz|_B\leq\rho\,|y-z|_B\,.$$

Since $0<\rho<1$, $T:B\to B$ is a contraction. Therefore, by the well known Banach fixed point theorem, there is a unique piecewise right continuous and bounded solution of Eq.(2.3). □

Lemma 4.1. *Under the hypotheses of Theorem 4.1, the solution* y *of the integral equation* (2.3) *satisfies the following adjoint equation:*

$$\left\{\begin{array}{l} y'(t)=-y(t)\,a(t),\ t\neq n,\ t\geq 0,\\ \Delta y(n)=\int\limits_{n-1}^{n}y(s)\,a(s)\,ds,\ n\in\mathbb{Z}^{+}.\end{array}\right.\tag{4.1}$$

Proof. Taking the derivative of (2.3) for $t \in (n, n+1)$, $n \in \mathbb{Z}^+$, we obtain

$$y'(t) = -y(t)\,a(t).$$

Moreover, we have

$$\begin{aligned} \Delta y(n) &= y\left(n^+\right) - y\left(n^-\right) \\ &= 1 - \left(1 - \int\limits_{n-1}^{n} y(s)\,a(s)\,ds\right) \\ &= \int\limits_{n-1}^{n} y(s)\,a(s)\,ds. \end{aligned}$$

So, the proof is complete. □

Now, let us denote the function

$$C(t) = y(t)\,x(t) + \int\limits_{[t]}^{t} y(s)\,a(s)\,x([s+1])\,ds,\ \ t \geq 0, \tag{4.2}$$

where y *is the solution of Eq.*(2.3) *and* x *is the solution of* $(1.1) - (1.3)$.

Lemma 4.2. *If the hypotheses of Theorem 4.1 hold, then*

$$C(t) = C(0) + \int\limits_{0}^{t} y(s)\,f(s)\,ds + \sum_{i=1}^{[t]} y\left(i^-\right)d(i). \tag{4.3}$$

Proof. To obtain (4.3), we should prove that $C(t)$ defined by (4.2) satisfies

$$\begin{cases} C'(t) = y(t)\,f(t),\ \ t \neq n,\ \ t \geq 0, \\ \Delta C(n) = y(n^-)\,d(n),\ \ n \in \mathbb{Z}^+. \end{cases} \tag{4.4}$$

For $t \in (n, n+1)$, $n \in \mathbb{Z}^+$, (4.2) can be written as

$$C(t) = y(t)\,x(t) + \left(\int\limits_{n}^{t} y(s)\,a(s)\,ds\right) x(n+1). \tag{4.5}$$

Differentiating (4.5), for $(n, n+1)$ we get

$$\begin{aligned} C'(t) &= y'(t)\,x(t) + y(t)\,x'(t) + y(t)\,a(t)\,x(n+1) \\ &= -y(t)\,a(t)\,x(t) + y(t)\,(a(t)\,(x(t) - x(n+1)) + f(t)) + y(t)\,a(t)\,x(n+1) \\ &= y(t)\,f(t). \end{aligned}$$

Moreover, from (4.2),

$$\begin{aligned} \Delta C(n) &= C\left(n^+\right) - C\left(n^-\right) \\ &= y(n)\,x(n) - y\left(n^-\right)x\left(n^-\right) - \left(\int\limits_{n-1}^{n} y(s)\,a(s)\,ds\right) x(n). \end{aligned} \tag{4.6}$$

Since

$$y(n)=1,\quad y\left(n^{-}\right)=1-\int_{n-1}^{n} y(s)\,a(s)\,ds$$

and

$$x\left(n^{-}\right)=x(n)-d(n),$$

we obtain from (4.6) $\Delta C(n)=\left(1-\int_{n-1}^{n} y(s)\,a(s)\,ds\right)d(n)=y(n^{-})\,d(n)$, which completes the proof of (4.4).

Now, integrating both sides of (4.4) from 0 to t, we obtain (4.3). □

We are now ready to prove the second main result.

Proof of Theorem 2.2. *Let $x(t)$ be the solution of $(1.1)-(1.3)$. It is sufficient to show that*

$$\lim_{t\to\infty} x(t)=C(0)+\int_{0}^{\infty} y(s)\,f(s)\,ds+\sum_{i=1}^{\infty} y\left(i^{-}\right)d(i) \tag{4.7}$$

where $C(0)=x(0)=x_0$ by (4.2). Indeed, the second hand of (4.7) is the same as $\ell(x_0)$ in (2.2).

