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EQUIVALENT CONTINUOUS G-FRAMES IN HILBERT C*-MODULES

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Abstract. In this paper, we investigate the mapping of continuous g-frames in Hilbert C*-module under bounded operators. So, operators that preserve continuous g-frames in Hilbert C*-module were characterized. Then, we introduce equivalent continuous g-frames in Hilbert C*-module by the mapping of continuous g-frames in Hilbert C*-module under bounded operators.

We show that every continuous g-frame in Hilbert C*-module is equivalent by a continuous Parseval g-frame in this space. We also verify the relation between the mapping of continuous g-frame in Hilbert C*-module and continuous g-frame operator in Hilbert C*-module. Then, we conclude if two continuous Parseval g-frames in Hilbert C*-module are equivalent then they are unitary equivalent.

1. Introduction

Frames for Hilbert spaces were first introduced in 1952 by Duffin and Schaeffer [11] for study of nonharmonic Fourier series. They were reintroduced and development in 1986 by Daubechies, Grossmann and Meyer[9], and popularized from then on. For basic results on frames, see [6, 7, 8, 13]. If H be a Hilbert space, and I a set which is finite or countable. A system $\{f_i\}_{i\in I} \subseteq H$ is called a frame for H if there exist the constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2 \tag{1}$$

for all $f \in H$. The constants A and B are called frame bounds. If $A = B$ we call this frame a tight frame and if A = B =1 it is called a Parseval frame.

Wenchang Sun [18] introduced a generalization of frames and showed that this includes more other cases of generalizations of frame concept and proved that many basic properties can be derived within this more general context.

In other hand, the concept of frames especially the g-frames was introduced in Hilbert C*- modules, and some of their properties were investigated in [12, 14, 15].

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Frank and Larson [12] defined the standard frames in Hilbert C*-modules in 1998 and got a series of result for standard frames in finitely or countably generated Hilbert C*-modules over unital C*-algebras. As for Hilbert C*-module, it is a generalizations of Hilbert spaces in that it allows the inner product to take values in a C*-algebra rather than the field of complex numbers. There are many differences between Hilbert C*-modules and Hilbert spaces. For example, we know that any closed subspaces in a Hilbert space has an orthogonal complement, but it is not true for Hilbert C*-module. And we can't get the analogue of the Riesz representation theorem of continuous functionals in Hilbert C*-modules generally. Thus it is more difficult to make a discussion of the theory of Hilbert C*-modules than that of Hilbert spaces in general. We refer readers to [16] for more details on Hilbert C*-modules. In [4, 15, 19] the authors made a discussion of some properties of g-frame in Hilbert C*-module in some aspects.

The concept of a generalization of frames to a family indexed by some locally compact space endowed with a Radon measure was proposed by G. Kaiser [13] and independently by Ali, Antoine and Gazeau [3]. These frames are known as continuous frames. The continuous g-frames in Hilbert C*-modules were firstly proposed by Rashidi and Nazari in [17], which are an extension to g-frames and continuous frames in Hilbert space and Hilbert C*-modules.

In this paper, we investigate the mapping of continuous g-frames in Hilbert C*-module under bounded operators and the concept of equivalent continuous g-frames in Hilbert C*-module was introduced such that they are an extension of some results in frame and continuous frame on Hilbert space [1, 5, 10] to continuous g-frame on Hilbert C*-module.

The paper is organized as follows. In Sections 2 we recall the basic definitions and some notations about continuous g-frames in Hilbert C*-module, we also give some basic properties of them which we will use in the later sections. In Section 3, operators that preserve continuous g-frames in Hilbert C*-module were characterized. Then, we introduce and verify equivalent continuous g-frames in Hilbert C*-module by the mapping of continuous g-frames in Hilbert C*-module under bounded operators.

2. Preliminaries

In the following we review some definitions and basic properties of Hilbert C*-modules and g-frames in Hilbert C*-module, we first introduce the definition of Hilbert C*-modules.

