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ON CONVERGENCE OF A MULTI-STEP ITERATION PROCESS WITH ERRORS FOR A FINITE FAMILY OF QUASI-NONEXPANSIVE MULTI-VALUED MAPPINGS

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Abstract. In this paper, we introduce a general iterative process with errors for a finite family of quasi-nonexpansive multi-valued mappings and prove some strong and weak convergence theorems for such mappings in uniformly convex Banach spaces. Our results are generalizations as well as refinement of several known results in the current literature.

1. Introduction and Preliminaries

Let E be a nonempty and convex subset of a Banach space X. The set E is called proximal if for each $x \in X$, there exists an element $y \in E$ such that

$$\|x-y\| = d(x, E) = \inf\{\|x-z\| : z \in E\}.$$

It is known that every closed convex subset of a uniformly convex Banach space is proximal. We denote by $CB(E), K(E)$ and $P(E)$ the collection of all nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximal bounded subsets of E respectively. The Hausdorff metric H on $CB(X)$ is defined by

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}$$

for all $A, B \in CB(X)$. Let $T : X \to 2^X$ be a multi-valued mapping. An element $x \in X$ is said to be a fixed point of T, if $x \in Tx$. The set of fixed points of T will be denote by $F(T)$.

Definition 1.1. *A multi-valued mapping $T : X \longrightarrow CB(X)$ is called*

(i) nonexpansive if $H(Tx,Ty) \le \|x-y\|$, for all $x, y \in X$;

(ii) quasi-nonexpansive if $F(T) \neq \emptyset$ and $H(Tx,Tp) \le \|x-p\|$, for all $x \in X$ and all $p \in F(T)$;

(iii) L-Lipschitzian if there exists a constant $L > 0$ such that $H(Tx,Ty) \le L\|x-y\|$, for all $x, y \in X$;

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(iv) hemi-compact if, for any sequence $\{x_n\}$ in X such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in X$. We note that if X is compact, then every multi-valued mappings $T : X \longrightarrow CB(X)$ is hemi-compact.

It is clear that every nonexpansive multi-valued mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive. But there exist quasi-nonexpansive mappings that are not nonexpansive (see [12]). It is known that if T is a quasi-nonexpansive multi-valued mapping, then $F(T)$ is closed.

The study of fixed points for multi-valued contractions and nonexpansive mappings using the Hausdorff metric was initiated by Markin [6] and Nadler [7]. Since then the theory of multi-valued mappings has applications in control theory, convex optimization, differential equations and economics. Theory of multi-valued nonexpansive mappings is harder than the corresponding theory of single-valued nonexpansive mappings. Different iterative processes have been used to approximate fixed points of multi-valued nonexpansive mappings.

Among these iterative processes, Sastry and Babu [10] considered the following.

Let E be a nonempty convex subset of a Banach space X, $T : E \longrightarrow P(E)$ a multi-valued mapping with $p \in Tp$.

(i) The sequence of Mann iterates is defined by $x_1 \in E$,

$$x_{n+1} = (1 - a_n)x_n + a_n y_n, \ n \geq 1, \tag{1.1}$$

where $y_n \in Tx_n$;

(ii) The sequence of Ishikawa iterates is defined by $x_1 \in E$,

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n z_n, \ n \geq 1, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n u_n, \ n \geq 1, \end{cases} \tag{1.2}$$

where $z_n \in Tx_n$ and $u_n \in Ty_n$.

They proved that the Mann and Ishikawa iteration processes for multi-valued mapping T with a fixed point p converge to a fixed point q of T under certain conditions. They also claimed that the fixed point q may be different from p. Panyanak [9] extended result of Sastry and Babu [10] to uniformly convex Banach spaces. After, Song and Wang [11] noted that there was a gap in the proof of the main result in [9]. They further revised the gap and also gave the affirmative answer to Panyanak's open question. Shazad and Zegeye [12] extended and improved results already appeared in the papers [9, 10, 11]. Moreover, the existence of fixed points for multivalued mappings in Banach spaces was showed by many other authors [1, 13, 15]. Recently, Cholamjiak and Suantai [4] introduced the following Ishikawa iteration with errors for two quasi-nonexpansive multi-valued mappings and prove some convergence theorems for such mappings.

Let E be a nonempty convex subset of a Banach space X and $T_1, T_2 : E \longrightarrow CB(E)$ be two quasi-nonexpansive multi-valued mappings. Then for $x_1 \in E$,

$$\begin{cases} y_n = (1 - a_n - b_n)x_n + a_n w_n + b_n u_n, \ n \geq 1, \\ x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n z_n + \beta_n v_n, \ n \geq 1, \end{cases} \tag{1.3}$$

where $w_n \in T_2 x_n$ and $z_n \in T_1 y_n$, $\{a_n\}, \{b_n\}, \{\alpha_n\}, \{\beta_n\} \in [0, 1]$ and $\{u_n\}, \{v_n\}$ are bounded sequences in E.

Next, they [3] introduced the following iteration process:

Let E be a nonempty convex subset of a Banach space X and $T_1, T_2 : E \longrightarrow CB(E)$ be two quasi-nonexpansive multi-valued mappings and

$$P_{T_i}(x) = \{y \in T_i(x) : \|x - y\| = d(x, T_i(x))\}, i = 1, 2.$$

Then for $x_1 \in E$,

$$\begin{cases} y_n = (1 - a_n - b_n)x_n + a_n w_n + b_n u_n, \ n \geq 1, \\ x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n z_n + \beta_n v_n, \ n \geq 1, \end{cases} \tag{1.4}$$

where $w_n \in P_{T_2}x_n$ and $z_n \in P_{T_1}y_n$, $\{a_n\}, \{b_n\}, \{\alpha_n\}, \{\beta_n\} \in [0,1]$ and $\{u_n\}, \{v_n\}$ are bounded sequences in E.

Very recently, Eslamian and Homaeipour [5] introduced a new three-step iterative process for multi-valued mappings in Banach spaces. And they proved some convergence theorems for multi-valued mappings in uniformly convex Banach spaces. Their iteration process is a generalization of Noor iteration process with errors as follows:

Let E be a nonempty convex subset of a Banach space X and $T_1, T_2, T_3 : E \longrightarrow CB(E)$ be three multi-valued mappings. Then for $x_1 \in E$,

$$\begin{cases} w_n = (1 - a_n - b_n)x_n + a_n z_n + b_n s_n, \ n \geq 1, \\ y_n = (1 - c_n - d_n)x_n + c_n u_n + d_n s_n^{'}, \ n \geq 1, \\ x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n v_n + \beta_n s_n^{''}, \ n \geq 1, \end{cases} \tag{1.5}$$

where $z_n \in T_1 x_n, u_n \in T_2 w_n$ and $v_n \in T_3 y_n$, $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\alpha_n\}, \{\beta_n\} \in [0,1]$ and $\{s_n\}, \left\{s_n^{'}\right\}$ and $\left\{s_n^{''}\right\}$ are bounded sequences in E.

Also, their iteration process contains the following iteration process.