By (4.3), we have for $t\geq 0$

$$\begin{aligned} x(t)-C(0)-\int_{0}^{\infty} y(s)\,f(s)\,ds-\sum_{i=1}^{\infty} y(i^{-})\,d(i) \\ =x(t)-\left(C(0)+\int_{0}^{t} y(s)\,f(s)\,ds+\sum_{i=1}^{[t]} y(i^{-})\,d_i\right)-\int_{t}^{\infty} y(s)\,f(s)\,ds-\sum_{i=[t]+1}^{\infty} y(i^{-})\,d(i) \\ =x(t)-C(t)-\int_{t}^{\infty} y(s)\,f(s)\,ds-\sum_{i=[t]+1}^{\infty} y(i^{-})\,d(i). \end{aligned}$$

Using (4.2), it follows for $t\geq 0$

$$\begin{aligned} x(t)-C(0)-\int_{0}^{\infty} y(s)\,f(s)\,ds-\sum_{i=1}^{\infty} y(i^{-})\,d(i) \\ =x(t)-y(t)\,x(t)-\int_{[t]}^{t} y(s)\,a(s)\,x([s+1])\,ds \\ -\int_{t}^{\infty} y(s)\,f(s)\,ds-\sum_{i=[t]+1}^{\infty} y(i^{-})\,d(i). \end{aligned} \tag{4.8}$$

On the other hand, multiplying (2.3) by $x(t)$, we obtain for $t\geq 0$

$$x(t)=y(t)\,x(t)+\int_{[t]}^{t} y(s)\,a(s)\,x(t)\,ds.$$

Substituting the last expression into (4.8), *we find for* $t \geq 0$

$$
\begin{aligned}
x(t) - C(0) - \int_0^\infty y(s) f(s) ds - \sum_{i=1}^{\infty} y(i^-) d(i) \\
&= \int_{[t]}^{t} y(s) a(s) (x(t) - x([s+1])) ds \\
&\quad - \int_t^\infty y(s) f(s) ds - \sum_{i=[t]+1}^{\infty} y(i^-) d(i).
\end{aligned}
\tag{4.9}
$$

From (4.9), *together with* (3.4) *and the boundedness of* $y(t)$ *on* $[0, \infty)$, *we get for* $t \geq 0$

$$
\begin{aligned}
&\left| x(t) - C(0) - \int_0^\infty y(s) f(s) ds - \sum_{i=1}^{\infty} y(i^-) d(i) \right| \\
&\leq |y|_B (2M) \int_{[t]}^{t} |a(s)| ds + |y|_B \int_t^\infty |f(s)| ds + |y|_B \sum_{i=[t]+1}^{\infty} |d(i)|.
\end{aligned}
$$

Taking into account (4.2), *it is easily verified that the limit relation* (4.7) *reduce to* (2.2). *Limiting both sides of the last inequality, as* $t \to +\infty$, *we have* (4.7). *So, the proof is completed.*

To illustrate our main results we can give the following example.

Example 4.1. *Consider the problem*

$$
x'(t) = a(t)(x(t) - x([t+1])) + \frac{1}{(2+3t)^2}, \; t \neq n \in \mathbb{Z}^+, \; t \geq 0 \tag{4.10}
$$

$$
\Delta x(n) = \frac{1}{3^n}, \; n \in \mathbb{Z}^+ \tag{4.11}
$$

$$
x(0) = 3, \tag{4.12}
$$

where $a(t) = \dfrac{1}{(2+3t)^2}$. *This is a special case of* $(1.1)-(1.3)$. *Here, all hypotheses of Theorem 2.1 and 2.2 are satisfied. So, as* $t \to \infty$, *the solution* $x(t)$ *of* $(4.10)-(4.12)$ *tends to a real constant, say* $\ell(3)$, *which can be calculated by* (2.2) *as*