Definition 2.1. *Let A be a C^*-algebra with involution $*$. An inner product A-module (or pre Hilbert A-module) is a complex linear space H which is a left A-module with map $\langle .,. \rangle : H \times H \to A$ which satisfies the following properties:*
1) $\langle \alpha f + \beta g, h\rangle = \alpha < f, h > + \beta < g, h >$ *for all* $f, g, h \in H$ *and* $\alpha, \beta \in \boldsymbol{C}$;
2) $\langle af, g\rangle = a\langle f, g\rangle$ *for all* $f, g \in H$ *and* $a \in A$;
3) $\langle f, g\rangle = \langle g, f\rangle^*$ *for all* $f, g \in H$;
4) $\langle f, f\rangle \geq 0$ *for all* $f \in H$ *and* $\langle f, f\rangle = 0$ *iff* $f = 0$.
For $f \in H$*,we define a norm on* H *by* $\|f\|_H = \|\langle f, f\rangle\|_A^{1/2}$. *If* H *is complete with this norm, it is called a Hilbert* C^**-module over* A *or a Hilbert* A*-module.*

An element a of a C*-algebra A is positive if a* = a and spectrum of a is a subset of positive real number. We write $a \geq 0$ to mean that a is positive. It is easy to see that $\langle f, f\rangle \geq 0$ for every $f \in H$, hence we define $|f| = \langle f, f\rangle^{1/2}$.

Frank and Larson in [12] defined the standard frames in Hilbert C*-modules. If H be a Hilbert C*-module, and I a set which is finite or countable. A system $\{f_i\}_{i\in I} \subseteq H$ is called a frame for H if there exist the constants $A, B > 0$ such that

$$A\langle f, f\rangle \leq \sum_{i\in I} \langle f, f_i\rangle\langle f_i, f\rangle \leq B\langle f, f\rangle \tag{2}$$

for all $f \in H$. The constants A and B are called frame bounds.

Khosravi and Khosravi in [15] defined g-frame in Hilbert C*-module. Let U and V be two Hilbert C*-module and $\{V_i : i \in I\}$ is a sequence of subspaces of V, where I is a subset of Z and $End^*_A(U, V_i)$ is the collection of all adjointable A-linear maps from U into V_i i.e. $\langle Tf, g\rangle = \langle f, T^*g\rangle$ for all $f, g \in H$ and $T \in End^*_A(U, V_i)$. We call a sequence $\{\Lambda_i \in End^*_A(U, V_i) : i \in I\}$ a generalized frame, or simply a g-frame, for Hilbert C*-module U with respect to $\{V_i : i \in I\}$ if there are two positive constants A and B such that

$$A\langle f, f\rangle \leq \sum_{i\in I} \langle \Lambda_i f, \Lambda_i f\rangle \leq B\langle f, f\rangle \tag{3}$$

for all $f \in U$. The constants A and B are called g-frame bounds. If $A = B$ we call this g-frame a tight g-frame and if A = B =1 it is called a Parseval g-frame.

Let $(M; \mathcal{S}; \mu)$ be a measure space, U and V be two Hilbert C*-module, $\{V_m : m \in M\}$ is a sequence of subspaces of V and $End^*_A(U, V_m)$ is the collection of all adjointable A-linear maps from U into V_m.

Definition 2.2. *(see [17]) We call a net $\{\Lambda_m \in End^*_A(U, V_m) : m \in M\}$ a continuous generalized frame, or simply a continuous g-frame, for Hilbert C*-module U with respect to $\{V_m : m \in M\}$ if:*
(a) for any $f \in U$, the function $\tilde{f} : M \to V_m$ defined by $\tilde{f}(m) = \Lambda_m f$ is measurable,
(b) there is a pair of constants $0 < A, B$ such that, for any $f \in U$,

$$A\langle f, f\rangle \leq \int_M \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m) \leq B\langle f, f\rangle. \tag{4}$$

The constants A and B are called continuous g-frame bounds. If A = B we call this continuous g-frame a continuous tight g-frame and if A = B =1 it is called a continuous Parseval g-frame. If only the right-hand inequality of (6) is satisfied, we call $\{\Lambda_m : m \in M\}$ the continuous g-Bessel for U with respect to $\{V_m : m \in M\}$ with Bessel bound B.

If $M = \mathbf{N}$ and μ is the counting measure, the continuous g-frame for U with respect to $\{V_m : m \in M\}$ is a g-frame for U with respect to $\{V_m : m \in M\}$.