Let E be a nonempty convex subset of a Banach space X and $T_1, T_2, T_3 : E \longrightarrow CB(E)$ be three multi-valued mappings. Then for $x_1 \in E$,

$$\begin{cases} w_n = (1 - a_n - b_n)x_n + a_n z_n + b_n s_n, \ n \geq 1, \\ y_n = (1 - c_n - d_n)x_n + c_n u_n + d_n s_n^{'}, \ n \geq 1, \\ x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n v_n + \beta_n s_n^{''}, \ n \geq 1, \end{cases} \tag{1.6}$$

where $z_n \in P_{T_1}x_n, u_n \in P_{T_2}w_n$ and $v_n \in P_{T_3}y_n$, $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\alpha_n\}, \{\beta_n\} \in [0,1]$ and $\{s_n\}, \left\{s_n^{'}\right\}$ and $\left\{s_n^{''}\right\}$ are bounded sequences in E.

Finding common fixed points of a finite family $\{T_i : i = 1, 2, ..., k\}$ of mappings acting on a Hilbert space is a problem that often arises in applied mathematics. In fact, many algorithms have been introduced for different classes of mappings with a nonempty set of common fixed points. Unfortunately, the existence results of common fixed points of a family of mappings are not known in many situations. Therefore, it is natural to consider approximation results for these classes of mappings.

2. Main Results

In this section, we use the following iteration processes.

(A): Let E be a nonempty convex subset of a Banach space X and $T_i : E \longrightarrow CB(E)$ $(i = 1, 2, ..., k)$ be a finite family of multi-valued mappings. Then for $x_1 \in E$,

$$\begin{cases} y_{1n} = (1 - \alpha_{1n} - \beta_{1n})\, x_n + \alpha_{1n} z_{n,1} + \beta_{1n} u_{1n},\ n \geq 1, \\ y_{2n} = (1 - \alpha_{2n} - \beta_{2n})\, x_n + \alpha_{2n} z_{n,2} + \beta_{2n} u_{2n},\ n \geq 1, \\ \vdots \\ y_{(k-1)n} = \left(1 - \alpha_{(k-1)n} - \beta_{(k-1)n}\right) x_n + \alpha_{(k-1)n} z_{n,k-1} + \beta_{(k-1)n} u_{(k-1)n},\ n \geq 1, \\ x_{n+1} = (1 - \alpha_{kn} - \beta_{kn})\, x_n + \alpha_{kn} z_{n,k} + \beta_{kn} u_{kn},\ n \geq 1, \end{cases}$$

where $z_{n,1} \in T_1(x_n)$ and $z_{n,i} \in T_i\left(y_{(i-1)n}\right)$ for $i = 2, 3, ..., k$ and $\{\alpha_{in}\}, \{\beta_{in}\} \in [0, 1]$ and $\{u_{in}\}$ are bounded sequences in E.

(B): Let E be a nonempty convex subset of a Banach space X and $T_i : E \longrightarrow CB(E)$ $(i = 1, 2, ..., k)$ be a finite family of multi-valued mappings and

$$P_{T_i}(x) = \{y \in T_i(x) : \|x - y\| = d(x, T_i(x))\}.$$

Then for $x_1 \in E$, we consider the following iterative process:

$$\begin{cases} y_{1n} = (1 - \alpha_{1n} - \beta_{1n})\, x_n + \alpha_{1n} z_{n,1} + \beta_{1n} u_{1n},\ n \geq 1, \\ y_{2n} = (1 - \alpha_{2n} - \beta_{2n})\, x_n + \alpha_{2n} z_{n,2} + \beta_{2n} u_{2n},\ n \geq 1, \\ \vdots \\ y_{(k-1)n} = \left(1 - \alpha_{(k-1)n} - \beta_{(k-1)n}\right) x_n + \alpha_{(k-1)n} z_{n,k-1} + \beta_{(k-1)n} u_{(k-1)n},\ n \geq 1, \\ x_{n+1} = (1 - \alpha_{kn} - \beta_{kn})\, x_n + \alpha_{kn} z_{n,k} + \beta_{kn} u_{kn},\ n \geq 1, \end{cases}$$

where $z_{n,1} \in P_{T_1}(x_n)$ and $z_{n,i} \in P_{T_i}\left(y_{(i-1)n}\right)$ for $i = 2, 3, ..., k$ and $\{\alpha_{in}\}, \{\beta_{in}\} \in [0, 1]$ and $\{u_{in}\}$ are bounded sequences in E.

Clearly, this iteration processes generalize the Mann iteration (1.1), the Ishikawa iteration (1.2), the Ishikawa iteration with errors (1.3)-(1.4) and the three-step iteration process (1.5)-(1.6) from one mapping to the finite family of mappings $\{T_i : i = 1, 2, ..., k\}$.

Definition 2.1. *A mapping $T : E \longrightarrow CB(E)$ is said to satisfy condition (I) if there is a nondecreasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0,\ g(r) > 0$ for all $r \in (0, \infty)$ such that*

$$d(x, Tx) \geq g(d(x, F(T))).$$

Let $T_i : E \longrightarrow CB(E)$ $(i = 1, 2, ..., k)$ be a finite family of mappings. The mappings T_i for all i $(i = 1, 2, ..., k)$ are said to satisfy condition (II) if there exist a nondecreasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0,\ g(r) > 0$ for all $r \in (0, \infty)$ such that

$$\sum_{i=1}^{k} d(x, T_i x) \geq g(d(x, \mathcal{F})),$$

where $\mathcal{F} = \cap_{i=1}^{k} F(T_i)$.

Throughout this paper, we denote the weak convergence and the strong convergence by $\rightharpoonup$ and $\to$, respectively.

Definition 2.2. *A Banach space E is said to satisfy Opial's condition* [8] *if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that*

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$.

Examples of Banach spaces satisfying this condition are Hilbert spaces and all l^p spaces $(1 < p < \infty)$. On the other hand, $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial's condition.

The mapping $T : E \longrightarrow CB(E)$ is called *demi-closed* if for every sequence $\{x_n\} \subset E$ and any $y_n \in Tx_n$ such that $x_n \rightharpoonup x$ and $y_n \to y$, we have $x \in E$ and $y \in Tx$.

We use the following lemmas to prove our main results.

Lemma 2.1. [14] *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequence of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n) a_n + b_n.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to 0, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2. [2] *Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous, strictly increasing and convex function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that*

$$\|\alpha x + \beta y + \gamma z\|^2 \leq \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha\beta\varphi(\|x - y\|),$$

for all $x, y, z \in B_r$ and all $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$.

Lemma 2.3. *Let E be a nonempty closed convex subset of a uniformly convex Banach space X and $T_i : E \longrightarrow CB(E)$, $(i = 1, 2, ..., k)$ be a finite family of quasi-nonexpansive multi-valued mappings. Assume that $\mathcal{F} = \cap_{i=1}^{k} F(T_i) \neq \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, ..., k)$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be the iterative process defined by (A) and $\sum_{n=1}^{\infty} \beta_{in} < \infty$ for each $i = 1, 2, ..., k$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in \mathcal{F}$.*

Proof. Let $p \in \mathcal{F}$. Since the sequences $\{u_{in}\}$ are bounded for $i = 1, 2, ..., k$, there exists $M > 0$ such that

$$\max\left\{\sup_{n\geq 1} \|u_{1n} - p\|, \sup_{n\geq 1} \|u_{2n} - p\|, ..., \sup_{n\geq 1} \|u_{kn} - p\|\right\} \leq M.$$

