Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 4 Issue 4(2012), Pages 108-115.

NULL SECTIONAL CURVATURE PINCHING FOR CR-LIGHTLIKE SUBMANIFOLDS OF SEMI-RIEMANNIAN MANIFOLDS

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Abstract. In this article we obtain the pinching of the null sectional curvature of CR- lightlike submanifolds of an indefinite Hermitian manifold. As a result of this inequality we conclude some non-existence results of such lightlike submanifolds.Moreover using the Index form we prove more non-existence results for CR-lightlike submanifolds.

1. Introduction

The theory of submanifolds of a Riemannian or semi-Riemannian manifold is well-known .(see for example, [1] and [6]). However the geometry of lightlike (null) submanifolds (for which the geometry is different from the non-degenerate case) is highly interesting and in a developing stage. In particular, curvature pinching relations are of substantial interest as they give the bounds for the curvature.Analogous to sectional curvature in Riemannian case, Duggal [2] defined the null sectional curvature for lightlike submanifolds. Earlier on, A. Gray [4] investigated different pinchings for sectional and bisectional curvature under certain conditions in case of Kaehler manifolds. In this article we would like to study CR-lightlike submanifold of an indefinite almost Hermitian manifold (for definite Hermitian manifolds see [5]) and hence obtain the null sectional curvature pinching

$$-K_\xi(JY) \leq K_\xi(X) \leq 3K_\xi(JY)$$

as our main theorem; where X, Y are two orthonormal vectors in some distribution of $S(TM)$ and $K_\xi(X)$ is the null sectional curvature [2] . With the help of this null curvature pinching we obtain some non-existence results for CR-lightlike submanifolds of an indefinite almost Hermitian manifold.

In the last we also study some applications of the Index form and Jacobi equation [6], to conclude some more non-existence results.

2. Preliminary

Let $(\bar{M}, \bar{g})$ be an $(m+n)-$dimensional semi-Riemannian manifold and $\bar{g}$ be a semi-Riemannian metric on $\bar{M}$. Let M be a lightlike submanifold of $\bar{M}$.

2000 *Mathematics Subject Classification.* 53C50, 53B30 .

Key words and phrases. CR-Lightlike submanifolds, Null sectional curvature, Index form, Variation of a curve.

Submitted April 25, 2012. Published October 8, 2012.

This Research Work is supported by UGC Major Research Project No: 33/112(2007)-SR.

Definition 2.1. [2] *A lightlike submanifold M of an indefinite almost Hermitian manifold $\bar{M}$ is said to be CR-lightlike submanifold if and only if the following two conditions are fulfilled:*

(a) $J(RadTM)$ is a distribution on M such that $RadTM \cap J(RadTM) = \{0\}$,

(b) there exists vector bundles $S(TM)$, $S(TM^{\perp})$, $ltr(TM)$, $D_\circ$ and D' over M such that

$$S(TM) = \{J(RadTM) \oplus D'\} \perp D_\circ \ ; \ \ JD_\circ = D_\circ; \ \ J(D') = L_1 \perp L_2$$

where $D_\circ$ is a non-degenerate distribution on M, and L_1 and L_2 are vector subbundles of $ltr(TM)$ and $S(TM^{\perp})$ respectively.

It is seen that there exists examples of CR-lightlike submanifolds of an indefinite Hermitian manifold. An example of such kind can be given as follows[2]:

Example1. Let M be a submanifold of codimension 2 of R^2_6 given by the equations

$$x^5 = x^1 \cos\alpha - x^2 \sin\alpha - f(x^3, x^4)\tan\alpha,$$

$$x^6 = x^1 \sin\alpha + x^2 \cos\alpha + f(x^3, x^4)$$

where $\alpha \in R - \left\{\frac{\pi}{2} + k\pi; k \in Z\right\}$ and f is a smooth function such that $\left(\frac{\partial f}{\partial x^3}, \frac{\partial f}{\partial x^4}\right) \neq (0,0)$. It is easily verified that the tangent bundle of M is spanned by

$$\begin{aligned}
\{\xi &= \frac{\partial}{\partial x^1} + \cos\alpha\frac{\partial}{\partial x^5} + \sin\alpha\frac{\partial}{\partial x^6};\\
X_\circ &= \frac{\partial}{\partial x^2} - \sin\alpha\frac{\partial}{\partial x^5} + \cos\alpha\frac{\partial}{\partial x^6};\\
X_1 &= \frac{\partial}{\partial x^3} - \frac{\partial f}{\partial x^3}\tan\alpha\frac{\partial}{\partial x^5} + \frac{\partial f}{\partial x^3}\frac{\partial}{\partial x^6};\\
X_2 &= \frac{\partial}{\partial x^4} - \frac{\partial f}{\partial x^4}\tan\alpha\frac{\partial}{\partial x^5} + \frac{\partial f}{\partial x^4}\frac{\partial}{\partial x^6}\}.
\end{aligned}$$