Let $\{\Lambda_m : m \in M\}$ be a continuous g-frame for U with respect to $\{V_m : m \in M\}$. Define the continuous g-frame operator S on U by

$$Sf = \int_M \Lambda^*_m \Lambda_m f d\mu(m).$$

Proposition 2.3. *(see [17]) The frame operator S is a bounded, positive, self-adjoint, and invertible.*

Proposition 2.4. *(see [17]) Let $\{\Lambda_m : m \in M\}$ be a continuous g-frame for U with respect to $\{V_m : m \in M\}$ with continuous g-frame operator S with bounds A and B. Then $\{\tilde{\Lambda}_m : m \in M\}$ defined by $\tilde{\Lambda}_m = \Lambda_m S^{-1}$ is a continuous g-frame for U with respect to $\{V_m : m \in M\}$ with continuous g-frame operator S^{-1} with bounds $1/B$ and $1/A$. That is called continuous canonical dual g-frame of $\{\Lambda_m : m \in M\}$.*

Next Theorem show that the continuous g-frame is equivalent to which the middle of (2.3) is norm bounded. The reader can find proof in [17].

Theorem 2.5. *Let $\Lambda_m \in End_A^*(U, V_m)$ for any $m \in M$. Then $\{\Lambda_m : m \in M\}$ is a continuous g-frame for U with respect to $\{V_m : m \in M\}$ if and only if there exist constants $A, B > 0$ such that for any $f \in U$*

$$A\|f\|^2 \leq \left\| \int_M \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m) \right\| \leq B\|f\|^2. \tag{5}$$

We define

$$\bigoplus_{m\in M} V_m = \left\{ g = \{g_m\} : g_m \in V_m \text{ and } \left\| \int_M |g_m|^2 d\mu(m) \right\| < \infty \right\}$$

For any $f = \{f_m : m \in M\}$ and $g = \{g_m : m \in M\}$, if the A-valued inner product is defined by $\langle f, g\rangle = \int_M \langle f_m, g_m\rangle d\mu(m)$, the norm is defined by $\|f\| = \|\langle f, f\rangle\|^{\frac{1}{2}}$, then $\bigoplus_{m\in M} V_m$ is a Hilbert A-module.(See [16]).

Let $\{\Lambda_m \in End_A^*(U, V_m) : m \in M\}$ is a continuous g-Bessel for U with respect to $\{V_m : m \in M\}$, we define synthesis operator $T : \bigoplus_{m\in M} V_m \to U$ by,

$$T(g) = \int_M \Lambda_m^* g_m d\mu(m) \quad \forall g = \{g_m\}_{m\in M} \in \bigoplus_{m\in M} V_m.$$

So analysis operator is defined for map $F : U \to \bigoplus_{m\in M} V_m$ by,

$$F(f) = \{\Lambda_m f : m \in M\}, \quad \forall f \in U.$$

3. Main Results

In this section, we investigate the mapping of continuous g-frames in Hilbert C*-module under bounded operators. Then, we introduce equivalent continuous g-frames in Hilbert C*-module by the mapping of continuous g-frames in Hilbert C*-module under bounded operators.

First, we give a lemma that will be used in the next section.

Lemma 3.1. *(see [2]) Let A be a C*-algebra, U and V be two Hilbert A-modules, and $T \in End_A^*(U, V)$, where $End_A^*(U, V)$ is the collection of all adjointable A-linear maps from U into V. Then the following statements are equivalent:*
(1) T is surjective.
(2) T^* is bounded below with respect to norm, i.e., there is $m > 0$ such that $\|T^* f\| \geq m\|f\|$ for all $f \in U$.
(3) T^* is bounded below with respect to the inner product, i.e, there is $m' > 0$ such that $\langle T^* f, T^* f\rangle \geq m'\langle f, f\rangle$.

Next theorem characterizes operators that preserve continuous g-frames in Hilbert C*- module.