Using (A) and quasi-nonexpansiveness of T_i $(i = 1, 2, ..., k)$ we have

$$\begin{aligned}
\|y_{1n} - p\| &\leq (1 - \alpha_{1n} - \beta_{1n}) \|x_n - p\| + \alpha_{1n} \|z_{n,1} - p\| + \beta_{1n} \|u_{1n} - p\| \\
&\leq (1 - \alpha_{1n} - \beta_{1n}) \|x_n - p\| + \alpha_{1n} d(z_{n,1}, T_1(p)) + \beta_{1n} \|u_{1n} - p\| \\
&\leq (1 - \alpha_{1n} - \beta_{1n}) \|x_n - p\| + \alpha_{1n} H(T_1(x_n), T_1(p)) + \beta_{1n} \|u_{1n} - p\| \\
&\leq (1 - \alpha_{1n} - \beta_{1n}) \|x_n - p\| + \alpha_{1n} \|x_n - p\| + \beta_{1n} \|u_{1n} - p\| \\
&= (1 - \beta_{1n}) \|x_n - p\| + \beta_{1n} \|u_{1n} - p\| \\
&\leq \|x_n - p\| + \beta_{1n} M
\end{aligned}$$

and

$$\begin{aligned}
\|y_{2n} - p\| &\leq (1 - \alpha_{2n} - \beta_{2n}) \|x_n - p\| + \alpha_{2n} \|z_{n,2} - p\| + \beta_{2n} \|u_{2n} - p\| \\
&\leq (1 - \alpha_{2n} - \beta_{2n}) \|x_n - p\| + \alpha_{2n} d(z_{n,2}, T_2(p)) + \beta_{2n} \|u_{2n} - p\| \\
&\leq (1 - \alpha_{2n} - \beta_{2n}) \|x_n - p\| + \alpha_{2n} H(T_2(y_{1n}), T_2(p)) + \beta_{2n} \|u_{2n} - p\| \\
&\leq (1 - \alpha_{2n} - \beta_{2n}) \|x_n - p\| + \alpha_{2n} \|y_{1n} - p\| + \beta_{2n} \|u_{2n} - p\| \\
&\leq (1 - \alpha_{2n} - \beta_{2n}) \|x_n - p\| + \alpha_{2n}(\|x_n - p\| + \beta_{1n} M) + \beta_{2n} \|u_{2n} - p\| \\
&= (1 - \beta_{2n}) \|x_n - p\| + \alpha_{2n}\beta_{1n} M + \beta_{2n} M \\
&\leq \|x_n - p\| + (\beta_{1n} + \beta_{2n}) M.
\end{aligned}$$

Similarly, we have

$$\left\|y_{(k-1)n}-p\right\| \leq \|x_n-p\|+\left(\beta_{1n}+\beta_{2n}+...+\beta_{(k-1)n}\right)M,$$

and also

$$\begin{aligned}
\|x_{n+1}-p\| &\leq (1-\alpha_{kn}-\beta_{kn})\|x_n-p\|+\alpha_{kn}\|z_{n,i}-p\|+\beta_{kn}\|u_{kn}-p\| \\
&\leq (1-\alpha_{kn}-\beta_{kn})\|x_n-p\|+\alpha_{kn}d(z_{n,k},T_k(p))+\beta_{kn}\|u_{kn}-p\| \\
&\leq (1-\alpha_{kn}-\beta_{kn})\|x_n-p\|+\alpha_{kn}H\left(T_k\left(y_{(k-1)n}\right),T_k(p)\right)+\beta_{kn}\|u_{kn}-p\| \\
&\leq (1-\alpha_{kn}-\beta_{kn})\|x_n-p\|+\alpha_{kn}\left\|y_{(k-1)n}-p\right\|+\beta_{kn}\|u_{kn}-p\| \\
&\leq (1-\alpha_{kn}-\beta_{kn})\|x_n-p\|+\alpha_{kn}\left[\|x_n-p\|\right. \\
&\quad \left.+\left(\beta_{1n}+\beta_{2n}+...+\beta_{(k-1)n}\right)M\right]+\beta_{kn}\|u_{kn}-p\| \\
&\leq (1-\beta_{kn})\|x_n-p\|+\alpha_{kn}\left(\beta_{1n}+\beta_{2n}+...+\beta_{(k-1)n}\right)M+\beta_{kn}M \\
&\leq \|x_n-p\|+\left(\beta_{1n}+\beta_{2n}+...+\beta_{(k-1)n}+\beta_{kn}\right)M \\
&= \|x_n-p\|+\theta_n \qquad (2.1)
\end{aligned}$$

where $\theta_n = M\left(\beta_{1n}+\beta_{2n}+...+\beta_{(k-1)n}+\beta_{kn}\right)$. By assumption we have $\sum_{n=1}^{\infty}\theta_n < \infty$. Therefore by Lemma 2.1 it follows that $\lim_{n\to\infty}\|x_n-p\|$ exist for any $p \in \mathcal{F}$. □

Theorem 2.1. *Let E be a nonempty closed convex subset of a uniformly convex Banach space X and $T_i : E \longrightarrow CB(E)$, $(i = 1, 2, ..., k)$ be a finite family of quasi-nonexpansive and L-Lipschitzian multi-valued mappings. Assume that $\mathcal{F} = \cap_{i=1}^{k} F(T_i) \neq \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, ..., k)$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be the iterative process defined by (A), and $\alpha_{in} + \beta_{in} \in [a, b] \subset (0, 1)$ for $i = 1, 2, ..., k$ and $\sum_{n=1}^{\infty}\beta_{in} < \infty$ for each i. Assume that T_i $(i = 1, 2, ..., k)$ satisfying the condition (II). Then $\{x_n\}$ converges strongly to a common fixed point of T_i for $i = 1, 2, ..., k$.*

Proof. Let $p \in \mathcal{F}$. From Lemma 2.3, we know that the sequences $\{y_{1n}\}, \{y_{2n}\}, ..., \left\{y_{(k-1)n}\right\}$ and $\{x_{n+1}\}$ are bounded. Therefore, we can find $r > 0$ depending on p such that $y_{1n}-p, y_{2n}-p, ..., y_{(k-1)n}-p, x_{n+1}-p \in B_r(0)$ for all $n \geq 1$. As the proof of Lemma 2.3, there exists $N > 0$ such that

$$\max\left\{\sup_{n\geq 1}\|u_{1n}-p\|^2, \sup_{n\geq 1}\|u_{2n}-p\|^2, ..., \sup_{n\geq 1}\|u_{kn}-p\|^2\right\} \leq N.$$

From Lemma 2.2, we get

$$\begin{aligned}
\|y_{1n}-p\|^2 &\leq (1-\alpha_{1n}-\beta_{1n})\|x_n-p\|^2+\alpha_{1n}\|z_{n,1}-p\|^2+\beta_{1n}\|u_{1n}-p\|^2 \\
&\quad -\alpha_{1n}(1-\alpha_{1n}-\beta_{1n})\varphi(\|x_n-z_{n,1}\|) \\
&\leq (1-\alpha_{1n}-\beta_{1n})\|x_n-p\|^2+\alpha_{1n}d(z_{n,1},T_1(p))^2+\beta_{1n}\|u_{1n}-p\|^2 \\
&\quad -\alpha_{1n}(1-\alpha_{1n}-\beta_{1n})\varphi(\|x_n-z_{n,1}\|) \\
&\leq (1-\alpha_{1n}-\beta_{1n})\|x_n-p\|^2+\alpha_{1n}H(T_1(x_n),T_1(p))^2+\beta_{1n}\|u_{1n}-p\|^2 \\
&\quad -\alpha_{1n}(1-\alpha_{1n}-\beta_{1n})\varphi(\|x_n-z_{n,1}\|) \\
&\leq (1-\beta_{1n})\|x_n-p\|^2+\beta_{1n}\|u_{1n}-p\|^2-\alpha_{1n}(1-\alpha_{1n}-\beta_{1n})\varphi(\|x_n-z_{n,1}\|) \\
&\leq \|x_n-p\|^2+\beta_{1n}N-\alpha_{1n}(1-\alpha_{1n}-\beta_{1n})\varphi(\|x_n-z_{n,1}\|)
\end{aligned}$$