Then M is a 1-lightlike submanifold with $Rad(TM) = span\{\xi\}$. Moreover by using $J(x_1, y_1, ..., x_m, y_m) = (-y_1, x_1, ..., -y_m, x_m)$ we obtain that $JRad(TM)$ is spanned by $X_\circ$ and therefore it is a distribution on M. Hence M is a CR-lightlike submanifold of codimension 2 of R^6_2.

Example 2. We consider a submanifold M of codimension 2 in R^8_2 given by the equations

$$x^7 = x^1\cos\alpha - x^2\sin\alpha - x^5x^6\tan\alpha,$$

$$x^8 = x^1\sin\alpha + x^2\cos\alpha + x^5x^6$$

where $\alpha \in R - \left\{\frac{\pi}{2} + k\pi; k \in Z\right\}$. Then TM is spanned by

$$\begin{aligned}
U_1 &= (1,0,0,0,0,0,\cos\alpha,\sin\alpha); U_2 = (0,1,0,0,0,0,-\sin\alpha,\cos\alpha);\\
U_3 &= (0,0,1,0,0,0,0,0); U_4 = (0,0,0,1,0,0,0,0);\\
U_5 &= (0,0,0,0,1,0,-x^6\tan\alpha, x^6); U_6 = (0,0,0,0,0,1,-x^5\tan\alpha, x^5).
\end{aligned}$$

It is easy to check that this submanifold is 1-lightlike submanifold of R^8_2 such that $Rad(TM) = span\{U_1\}$. Furthermore by using

$$J(x_1, y_1, ..., x_m, y_m) \ = (-y_1, x_1, ..., -y_m, x_m)$$

on R_2^8 we see that $U_2 = JU_1$. This shows that $JRad(TM)$ is a distribution on M. Hence M is a CR-lightlike submanifold.

Let $u \in M$ and ξ be a null vector in T_uM. A plane P of T_uM is called a null plane directed by ξ if it contains ξ, $g_u(\xi, X) = 0$ for any $X \in P$ and there exists $X_\circ \in P$ such that $g_u(X_\circ, X_\circ) \neq 0$. As in case of lightlike submanifolds the collection of null vectors is denoted by $Rad(TM)$ and non-null vectors by $S(TM)$ i.e. we always have $\xi \in Rad(TM)$ and $X_\circ \in S(TM)$. This means that in case of lightlike submanifolds null plane is spanned by a vector of $Rad(TM)$ and a vector of $S(TM)$.

Definition 2.2. [2] *The null sectional curvature of P with respect to ξ and ∇ is defined as the real number*

$$K_\xi(X) = \frac{g(R(X,\xi)\xi, X)}{g(X,X)}, \forall \xi \in Rad(TM), X \in S(TM).$$

The null sectional curvature of P with respect to ξ and $\bar{\nabla}$ is defined as the real number

$$\bar{K}_\xi(X) = \frac{\bar{g}(\bar{R}(X,\xi)\xi, X)}{\bar{g}(X,X)}, \forall \xi \in Rad(TM), X \in S(TM).$$

We denote by $Q_\xi(X)$, the quantity $g(R(X,\xi)\xi, X)$ i.e. $Q_\xi(X) = g(R(X,\xi)\xi, X)$ which gives

$$Q_\xi(X) = \|X\|^2 K_\xi(X). \tag{2.1}$$

Similarly we have $\bar{Q}_\xi(X)$ in case of $\bar{M}$.

In [3] Duggal and Jin defined totally umbilical lightlike submanifolds of a semi-Riemannian manifold.

Definition 2.3. [2] *A lightlike submanifold M of a semi-Riemannian manifold $\bar{M}$ is totally umbilical if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M called the transversal curvature vector field of M, such that for all $X, Y \in \Gamma(TM)$,*

$$h(X,Y) = Hg(X,Y).$$

A CR-lightlike submanifold which is totally umbilical is called totally umbilical CR-lightlike submanifold.