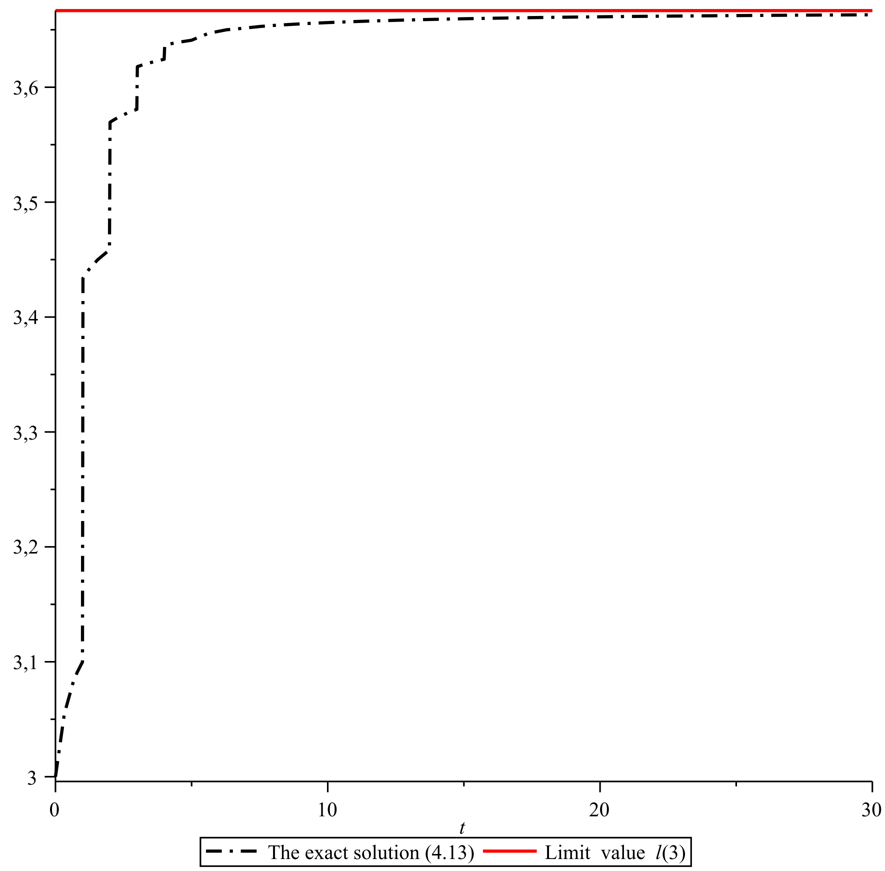
$$
\ell(3) = 3 + \int_0^\infty y(s) \frac{1}{(2+3s)^2} ds + \sum_{i=1}^{\infty} y(i^-) \frac{1}{3^i}
$$

where $y(t)$ *satisfies the integral equation* $y(t) = 1 - \int_{[t]}^{t} y(s) \dfrac{1}{(2+3s)^2} ds$.

Particularly, if we take $a(t) = 0$ *in* (4.10), *then we get* $\ell(3) = \dfrac{11}{3}$ *by* (2.2). *Also, this limit value can be found by the exact solution*

$$
x(t) = 3 + \frac{t - [t]}{(2+3t)(2+3[t])} + \sum_{r=0}^{[t-1]} \frac{1}{(2+3r)(5+3r)} + \sum_{r=1}^{[t]} \frac{1}{3^r}. \tag{4.13}
$$

Finally, for this example the following figure shows the behavior of the exact solution (4.13) *on the interval* $[0, 30]$.



5. Conclusion

In this paper, we stated the existence and uniqueness of the solution of the initial value problem $(1.1)-(1.3)$. *Moreover, we showed that the limit of this solution is a real constant and also we presented a formula for this limit value. It should be emphasized that using the exact solution* $x(t)$ *of* $(1.1)-(1.3)$ *to compute its limit value might be difficult. So, for the calculation of the limit of* $x(t)$, *it is better to use the formula* (2.2).

Furthermore, it is worth to say that in addition to the hypotheses of Theorem 2.2, if we assume $x_0+\int\limits_0^\infty y(s)\,f(s)\,ds+\sum\limits_{i=1}^\infty y(i^-)\,d(i)=0$, *then the solution of* $(1.1)-(1.3)$ $x(t)$ *goes to zero, as* $t\to\infty$.

On the other hand, if $d(n)=0$, *then the problem* $(1.1)-(1.3)$ *reduces to*

$$\begin{cases} x'(t)=a(t)\,(x(t)-x([t+1]))+f(t) \\ x(0)=x_0 \end{cases} \tag{5.1}$$

which is a continuous dynamical system, i.e. it does not include any impulses. Indeed, the solution $x(t)$ *of* (5.1) *can be obtained by putting* $d(n)=0$ *in* (1.11)

which is continuous on the interval $[0, \infty)$. *In this case, the limit value* $\ell(x_0)$ *is*

$$\lim_{t\to\infty} x(t) = \ell(x_0) = x_0 + \int_0^\infty y(s) f(s) ds$$

where $y(t)$ *is the solution of* $y(t) = 1 - \int_{[t]}^{t} y(s) a(s) ds$.

Finally, we note that if $f(t) \equiv 0$, *then the unique solution of* (5.1) *is constant,* $x(t) \equiv x_0$.

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