and

$$\begin{aligned}
\|y_{2n}-p\|^2 &\leq (1-\alpha_{2n}-\beta_{2n})\|x_n-p\|^2+\alpha_{2n}\|z_{n,2}-p\|^2+\beta_{2n}\|u_{2n}-p\|^2 \\
&\quad -\alpha_{2n}(1-\alpha_{2n}-\beta_{2n})\varphi(\|x_n-z_{n,2}\|) \\
&\leq (1-\alpha_{2n}-\beta_{2n})\|x_n-p\|^2+\alpha_{2n}d(z_{n,2},T_2(p))^2+\beta_{2n}\|u_{2n}-p\|^2 \\
&\quad -\alpha_{2n}(1-\alpha_{2n}-\beta_{2n})\varphi(\|x_n-z_{n,2}\|) \\
&\leq (1-\alpha_{2n}-\beta_{2n})\|x_n-p\|^2+\alpha_{2n}H(T_2(y_{1n}),T_2(p))^2+\beta_{2n}\|u_{2n}-p\|^2 \\
&\quad -\alpha_{2n}(1-\alpha_{2n}-\beta_{2n})\varphi(\|x_n-z_{n,2}\|) \\
&\leq (1-\alpha_{2n}-\beta_{2n})\|x_n-p\|^2+\alpha_{2n}\Big[\|x_n-p\|^2+\beta_{1n}N-\alpha_{1n}(1-\alpha_{1n}-\beta_{1n}) \\
&\quad \varphi(\|x_n-z_{n,1}\|)]+\beta_{2n}\|u_{2n}-p\|^2-\alpha_{2n}(1-\alpha_{2n}-\beta_{2n})\varphi(\|x_n-z_{n,2}\|) \\
&\leq (1-\beta_{2n})\|x_n-p\|^2+\alpha_{2n}\beta_{1n}N-\alpha_{1n}\alpha_{2n}(1-\alpha_{1n}-\beta_{1n})\varphi(\|x_n-z_{n,1}\|) \\
&\quad +\beta_{2n}N-\alpha_{2n}(1-\alpha_{2n}-\beta_{2n})\varphi(\|x_n-z_{n,2}\|) \\
&\leq \|x_n-p\|^2+(\beta_{1n}+\beta_{2n})N-\alpha_{1n}\alpha_{2n}(1-\alpha_{1n}-\beta_{1n})\varphi(\|x_n-z_{n,1}\|) \\
&\quad -\alpha_{2n}(1-\alpha_{2n}-\beta_{2n})\varphi(\|x_n-z_{n,2}\|).
\end{aligned}$$

It follows from Lemma 2.2 that

$$\begin{aligned}
\left\|y_{(k-1)n}-p\right\|^2 &\leq \|x_n-p\|^2+\left(\beta_{1n}+\beta_{2n}+...+\beta_{(k-1)n}\right)N-\alpha_{1n}\alpha_{2n}...\alpha_{(k-1)n}(1-\alpha_{1n}-\beta_{1n}) \\
&\quad \varphi(\|x_n-z_{n,1}\|)-\alpha_{2n}\alpha_{3n}...\alpha_{(k-1)n}(1-\alpha_{2n}-\beta_{2n})\varphi(\|x_n-z_{n,2}\|) \\
&\quad -...-\alpha_{(k-1)n}\left(1-\alpha_{(k-1)n}-\beta_{(k-1)n}\right)\varphi(\|x_n-z_{n,k-1}\|) \\
&\leq \|x_n-p\|^2+\left(\beta_{1n}+\beta_{2n}+...+\beta_{(k-1)n}\right)N \\
&\quad -\prod_{i=1}^{k-1}\alpha_{in}\left[\sum_{i=1}^{k-1}(1-\alpha_{in}-\beta_{in})\varphi(\|x_n-z_{n,i}\|)\right].
\end{aligned}$$

By another application Lemma 2.2 we obtain that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1-\alpha_{kn}-\beta_{kn})\|x_n-p\|^2+\alpha_{kn}\|z_{n,k}-p\|^2+\beta_{kn}\|u_{kn}-p\|^2 \\
&\quad -\alpha_{kn}(1-\alpha_{kn}-\beta_{kn})\varphi(\|x_n-z_{n,k}\|) \\
&\leq (1-\alpha_{kn}-\beta_{kn})\|x_n-p\|^2+\alpha_{kn}d(z_{n,k},T_k(p))^2+\beta_{kn}\|u_{kn}-p\|^2 \\
&\quad -\alpha_{kn}(1-\alpha_{kn}-\beta_{kn})\varphi(\|x_n-z_{n,k}\|) \\
&\leq (1-\alpha_{kn}-\beta_{kn})\|x_n-p\|^2+\alpha_{kn}H\left(T_k\left(y_{(k-1)n}\right),T_k(p)\right)^2+\beta_{kn}\|u_{kn}-p\|^2 \\
&\quad -\alpha_{kn}(1-\alpha_{kn}-\beta_{kn})\varphi(\|x_n-z_{n,k}\|) \\
&\leq (1-\alpha_{kn}-\beta_{kn})\|x_n-p\|^2+\alpha_{kn}\left\|y_{(k-1)n}-p\right\|^2+\beta_{kn}N \\
&\quad -\alpha_{kn}(1-\alpha_{kn}-\beta_{kn})\varphi(\|x_n-z_{n,k}\|) \\
&\leq \|x_n-p\|^2+\left(\beta_{1n}+\beta_{2n}+...+\beta_{(k-1)n}+\beta_{kn}\right)N \\
&\quad -\prod_{i=1}^{k}\alpha_{in}\left[\sum_{i=1}^{k}(1-\alpha_{in}-\beta_{in})\varphi(\|x_n-z_{n,i}\|)\right].
\end{aligned}$$

Since $\alpha_{in}+\beta_{in}\in[a,b]\subset(0,1)$ for $i=1,2,...,k$, we have

$$\begin{aligned} a^k\sum_{i=1}^{k}(1-b)\varphi(\|x_n-z_{n,i}\|) &\leq \prod_{i=1}^{k}\alpha_{in}\left[\sum_{i=1}^{k}(1-\alpha_{in}-\beta_{in})\varphi(\|x_n-z_{n,i}\|)\right] \\ &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\left(\beta_{1n}+\beta_{2n}+...+\beta_{(k-1)n}+\beta_{kn}\right)N. \end{aligned}$$

This implies that

$$\sum_{n=1}^{\infty}\left[a^k\sum_{i=1}^{k}(1-b)\varphi(\|x_n-z_{n,i}\|)\right]\leq\|x_1-p\|^2+\sum_{n=1}^{\infty}\left(\beta_{1n}+\beta_{2n}+...+\beta_{(k-1)n}+\beta_{kn}\right)N<\infty$$

from which it follows that $\lim_{n\to\infty}\varphi(\|x_n-z_{n,i}\|)=0$. Since φ is continuous at 0 and is strictly increasing, we have

$$\lim_{n\to\infty}\|x_n-z_{n,i}\|=0.$$

Hence for $i=1,2,...,k$, we have

$$\lim_{n\to\infty}\|x_n-z_{n,1}\|=\lim_{n\to\infty}\|x_n-z_{n,2}\|=...=\lim_{n\to\infty}\|x_n-z_{n,k}\|=0. \tag{2.2}$$