The following lemma is an important result regarding the null sectional curvature of totally umbilical CR-lightlike submanifold:

Lemma 2.4. *Let (M, g) be a totally umbilical CR-lightlike submanifold of an almost Hermitian manifold $(\bar{M}, \bar{g})$. Then the null sectional curvature of M is equal to the null sectional curvature of $\bar{M}$.*

Proof. Let (M, g) be a totally umbilical CR-lightlike submanifold of $(\bar{M}, \bar{g})$. Then from [2] we can write

$$\bar{g}(\bar{R}(X,\xi)\xi, X) = g(R(X,\xi)\xi, X) + \bar{g}(h^s(X,\xi), h^s(\xi, X)) - \bar{g}(h^s(X,X), h^s(\xi,\xi)), \tag{2.2}$$

$\forall\ X \in D_\circ$ and $\xi \in Rad(TM)$.Since M is totally umbilical we have,

$$\bar{K}_\xi(X) = K_\xi(X).$$

□

3. The Pinching theorem

Curvature pinching relations are an important tool to study the geometry of a manifold (or submanifold) which is evident from many interesting articles in the litreture (for example see [4]). We prove the null sectional curvature pinching theorem in case of CR-lightlike submanifolds of an indefinite almost Hermitian manifold.

Theorem 3.1. *Let (M, g) be a CR-lightlike submanifold of an indefinite almost Hermitian manifold $(\bar{M}, \bar{g})$ with non-zero null sectional curvature. Also suppose that X, Y be any two orthonormal vectors in $D_\circ$ such that $g(X, JY) = \cos\theta$. Then either*

$$-K_\xi(JY) \leq K_\xi(X) \leq 3K_\xi(JY) \tag{3.1}$$

or

$$\cos\theta = \frac{1}{2}.$$

Proof. From the definition of $Q_\xi(X)$ and the linearity of the curvature tensor R we conclude that

$$Q_\xi(X + JY) = g(R(X, \xi)\xi, X) + g(R(X, \xi)\xi, JY) + g(R(JY, \xi)\xi, X) + g(R(JY, \xi)\xi, JY). \tag{3.2}$$

Similarly we have

$$Q_\xi(X - JY) = g(R(X, \xi)\xi, X) - g(R(X, \xi)\xi, JY) - g(R(JY, \xi)\xi, X) + g(R(JY, \xi)\xi, JY). \tag{3.3}$$

Combining equations 3.2 and 3.3, we derive

$$g(R(X, \xi)\xi, X) = Q_\xi(X + JY) + Q_\xi(X - JY) - 2Q_\xi(JY) - Q_\xi(X) \tag{3.4}$$

Let X and JY be any two vectors of $D_\circ$ such that $g(X, JY) = \cos\theta$, then as a consequence of equations 2.2 and 3.4, it follows that

$$\begin{aligned} K_\xi(X) &= \|X + JY\|^2 K_\xi(X + JY) + \|X - JY\|^2 K_\xi(X - JY) \\ &\quad -2\|JY\|^2 K_\xi(JY) - \|X\|^2 K_\xi(X) \\ &= \{\|X\|^2 + 2\cos\theta + \|JY\|^2\}K_\xi(X + JY) + \{\|X\|^2 - 2\cos\theta \\ &\quad + \|JY\|^2\}K_\xi(X - JY) - 2\|JY\|^2 K_\xi(JY) - \|X\|^2 K_\xi(X) \end{aligned} \tag{3.5}$$

Now we consider the cases depending on the signature of vector fields:

Case (a) If X and JY are spacelike vectors i.e. $\|X\|^2 = \|JY\|^2 = 1$, then from equation 3.5 we have

$$= 2(1 + \cos\theta)K_\xi(X + JY) + 2(1 - \cos\theta)K_\xi(X - JY) - 2K_\xi(JY) - K_\xi(X).$$

Using the linearity of the tensor K_ξ we calculate the above equation as

$$K_\xi(X) = (1 - 2\cos\theta)K_\xi(JY), \quad \forall X, Y \in D_\circ. \tag{3.6}$$