Theorem 3.2. *Let $\{\Lambda_m : m \in M\}$ be a continuous g-frame in Hilbert C*-module U with respect to $\{V_m : m \in M\}$ with frame bounds A and B, and F is a bounded linear operator in $End_A^*(U,U)$. $\{\Lambda_m F : m \in M\}$ is a continuous g-frame in Hilbert C*-module U with respect to $\{V_m : m \in M\}$ if and only if F is bounded below respect to norm, i.e., if and only if*

$$\|Ff\|^2 \geq \gamma \|f\|^2, \;\; \forall f \in U. \tag{6}$$

Proof. To prove sufficiency observe that if the relation (3.1) satisfies then

$$\gamma A\|f\|^2 \leq A\|Ff\|^2 \leq \left\| \int_M \langle \Lambda_m Ff, \Lambda_m Ff \rangle d\mu(m) \right\| \leq B\|Ff\|^2 \leq B\|F\|^2\|f\|^2,$$

i.e., $\{\Lambda_m F : m \in M\}$ is a continuous g-frame in Hilbert C*-module U with respect to $\{V_m : m \in M\}$ with bounds γA and $B\|U\|^2$.

To prove the necessity, assume $\{\Lambda_m F : m \in M\}$ is a continuous g-frame in Hilbert C*-module U with respect to $\{V_m : m \in M\}$ with frame bounds C and D. Then, for all $f \in U$,

$$C\|f\|^2 \leq \left\| \int_M \langle \Lambda_m Ff, \Lambda_m Ff \rangle d\mu(m) \right\| \leq B\|Ff\|^2.$$

Thus (3.1) holds for $\gamma = C/B$ □

Corollary 3.3. *Let $\{\Lambda_m : m \in M\}$ be a continuous g-frame in Hilbert C*-module U with respect to $\{V_m : m \in M\}$ and F is a bounded linear operator in $End_A^*(U,U)$. $\{\Lambda_m F : m \in M\}$ is a continuous g-frame in Hilbert C*-module U with respect to $\{V_m : m \in M\}$ if and only if F^* is surjective.*

Proof. Using Lemma 3.1. □

Proposition 3.4. *Let F be an invertible bounded operator in $End_A^*(U,U)$ and let $\Lambda = \{\Lambda_m : m \in M\}$ and $\Gamma = \{\Gamma_m : m \in M\}$ be two continuous g-frames in Hilbert C*-module U with respect to $\{V_m : m \in M\}$ such that $\Gamma_m = \Lambda_m F$ for all $m \in M$. Then the following assertions are true:*
(a) $T_\Gamma = F^* T_\Lambda$,
(b) $S_\Gamma = F^* S_\Lambda F$,
where T_Λ and T_Γ are analysis operators and S_Λ and S_Γ are continuous g-frame operators corresponding to Λ and Γ, respectively.

Proof. (a) Let $f = \{f_m\}_{m\in M} \in \bigoplus_{m\in M} V_m$. Then for all $m \in M$,

$$T_\Gamma f = \int_M \Gamma_m^* f_m d\mu(m) = \int_M F^* \Lambda_m^* f_m d\mu(m) = F^* \int_M \Lambda_m^* f_m d\mu(m) = F^* T_\Gamma f.$$

Then $T_\Gamma = F^* T_\Lambda$ which proves (a).
(b) We have

$$S_\Gamma = T_\Gamma T_\Gamma^* = F^* T_\Lambda T_\Lambda^* F = F^* S_\Lambda F.$$

□

Proposition 3.5. *Let $\Lambda = \{\Lambda_m : m \in M\}$ and $\Gamma = \{\Gamma_m : m \in M\}$ be two continuous g-frames in Hilbert C*-module U with respect to $\{V_m : m \in M\}$. Then, there is a bounded linear operator,*

$$F : \bigoplus_{m\in M} V_m \to \bigoplus_{m\in M} V_m,$$

such that for any $f \in U$,

$$T_\Gamma^* f = FT_\Lambda^* f.$$

Proof. Since Λ is a continuous g-frames in Hilbert C*-module U with respect to $\{V_m : m \in M\}$, then $X = T_\Lambda^*(U)$ is a closed subspace of $\bigoplus_{m\in M} V_m$. Therefore, $T_\Lambda^* : U \to X$ is bijective and so it is invertible. $(T_\Lambda^*)^{-1}$ is bounded. Let $F_0 = T_\Gamma^*(T_\Lambda^*)^{-1} : X \to \bigoplus_{m\in M} V_m$. It is obvious that F_0 is bounded on X and we can extend it on $\bigoplus_{m\in M} V_m$ by:

$$F(x) := \begin{cases} F_0(x) & \text{if } x \in X, \\ 0 & \text{if } x \in X^\perp. \end{cases}$$

Then, for each $f \in U$ we have,

$$FT_\Lambda^* f = T_\Gamma^*(T_\Lambda^*)^{-1} T_\Lambda^* f = T_\Gamma^* f.$$

□

Now, we introduce equivalent continuous g-frames in Hilbert C*-module.