Also, using (A), (2.2) and $\sum_{n=1}^{\infty}\beta_{in}<\infty$ for each i, we have

$$\begin{aligned} \lim_{n\to\infty}\|y_{1n}-x_n\| &= \lim_{n\to\infty}(\alpha_{1n}\|z_{n,1}-x_n\|+\beta_{1n}\|u_{1n}-x_n\|)=0, \quad (2.3)\\ \lim_{n\to\infty}\|y_{2n}-x_n\| &= \lim_{n\to\infty}(\alpha_{2n}\|z_{n,2}-x_n\|+\beta_{2n}\|u_{2n}-x_n\|)=0, \\ &\vdots \\ \lim_{n\to\infty}\|x_{n+1}-x_n\| &= \lim_{n\to\infty}(\alpha_{kn}\|z_{n,k}-x_n\|+\beta_{kn}\|u_{kn}-x_n\|)=0. \end{aligned}$$

Therefore from (2.2) and (2.3) we have

$$\begin{aligned} d(x_n,T_1(x_n)) &\leq d(x_n,T_1(x_n))+H(T_1(x_n),T_1(x_n)) \\ &\leq d(x_n,T_1(x_n))+L\|x_n-x_n\| \\ &\leq \|x_n-z_{n,1}\|\to 0 \text{ as } n\to\infty, \end{aligned}$$

and

$$\begin{aligned} d(x_n,T_2(x_n)) &\leq d(x_n,T_2(y_{1n}))+H(T_2(y_{1n}),T_2(x_n)) \\ &\leq d(x_n,T_2(y_{1n}))+L\|y_{1n}-x_n\| \\ &\leq L\|y_{1n}-x_n\|+\|x_n-z_{n,2}\|\to 0 \text{ as } n\to\infty. \end{aligned}$$

In a similar way, we can show that $d(x_n,T_i(x_n))\to 0$ as $n\to\infty$ for $i=1,2,...,k$. From $\lim_{n\to\infty}d(x_n,T_i(x_n))=0$ and condition (II), we obtain that $\lim_{n\to\infty}d(x_n,\mathcal{F})=0$. Therefore, we can choose a sequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence p_j in $\mathcal{F}$ such that for all $j\in\mathbb{N}$

$$\|x_{n_j}-p_j\|<\frac{1}{2^j}.$$

Therefore by inequality (2.1) we get

$$\begin{aligned}\left\|x_{n_{j+1}}-p\right\| &\leq \left\|x_{n_{j+1}-1}-p\right\|+\theta_{n_{j+1}-1} \\ &\leq \left\|x_{n_{j+1}-2}-p\right\|+\theta_{n_{j+1}-2}+\theta_{n_{j+1}-1} \\ &\vdots \\ &\leq \left\|x_{n_j}-p\right\|+\sum_{l=1}^{n_{j+1}-n_j-1}\theta_{n_j+l}\end{aligned}$$

for all $p \in \mathcal{F}$. This implies that

$$\begin{aligned}\left\|x_{n_{j+1}}-p\right\| &\leq \left\|x_{n_j}-p_j\right\|+\sum_{l=1}^{n_{j+1}-n_j-1}\theta_{n_j+l} \\ &< \frac{1}{2^j}+\sum_{l=1}^{n_{j+1}-n_j-1}\theta_{n_j+l}.\end{aligned}$$

Now, we show that $\{p_j\}$ is Cauchy sequence in E. Note that

$$\begin{aligned}\|p_{j+1}-p_j\| &\leq \left\|p_{j+1}-x_{n_{j+1}}\right\|+\left\|x_{n_{j+1}}-p_j\right\| \\ &< \frac{1}{2^{j+1}}+\frac{1}{2^j}+\sum_{l=1}^{n_{j+1}-n_j-1}\theta_{n_j+l} \\ &< \frac{1}{2^{j-1}}+\sum_{l=1}^{n_{j+1}-n_j-1}\theta_{n_j+l}.\end{aligned}$$

Consequently, we conclude that $\{p_j\}$ is Cauchy sequence in E and hence converges to $q \in E$. Since for $i = 1, 2, ..., k$

$$d\left(p_j, T_i\left(q\right)\right) \leq H\left(T_i\left(p_j\right), T_i\left(q\right)\right) \leq \|p_j - q\|$$

and $p_j \to q$ as $j \to \infty$, it follows that $d\left(q, T_i\left(q\right)\right) = 0$ for $i = 1, 2, ..., k$. Hence $q \in \mathcal{F}$ and $\left\{x_{n_j}\right\}$ converges strongly to q. Since $\lim_{n\to\infty} \|x_n - q\|$ exists, we conclude that $\{x_n\}$ converges strongly to q. □

Theorem 2.2. *Let E be a nonempty closed convex subset of a uniformly convex Banach space X and $T_i : E \longrightarrow CB(E)$, $(i = 1, 2, ..., k)$ be a finite family of quasi-nonexpansive and L-Lipschitzian multi-valued mappings. Assume that $\mathcal{F} = \cap_{i=1}^{k} F\left(T_i\right) \neq \emptyset$ and $T_i\left(p\right) = \{p\}, (i = 1, 2, ..., k)$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be the iterative process defined by (A), and $\alpha_{in} + \beta_{in} \in [a, b] \subset (0, 1)$ for $i = 1, 2, ..., k$ and $\sum_{n=1}^{\infty} \beta_{in} < \infty$ for each i. Assume that one of the multi-valued mappings $\{T_i : i = 1, 2, ..., k\}$ is hemi-compact. Then $\{x_n\}$ converges strongly to a common fixed point of T_i for $i = 1, 2, ..., k$.*

Proof. From the proof of Theorem 2.1, we know that $\lim_{n\to\infty} d\left(x_n, T_i\left(x_n\right)\right) = 0$ for each i. We suppose that one of the multi-valued mappings $\{T_i : i = 1, 2, ..., k\}$ is hemi-compact. Then there exists a subsequence $\left\{x_{n_j}\right\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} x_{n_j} = z$ for some $z \in E$. For $i = 1, 2, ..., k$ we have

$$\begin{aligned}d\left(z, T_i\left(z\right)\right) &\leq \left\|z-x_{n_j}\right\|+d\left(x_{n_j}, T_i\left(z\right)\right) \\ &\leq \left\|z-x_{n_j}\right\|+d\left(x_{n_j}, T_i\left(x_{n_j}\right)\right)+H\left(T_i\left(x_{n_j}\right), T_i\left(z\right)\right) \\ &\leq d\left(x_{n_j}, T_i\left(x_{n_j}\right)\right)+(1+L)\left\|z-x_{n_j}\right\| \to 0 \text{ as } j\to\infty,\end{aligned}$$

this implies that $z \in \mathcal{F}$. Since $\{x_{n_j}\}$ converges strongly to z and the limit $\lim_{n\to\infty}\|x_n - z\|$ exists (as in the proof Theorem 2.1), this implies that $\{x_n\}$ converges strongly to z. □