Since $-1 \leq \cos\theta \leq 1$ we obtain that

$$-K_\xi(JY) \leq K_\xi(X) \leq 3K_\xi(JY), \quad \forall X, Y \in D_\circ$$

Case (b) If X and JY are timelike vectors i.e. $\|X\|^2 = \|JY\|^2 = -1$, then from equation 3.5 we find

$$2K_\xi(X) = (1 + 2\cos\theta)K_\xi(JY), \quad \forall X, Y \in D_\circ.$$

The above equation implies that

$$-\frac{1}{2}K_\xi(JY) \leq K_\xi(X) \leq \frac{3}{2}K_\xi(JY)$$

or we can say

$$-K_\xi(JY) \leq K_\xi(X) \leq 3K_\xi(JY).$$

Case (c) If $\|X\|^2 = 1$ and $\|JY\|^2 = -1$, then from equation 3.5 we derive

$$K_\xi(X) = (1 - 2\cos\theta)K_\xi(JY), \quad \forall X, Y \in D_\circ.$$

which again gives

$$-K_\xi(JY) \leq K_\xi(X) \leq 3K_\xi(JY).$$

Case (d) If $\|X\|^2 = -1$ and $\|JY\|^2 = 1$, then equation 3.5 simplifies to

$$K_\xi(JY) = 2\cos\theta K_\xi(JY).$$

This shows that

$$\cos\theta = \frac{1}{2}$$

since M is with non-zero null sectional curvature. $\square$

Remark :- *We see here that one cannot consider the entire $S(TM)$ for the above pinching of null sectional curvature since from the definition of CR-lightlike submanifolds $S(TM) = \{J(RadTM) \oplus D'\} \perp D_\circ$ and $JD' \subset ltr(TM) \perp S(TM^\perp)$.*

From lemma-1 and the above theorem, we immediately have:

Corollary 3.2. *Let (M, g) be a totally umbilical CR-lightlike submanifold of an indefinite almost Hermitian manifold $(\bar{M}, \bar{g})$ with non-zero null sectional curvature. Also suppose that X, Y be any two orthonormal vectors in $D_\circ$ such that $g(X, JY) = \cos\theta$. Then either*

$$-\bar{K}_\xi(JY) \leq \bar{K}_\xi(X) \leq 3\bar{K}_\xi(JY), \quad \forall X, Y \in D_\circ,$$

or

$$\cos\theta = \frac{1}{2}.$$

We also note the following:

Corollary 3.3. *Let (M, g) be a CR-lightlike submanifold of an indefinite almost Hermitian manifold $(\bar{M}, \bar{g})$. Also suppose X and JY are both spacelike or timelike orthonormal vectors where $X, Y \in D_\circ$. Then there exists no such submanifolds with negative null sectional curvature.*

Proof. Putting $\theta = \frac{\pi}{2}$ in equation 3.6 we have

$$K_\xi(X) = K_\xi(JY).$$

Hence from pinching 3.1 we find

$$-K_\xi(X) \leq K_\xi(X) \leq 3K_\xi(X), \quad \forall X, Y \in D_\circ$$

which gives that

$$K_\xi(X) \geq 0.$$

Hence the result. □

Corollary 3.4. *Let (M, g) be a totally umbilical CR-lightlike submanifold of an indefinite almost Hermitian manifold $(\bar{M}, \bar{g})$. Also suppose X and JY are both spacelike or timelike orthonormal vectors where $X, Y \in D_\circ$. Then $\bar{M}$ cannot be with negative null sectional curvature.*

4. Index form and application of null curvature pinching

In the present section we deal with the application of Index form of non null geodesics of CR-lightlike submanifold of an indefinite almost Hermitian manifold. First we give a brief idea of the variation of a curve.

Let M be a CR-lightlike submanifold of an indefinite almost Hermitian manifold $\bar{M}$, then since the non-degenerate metric of $\bar{M}$ induce the degenerate metric on M (c.f. [2]; page-1), we can define the variation of any non-null curve α in M of sign ε, as defined in [6].

Definition 4.1. *A variation of a curve segment $\alpha : [a, b] \longrightarrow M$ is a two parameter mapping*

$$x : [a, b] \times (-\delta, \delta) \longrightarrow M$$

such that $\alpha(u) = x(u, 0)$ for all $a \leq u \leq b$. The vector field V on α given by $V(u) = x_v(u, 0)$ is called the variation vector field of x. Similarly the vector field $A(u) = x_{vv}(u, 0)$ gives the acceleration and we call it the transverse acceleration vector field of x.

We note that the variation vector field V on $\alpha \subset M$ may or may not be a null vector field since in the definition it is not bound to have special causal character.

To find out the change in arc length of a curve segment under small displacements let $x : [a, b] \times (-\delta, \delta) \longrightarrow M$ be a variation of a curve segment. For each $v \in (-\delta, \delta)$, let $L_x(v)$ be the length of the longitudinal curve $u \longrightarrow x(u, v)$.Then it is easy to see that the first variation of the arc length function $L_x(v)$ is given by

$$L'_x(0) = \varepsilon \int_a^b g(\frac{\alpha'}{|\alpha'|}, V')du. \tag{4.1}$$

The second variation of arc length of $L_x(v)$ is possible in case α is a geodesic and is given by

$$L''_x(0) = \frac{\varepsilon}{c} \int_a^b \{g(V', V') - g(R(V, \alpha')V, \alpha')\}du + \frac{\varepsilon}{c}[g(\alpha', A)]_a^b$$

where $\|\alpha'\| = c$.