Definition 3.6. *Two continuous g-frames $\Lambda = \{\Lambda_m : m \in M\}$ and $\Gamma = \{\Gamma_m : m \in M\}$ on Hilbert C*-module U with respect to $\{V_m : m \in M\}$ are partial equivalent if there is an operator $F : U \to U$ in $End_A^*(U, U)$ such that*

$$\Gamma_m = \Lambda_m F, \qquad \forall m \in M.$$

They are equivalent if F is invertible. So, we say they are unitarily (isometrical) equivalent if F is a unitary (isometry) operator.

Corollary 3.7. *Let F be a bounded linear operator in $End_A^*(U, U)$ and $\{\Lambda_m : m \in M\}$ be a continuous g-frame for U with respect to $\{V_m : m \in M\}$. If F is bounded below or F^* is surjective then $\{\Lambda_m : m \in M\}$ and $\{\Lambda_m F : m \in M\}$ are partial equivalent.*

Theorem 3.8. *If $\{\Lambda_m : m \in M\}$ is a continuous g-frame in Hilbert C*-module U with respect to $\{V_m : m \in M\}$ with continuous g-frame operator S_1 and S_2 is any positive, self-adjoint, invertible operator, then there is an invertible operator F so that the continuous g-frame operator $\{\Lambda_m F : m \in M\}$ is S_2.*

Proof. If we let $F = S_1^{\frac{-1}{2}} S_2^{\frac{-1}{2}}$, then $T^* S_1 T = S_2$. □

Corollary 3.9. *If $\Lambda = \{\Lambda_m : m \in M\}$ and $\Gamma = \{\Gamma_m : m \in M\}$ are equivalent continuous Parseval g-frames, then they are unitarily equivalent.*

Proof. Let F be an invertible operator with $\Gamma_m = \Lambda_m F$. Then the continuous g-frame operator of $\{\Lambda_m F : m \in M\}$ is $F^* ST$ where $S = I$ is the continuous g-frame operator for $\{\Lambda_m : m \in M\}$. Since the continuous g-frame operator for $\{\Gamma_m : m \in M\}$ is also I we have

$$I = F^* SF = FIF^* = FF^*.$$

□

Theorem 3.10. *Let $\{\Lambda_m : m \in M\}$ be a continuous g-frame for U with respect to $\{V_m : m \in M\}$ and S is continuous g-frame operator of $\{\Lambda_m : m \in M\}$, then $\{\Lambda_m S^{\frac{-1}{2}} : m \in M\}$ is a continuous Parseval g-frame.*

Proof. From [17] we know that continuous g-frame operator S is positive self-adjoint and invertible, hence operators $S^{\frac{-1}{2}}$ and $S^{\frac{1}{2}}$ are well defined. Therefore every $f \in U$ can be written $f = S^{\frac{-1}{2}} S S^{\frac{-1}{2}} f$. Since $\langle Sf, f\rangle = \int_M \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m)$ we have

$$\langle f, f\rangle = \langle S^{\frac{-1}{2}} S S^{\frac{-1}{2}} f, f\rangle = \langle S S^{\frac{-1}{2}} f, S^{\frac{-1}{2}} f\rangle = \int_M \langle \Lambda_m S^{\frac{-1}{2}} f, \Lambda_m S^{\frac{-1}{2}} f\rangle d\mu(m),$$

which proof that $\{\Lambda_m S^{\frac{-1}{2}} : m \in M\}$ is a continuous Parseval g-frame in Hilbert C*-module U with respect to $\{V_m : m \in M\}$. □

Therefore every continuous g-frame with continuous g-frame operator S is $S^{\frac{-1}{2}}$-equivalent to a continuous Parseval g-frame.
So, we can say for any continuous g-frame there is an invertible operator T taking our continuous g-frame to another continuous g-frame with continuous g-frame operator I.

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