Theorem 2.3. *Let E be a nonempty closed convex subset of a uniformly convex Banach space X with the Opial property and $T_i : E \longrightarrow CB(E)$, $(i = 1, 2, ..., k)$ be a finite family of quasi-nonexpansive and L-Lipschitzian multi-valued mappings. Let $\{x_n\}$ be the iterative process defined by (A), and $\alpha_{in} + \beta_{in} \in [a, b] \subset (0, 1)$ for $i = 1, 2, ..., k$ and $\sum_{n=1}^{\infty} \beta_{in} < \infty$ for each i. Assume that $\mathcal{F} = \cap_{i=1}^{k} F(T_i) \neq \emptyset$, $T_i(p) = \{p\}, (i = 1, 2, ..., k)$ for each $p \in \mathcal{F}$ and $I - T_i$ is demiclosed with respect to zero for each $i = 1, 2, ..., k$. Then $\{x_n\}$ converges weakly to a common fixed point of T_i for $i = 1, 2, ..., k$.*

Proof. Let $p \in \mathcal{F}$. From the proof of Lemma 2.3, $\lim_{n\to\infty}\|x_n - p\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in $\mathcal{F}$. To prove this, let q and w be weak limits of the subsequences $\{x_{n_j}\}$ and $\{x_{n_m}\}$ of $\{x_n\}$, respectively. By Theorem 2.1, there exists $z_{n,i} \in T_i\left(y_{(i-1)n}\right)$ such that $\lim_{n\to\infty}\|x_n - z_{n,i}\| = 0$ and $I - T_i$ is demiclosed with respect to zero for each $i = 1, 2, ..., k$, therefore we obtain $q \in T_i q$ for each i. That is, $q \in \mathcal{F}$. Again in the same way, we can prove that $w \in \mathcal{F}$.

Next, we prove uniqueness. For this, suppose that $q \neq w$. Then by the Opial property of Banach space X, we have

$$\begin{aligned}
\lim_{n\to\infty}\|x_n - q\| &= \lim_{j\to\infty}\left\|x_{n_j} - q\right\| \\
&< \lim_{j\to\infty}\left\|x_{n_j} - w\right\| = \lim_{n\to\infty}\|x_n - w\| \\
&= \lim_{m\to\infty}\|x_{n_m} - w\| < \lim_{m\to\infty}\|x_{n_m} - q\| \\
&= \lim_{n\to\infty}\|x_n - q\|
\end{aligned}$$

which is a contradiction. Therefore $\{x_n\}$ converges weakly to a point in $\mathcal{F}$. □

The compactness assumption is quite strong, since it is easy to find a sequence in the domain which converges to a fixed point of the mapping. Therefore, we give the following result.

Corollary 2.1. *Let E be a nonempty closed convex subset of a uniformly convex Banach space X with the Opial property and $T_i : E \longrightarrow K(E)$, $(i = 1, 2, ..., k)$ be a finite family of quasi-nonexpansive and L-Lipschitzian multi-valued mappings. Assume that $\mathcal{F} = \cap_{i=1}^{k} F(T_i) \neq \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, ..., k)$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be the iterative process defined by (A), and $\alpha_{in} + \beta_{in} \in [a, b] \subset (0, 1)$ for $i = 1, 2, ..., k$ and $\sum_{n=1}^{\infty} \beta_{in} < \infty$ for each i. Then $\{x_n\}$ converges weakly to a common fixed point of T_i for $i = 1, 2, ..., k$.*

Now, we will remove the restriction that $T_i(p) = p$ for each $p \in \mathcal{F}$. Then we have the following result.

Theorem 2.4. *Let E be a nonempty closed convex subset of a uniformly convex Banach space X and $T_i : E \longrightarrow P(E)$, $(i = 1, 2, ..., k)$ be a finite family of multi-valued mappings such that P_{T_i} is quasi-nonexpansive and L-Lipschitzian for $i = 1, 2, ..., k$. Let $\{x_n\}$ be the iterative process defined by (B), and $\alpha_{in} + \beta_{in} \in [a, b] \subset (0, 1)$ for*

$i = 1, 2, ..., k$ and $\sum_{n=1}^{\infty} \beta_{in} < \infty$ for each i. Assume that T_i $(i = 1, 2, ..., k)$ satisfying the condition (II) and $\mathcal{F} = \cap_{i=1}^{k} F(T_i) \neq \emptyset$. Then $\{x_n\}$ converges strongly to a common fixed point of T_i for $i = 1, 2, ..., k$.

Proof. Firstly, we show that $\lim_{n\to\infty} \|x_n - p\|$ exist for any $p \in \mathcal{F}$. Let $p \in \mathcal{F}$. Then, for $i = 1, 2, ..., k$ we have $p \in P_{T_i}(p) = \{p\}$. Since $\{u_{in}\}$ are bounded for each i, there exists $M > 0$ such that

$$\max\left\{\sup_{n\geq 1}\|u_{1n} - p\|, \sup_{n\geq 1}\|u_{2n} - p\|, ..., \sup_{n\geq 1}\|u_{kn} - p\|\right\} \leq M.$$

From (A) and quasi-nonexpansiveness of T_i $(i = 1, 2, ..., k)$, we get

$$\begin{aligned}
\|y_{1n} - p\| &\leq (1 - \alpha_{1n} - \beta_{1n})\|x_n - p\| + \alpha_{1n}\|z_{n,1} - p\| + \beta_{1n}\|u_{1n} - p\| \\
&\leq (1 - \alpha_{1n} - \beta_{1n})\|x_n - p\| + \alpha_{1n} d(z_{n,1}, P_{T_1}(p)) + \beta_{1n}\|u_{1n} - p\| \\
&\leq (1 - \alpha_{1n} - \beta_{1n})\|x_n - p\| + \alpha_{1n} H(P_{T_1}(x_n), P_{T_1}(p)) + \beta_{1n}\|u_{1n} - p\| \\
&\leq (1 - \alpha_{1n} - \beta_{1n})\|x_n - p\| + \alpha_{1n}\|x_n - p\| + \beta_{1n}\|u_{1n} - p\| \\
&= (1 - \beta_{1n})\|x_n - p\| + \beta_{1n}\|u_{1n} - p\| \\
&\leq \|x_n - p\| + \beta_{1n} M
\end{aligned}$$

and

$$\begin{aligned}
\|y_{2n} - p\| &\leq (1 - \alpha_{2n} - \beta_{2n})\|x_n - p\| + \alpha_{2n}\|z_{n,2} - p\| + \beta_{2n}\|u_{2n} - p\| \\
&\leq (1 - \alpha_{2n} - \beta_{2n})\|x_n - p\| + \alpha_{2n} d(z_{n,2}, P_{T_2}(p)) + \beta_{2n}\|u_{2n} - p\| \\
&\leq (1 - \alpha_{2n} - \beta_{2n})\|x_n - p\| + \alpha_{2n} H(P_{T_2}(y_{1n}), P_{T_2}(p)) + \beta_{2n}\|u_{2n} - p\| \\
&\leq (1 - \alpha_{2n} - \beta_{2n})\|x_n - p\| + \alpha_{2n}\|y_{1n} - p\| + \beta_{2n}\|u_{2n} - p\| \\
&\leq (1 - \alpha_{2n} - \beta_{2n})\|x_n - p\| + \alpha_{2n}(\|x_n - p\| + \beta_{1n} M) + \beta_{2n}\|u_{2n} - p\| \\
&\leq (1 - \beta_{2n})\|x_n - p\| + \alpha_{2n}\beta_{1n} M + \beta_{2n} M \\
&\leq \|x_n - p\| + (\beta_{1n} + \beta_{2n}) M.
\end{aligned}$$