It is clear that for a fixed endpoint variation the last term of the above expression is zero and hence we have

$$L''_x(0) = \frac{\varepsilon}{c} \int_a^b \{g(V', V') - g(R(V, \alpha')V, \alpha')\}du. \tag{4.2}$$

Definition 4.2. *The index form I_α of a nonnull geodesic $\alpha \in \Omega(p,q)$,is the unique symmetric bilinear form*

$$I_\alpha : T_\alpha(\Omega) \times T_\alpha(\Omega) \longrightarrow R$$

such that if $V \in T_\alpha(\Omega)$, then $I_\alpha(V,V) = L''_x(0)$, where $\Omega(p,q)$ is the collection of all piecewise smooth curve segments $\alpha : [a,b] \longrightarrow M$ from p to q.

Let us assume α to be a nonnull geodesic in M of sign ε. Then we have the following theorem:

Theorem 4.3. *Let M^n be a CR-lightlike submanifold of an indefinite almost Hermitian manifold $\bar{M}$ with $\left\|\frac{d\xi}{du}\right\| > 0$, $\forall \xi \in RadTM$ and let X, Y be any two orthonormal vectors in $D_\circ$ such that $g(X, JY) = \cos\theta < \frac{1}{2}$. Then there exists no such submanifolds with non-negative null sectional curvature.*

Proof. Let α be any non-null geodesic The index form I_α for a nonnull geodesic α is given by

$$I_\alpha(V,V) = \frac{\varepsilon}{c}\int_a^b \{g(V',V') - g(R(V,\alpha')V,\alpha')\}du.$$

Let X, Y be any two orthonormal vectors in $D_\circ$ such that $g(X,JY) = \cos\theta < \frac{1}{2}$, then replacing α' by $X \in D_\circ$, V by $\xi \in Rad(TM)$ we get

$$I_\alpha(\xi,\xi) = \frac{\varepsilon}{c}\int_a^b \{g(\xi',\xi') + K_\xi(X)\}du.$$

It is easy to see that V can be lightlike vector since by definition of Index form $V \in T_\alpha(\Omega)$ and $T_\alpha(\Omega)$ is the tangent space to Ω at α which consists of all piecewise smooth vector fields on α [6]. Therefore from the last equation we find

$$I_\alpha(\xi,\xi) = \frac{\varepsilon}{c}\int_a^b \{g(\xi',\xi') + (1-2\cos\theta)K_\xi(JY)\}du. \tag{4.3}$$

Now from [2] (equation -2.36; page-160), we note that in general ξ' i.e. $\nabla_{\frac{\partial}{\partial u}}\xi$ or $\frac{d\xi}{du}$ is not purely lightlike in nature. Therefore $g(\xi',\xi') \neq 0$. Furthermore since $g(\xi,\xi) = 0$ implies that $g(\xi,\xi') = 0$ which shows that ξ is orthogonal to ξ'. Combining these facts, we can consider ξ' to be a vector in $D_\circ$ which is non-degenerate.

Furthermore if M^n is a CR-lightlike submanifold of an indefinite almost Hermitian manifold $\bar{M}$ with non-negative null sectional curvature then since $\|X\| = +1$ or -1 implies $\varepsilon = +1$ or -1 respectively and $\left\|\frac{d\xi}{du}\right\| > 0$, we see from the above equation 4.3 that $I_\alpha(\xi,\xi) > 0$. But then from [6](lemma-13; chapter-10) M^n is with index either zero or n , which being lightlike submanifold ,are not possible cases for M^n. Thus we get a contradiction and hence this proves the non-existence of M^n. □

We conclude the following corollary from the above theorem.

Corollary 4.4. *Let M^n be a CR-lightlike submanifold of an indefinite almost Hermitian manifold $\bar{M}$ with $\left\|\frac{d\xi}{du}\right\| > 0$, $\forall \xi \in RadTM$ and let X, Y be any two orthonormal vectors in $D_\circ$ such that $g(X, JY) = \cos\theta < \frac{1}{2}$. Then $\bar{M}$ cannot have negative null sectional curvature.*

Acknowledgments. The authors wish to thank the referees for their many valuable suggestions that improved the paper.

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