Similarly, we get

$$\left\|y_{(k-1)n} - p\right\| \leq \|x_n - p\| + \left(\beta_{1n} + \beta_{2n} + ... + \beta_{(k-1)n}\right) M,$$

and also

$$\begin{aligned}
\|x_{n+1} - p\| &\leq (1 - \alpha_{kn} - \beta_{kn})\|x_n - p\| + \alpha_{kn}\|z_{n,i} - p\| + \beta_{kn}\|u_{kn} - p\| \\
&\leq (1 - \alpha_{kn} - \beta_{kn})\|x_n - p\| + \alpha_{kn} d(z_{n,k}, P_{T_k}(p)) + \beta_{kn}\|u_{kn} - p\| \\
&\leq (1 - \alpha_{kn} - \beta_{kn})\|x_n - p\| + \alpha_{kn} H\left(P_{T_k}\left(y_{(k-1)n}\right), P_{T_k}(p)\right) + \beta_{kn}\|u_{kn} - p\| \\
&\leq (1 - \alpha_{kn} - \beta_{kn})\|x_n - p\| + \alpha_{kn}\left\|y_{(k-1)n} - p\right\| + \beta_{kn}\|u_{kn} - p\| \\
&\leq (1 - \alpha_{kn} - \beta_{kn})\|x_n - p\| + \alpha_{kn}\left[\|x_n - p\|\right. \\
&\quad \left. + \left(\beta_{1n} + \beta_{2n} + ... + \beta_{(k-1)n}\right) M\right] + \beta_{kn}\|u_{kn} - p\| \\
&\leq (1 - \beta_{kn})\|x_n - p\| + \alpha_{kn}\left(\beta_{1n} + \beta_{2n} + ... + \beta_{(k-1)n}\right) M + \beta_{kn} M \\
&\leq \|x_n - p\| + \left(\beta_{1n} + \beta_{2n} + ... + \beta_{(k-1)n} + \beta_{kn}\right) M \\
&= \|x_n - p\| + \theta_n
\end{aligned}$$

where $\theta_n = M\left(\beta_{1n} + \beta_{2n} + ... + \beta_{(k-1)n} + \beta_{kn}\right)$. Since $\sum_{n=1}^{\infty} \beta_{in} < \infty$ for each i, we have $\sum_{n=1}^{\infty} \theta_n < \infty$. Thus by Lemma 2.1, $\lim_{n\to\infty} \|x_n - p\|$ exist for any $p \in \mathcal{F}$. Since the sequences $\{y_{1n}\}, \{y_{2n}\}, ..., \left\{y_{(k-1)n}\right\}$ and $\{x_{n+1}\}$ are bounded, we can find $r > 0$ depending on p such that $y_{1n} - p, y_{2n} - p, ..., y_{(k-1)n} - p, x_{n+1} - p \in B_r(0)$

for all $n \geq 1$. Also, since $\{u_{in}\}$ are bounded for each i, there exists $N > 0$ such that

$$\max\left\{\sup_{n\geq 1}\|u_{1n}-p\|^2, \sup_{n\geq 1}\|u_{2n}-p\|^2, ..., \sup_{n\geq 1}\|u_{kn}-p\|^2\right\} \leq N.$$

From Lemma 2.2, we have

$$\begin{aligned}
\|y_{1n}-p\|^2 &\leq (1-\alpha_{1n}-\beta_{1n})\|x_n-p\|^2+\alpha_{1n}\|z_{n,1}-p\|^2+\beta_{1n}\|u_{1n}-p\|^2 \\
&\quad -\alpha_{1n}(1-\alpha_{1n}-\beta_{1n})\varphi(\|x_n-z_{n,1}\|) \\
&\leq (1-\alpha_{1n}-\beta_{1n})\|x_n-p\|^2+\alpha_{1n}d(z_{n,1},P_{T_1}(p))^2+\beta_{1n}\|u_{1n}-p\|^2 \\
&\quad -\alpha_{1n}(1-\alpha_{1n}-\beta_{1n})\varphi(\|x_n-z_{n,1}\|) \\
&\leq (1-\alpha_{1n}-\beta_{1n})\|x_n-p\|^2+\alpha_{1n}H(P_{T_1}(x_n),P_{T_1}(p))^2+\beta_{1n}\|u_{1n}-p\|^2 \\
&\quad -\alpha_{1n}(1-\alpha_{1n}-\beta_{1n})\varphi(\|x_n-z_{n,1}\|) \\
&\leq (1-\beta_{1n})\|x_n-p\|^2+\beta_{1n}\|u_{1n}-p\|^2-\alpha_{1n}(1-\alpha_{1n}-\beta_{1n})\varphi(\|x_n-z_{n,1}\|) \\
&\leq \|x_n-p\|^2+\beta_{1n}N-\alpha_{1n}(1-\alpha_{1n}-\beta_{1n})\varphi(\|x_n-z_{n,1}\|)
\end{aligned}$$

and

$$\begin{aligned}
\|y_{2n}-p\|^2 &\leq (1-\alpha_{2n}-\beta_{2n})\|x_n-p\|^2+\alpha_{2n}\|z_{n,2}-p\|^2+\beta_{2n}\|u_{2n}-p\|^2 \\
&\quad -\alpha_{2n}(1-\alpha_{2n}-\beta_{2n})\varphi(\|x_n-z_{n,2}\|) \\
&\leq (1-\alpha_{2n}-\beta_{2n})\|x_n-p\|^2+\alpha_{2n}d(z_{n,2},P_{T_2}(p))^2+\beta_{2n}\|u_{2n}-p\|^2 \\
&\quad -\alpha_{2n}(1-\alpha_{2n}-\beta_{2n})\varphi(\|x_n-z_{n,2}\|) \\
&\leq (1-\alpha_{2n}-\beta_{2n})\|x_n-p\|^2+\alpha_{2n}H(P_{T_2}(y_{1n}),P_{T_2}(p))^2+\beta_{2n}\|u_{2n}-p\|^2 \\
&\quad -\alpha_{2n}(1-\alpha_{2n}-\beta_{2n})\varphi(\|x_n-z_{n,2}\|) \\
&\leq (1-\alpha_{2n}-\beta_{2n})\|x_n-p\|^2+\alpha_{2n}\Big[\|x_n-p\|^2+\beta_{1n}N-\alpha_{1n}(1-\alpha_{1n}-\beta_{1n}) \\
&\quad \varphi(\|x_n-z_{n,1}\|)]+\beta_{2n}\|u_{2n}-p\|^2-\alpha_{2n}(1-\alpha_{2n}-\beta_{2n})\varphi(\|x_n-z_{n,2}\|) \\
&\leq (1-\beta_{2n})\|x_n-p\|^2+\alpha_{2n}\beta_{1n}N-\alpha_{1n}\alpha_{2n}(1-\alpha_{1n}-\beta_{1n})\varphi(\|x_n-z_{n,1}\|) \\
&\quad +\beta_{2n}N-\alpha_{2n}(1-\alpha_{2n}-\beta_{2n})\varphi(\|x_n-z_{n,2}\|) \\
&\leq \|x_n-p\|^2+(\beta_{1n}+\beta_{2n})N-\alpha_{1n}\alpha_{2n}(1-\alpha_{1n}-\beta_{1n})\varphi(\|x_n-z_{n,1}\|) \\
&\quad -\alpha_{2n}(1-\alpha_{2n}-\beta_{2n})\varphi(\|x_n-z_{n,2}\|).
\end{aligned}$$

Using Lemma 2.2, we obtain that

$$\begin{aligned}
\left\|y_{(k-1)n}-p\right\|^2 &\leq \|x_n-p\|^2+\left(\beta_{1n}+\beta_{2n}+...+\beta_{(k-1)n}\right)N-\alpha_{1n}\alpha_{2n}...\alpha_{(k-1)n}(1-\alpha_{1n}-\beta_{1n}) \\
&\quad \varphi(\|x_n-z_{n,1}\|)-\alpha_{2n}\alpha_{3n}...\alpha_{(k-1)n}(1-\alpha_{2n}-\beta_{2n})\varphi(\|x_n-z_{n,2}\|) \\
&\quad -...-\alpha_{(k-1)n}\left(1-\alpha_{(k-1)n}-\beta_{(k-1)n}\right)\varphi(\|x_n-z_{n,k-1}\|) \\
&\leq \|x_n-p\|^2+\left(\beta_{1n}+\beta_{2n}+...+\beta_{(k-1)n}\right)N \\
&\quad -\prod_{i=1}^{k-1}\alpha_{in}\left[\sum_{i=1}^{k-1}(1-\alpha_{in}-\beta_{in})\varphi(\|x_n-z_{n,i}\|)\right].
\end{aligned}$$

Again, we apply Lemma 2.2 to conclude that

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1-\alpha_{kn}-\beta_{kn})\|x_n-p\|^2+\alpha_{kn}\|z_{n,k}-p\|^2+\beta_{kn}\|u_{kn}-p\|^2 \\
&\quad -\alpha_{kn}(1-\alpha_{kn}-\beta_{kn})\varphi(\|x_n-z_{n,k}\|) \\
&\leq (1-\alpha_{kn}-\beta_{kn})\|x_n-p\|^2+\alpha_{kn}d(z_{n,k},P_{T_k}(p))^2+\beta_{kn}\|u_{kn}-p\|^2 \\
&\quad -\alpha_{kn}(1-\alpha_{kn}-\beta_{kn})\varphi(\|x_n-z_{n,k}\|) \\
&\leq (1-\alpha_{kn}-\beta_{kn})\|x_n-p\|^2+\alpha_{kn}H\left(P_{T_k}\left(y_{(k-1)n}\right),P_{T_k}(p)\right)^2+\beta_{kn}\|u_{kn}-p\|^2 \\
&\quad -\alpha_{kn}(1-\alpha_{kn}-\beta_{kn})\varphi(\|x_n-z_{n,k}\|) \\
&\leq (1-\alpha_{kn}-\beta_{kn})\|x_n-p\|^2+\alpha_{kn}\left\|y_{(k-1)n}-p\right\|^2+\beta_{kn}N \\
&\quad -\alpha_{kn}(1-\alpha_{kn}-\beta_{kn})\varphi(\|x_n-z_{n,k}\|) \\
&\leq \|x_n-p\|^2+\left(\beta_{1n}+\beta_{2n}+...+\beta_{(k-1)n}+\beta_{kn}\right)N \\
&\quad -\prod_{i=1}^{k}\alpha_{in}\left[\sum_{i=1}^{k}(1-\alpha_{in}-\beta_{in})\varphi(\|x_n-z_{n,i}\|)\right].
\end{aligned}
$$

Since $\alpha_{in}+\beta_{in}\in[a,b]\subset(0,1)$ for $i=1,2,...,k$, we have

$$
\begin{aligned}
a^k\sum_{i=1}^{k}(1-b)\varphi(\|x_n-z_{n,i}\|) &\leq \prod_{i=1}^{k}\alpha_{in}\left[\sum_{i=1}^{k}(1-\alpha_{in}-\beta_{in})\varphi(\|x_n-z_{n,i}\|)\right] \\
&\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\left(\beta_{1n}+\beta_{2n}+...+\beta_{(k-1)n}+\beta_{kn}\right)N.
\end{aligned}
$$

This implies that

$$
\sum_{n=1}^{\infty}\left[a^k\sum_{i=1}^{k}(1-b)\varphi(\|x_n-z_{n,i}\|)\right]\leq\|x_1-p\|^2+\sum_{n=1}^{\infty}\left(\beta_{1n}+\beta_{2n}+...+\beta_{(k-1)n}+\beta_{kn}\right)N<\infty
$$

and hence $\lim_{n\to\infty}\varphi(\|x_n-z_{n,i}\|)=0$. Since φ is continuous at 0 and is strictly increasing, we have

$$
\lim_{n\to\infty}\|x_n-z_{n,i}\|=0.
$$

As in the proof Theorem 2.1, we obtain that $\lim_{n\to\infty}d(x_n,\mathcal{F})=0$. Therefore, we can choose a sequence $\left\{x_{n_j}\right\}$ of $\{x_n\}$ and a sequence p_j in $\mathcal{F}$ such that for all $j\in\mathbb{N}$

$$
\left\|x_{n_j}-p_j\right\|<\frac{1}{2^j}.
$$

As in the proof Theorem 2.1, $\{p_j\}$ is Cauchy sequence in E and hence converges to $q\in E$. Since for $i=1,2,...,k$

$$
d(p_j,T_i(q))\leq d(p_j,P_{T_i}(q))\leq H(P_{T_i}(p_j),P_{T_i}(q))\leq\|p_j-q\|,
$$

and $p_j\to q$ as $j\to\infty$, it follows that $d(q,T_i(q))=0$ for $i=1,2,...,k$. Hence $q\in\mathcal{F}$ and $\left\{x_{n_j}\right\}$ converges strongly to q. Since $\lim_{n\to\infty}\|x_n-q\|$ exists, we conclude that $\{x_n\}$ converges strongly to q. □

Finally, using Theorem 2.4, we will give the following results.

Corollary 2.2. *Let E be a nonempty closed convex subset of a uniformly convex Banach space X and $T_i:E\longrightarrow P(E)$, $(i=1,2,...,k)$ be a finite family of multi-valued mappings such that P_{T_i} is quasi-nonexpansive and L-Lipschitzian for $i=1,2,...,k$. Let $\{x_n\}$ be the iterative process defined by (B), and* $\alpha_{in}+\beta_{in}\in[a,b]\subset$

$(0,1)$ for $i=1,2,...,k$ and $\sum_{n=1}^{\infty}\beta_{in}<\infty$ for each i. Assume that one of the multi-valued mappings $\{T_i : i=1,2,...,k\}$ is hemi-compact and $\mathcal{F}=\cap_{i=1}^{k}F(T_i)\neq\emptyset$. Then $\{x_n\}$ converges strongly to a common fixed point of T_i for $i=1,2,...,k$.

Corollary 2.3. *Let E be a nonempty closed convex subset of a uniformly convex Banach space X and $T_i : E \longrightarrow P(E)$, $(i=1,2,...,k)$ be a finite family of multi-valued mappings such that P_{T_i} is quasi-nonexpansive and L-Lipschitzian for $i=1,2,...,k$. Let $\{x_n\}$ be the iterative process defined by (B), and $\alpha_{in}+\beta_{in}\in[a,b]\subset(0,1)$ for $i=1,2,...,k$ and $\sum_{n=1}^{\infty}\beta_{in}<\infty$ for each i. Assume that $I-T_i$ is demi-closed at 0 for each $i=1,2,...,k$ and $\mathcal{F}=\cap_{i=1}^{k}F(T_i)\neq\emptyset$. Then $\{x_n\}$ converges weakly to a common fixed point of T_i for $i=1,2,...,k$.*

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