Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 5 Issue 4 (2013), Pages 65-77.

EXTENDED GENERALIZED HYPERGEOMETRIC FUNCTIONS AND THEIR APPLICATIONS

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Abstract. By means of the extended beta function $\mathcal{B}_{b,d}^{(\kappa_l)}$, we introduce new extensions of the generalized hypergeometric functions and present some new integral and series representations (including the one obtained by adopting the well-known Ramanujan's Master Theorem). Further generalizations are also considered. We point out the usefulness of some of the the results by showing their connections with other special functions and with a class of fractional calculus operators.

1. Introduction and Prelimiaries

New extensions of some of the well-known special functions (e.g. gamma function, beta function, Gauss hypergeometric function, etc.) have been extensively studied in recent past. By inserting a regularization factor $e^{-bt^{-1}}$, Chaudhry and Zubair [3] have introduced the following extension of the gamma function:

$$\Gamma_b(x) = \int_0^\infty t^{x-1} \exp\left(-t - \frac{b}{t}\right) \mathrm{d}t, \ \Re(b) > 0, \tag{1.1}$$

and Chaudhry et al. [4] considered the extension of Euler's beta function in the following form:

$$B_b(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} \exp\left(-\frac{b}{t(1-t)}\right) \mathrm{d}t, \ \Re(b) > 0. \tag{1.2}$$

Later, Chaudhry et al. [5] used $B_b(x,y)$ to extend the Gauss hypergeometric function given by

$$F_b(\alpha,\beta;\gamma;z) = \sum_{m=0}^\infty (\alpha)_m \frac{B_b(\beta+m,\gamma-\beta)}{B(\beta,\gamma-\beta)} \frac{z^m}{m!}, \tag{1.3}$$

$$(b \geq 0; \ |z| < 1; \ \Re(\gamma) > \Re(\beta) > 0)$$

2010 *Mathematics Subject Classification.* 33A65, 33C90.

Key words and phrases. Extended beta function, Generalized hypergeometric functions, Ramanujan's Master Theorem.

Submitted July 2, 2013. Published November 4, 2013.

where $(\alpha)_m$ denotes the Pochhammer symbol defined in terms of gamma functions by

$$(\alpha)_m = \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)} = \begin{cases} 1 & m=0;\ \alpha\in\mathbb{C}\setminus\{0\} \\ \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+m-1) & m\in\mathbb{N};\ \alpha\in\mathbb{C}. \end{cases}$$

For $b=0$, the function (1.3) evidently reduces to the usual Gauss hypergeometric function.

Subsequently, Ozergin et al. [12] introduced generalizations of the gamma and Euler's beta functions given by

$$\Gamma_b^{(\alpha,\beta)}(x) = \int_0^\infty t^{x-1}\,{}_1F_1\left(\alpha;\beta;-t-\frac{b}{t}\right)\mathrm{d}t \tag{1.4}$$
$$(\Re(\alpha)>0,\ \Re(\beta)>0, \Re(b)>0, \Re(x)>0)$$

and

$$B_b^{(\alpha,\beta)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;\frac{-b}{t(1-t)}\right)\mathrm{d}t \tag{1.5}$$
$$(\Re(\alpha)>0,\ \Re(\beta)>0, \Re(b)>0, \Re(x)>0, \Re(y)>0).$$

By applying $B_b^{(\alpha,\beta)}(x,y)$, a slightly generalized version of (1.3) was given in [12].

In a recent interesting paper, Srivastava et al. [15] introduced a family of generalized incomplete hypergeometric functions by replacing one of the Pochhammer symbols involving in the coefficient of the generalized hypergeometric function with the *incomplete* Pochhammer symbols. For further works on the subject of the incomplete Pochhammer symbols and the generalized incomplete hypergeometric functions, one may also refer to the papers [14], [17], [18] and [19].

In the sequel, we shall be employing in our extended results the following definition due to Srivastava et al. [16] (see also [14] and [18]).

Definition 1.1. *(*[16, p.243]*) Let a function $\Theta(\kappa_l;z)$ be analytic within the disk $|z|<R$ $(0<R<\infty)$ and let its Taylor-Maclaurin coeficients be explicitly denoted by the sequence $\{\kappa_l\}_{l\in\mathbb{N}_0}$. Suppose also that the function $\Theta(\kappa_l;z)$ can be continued analytically in the right half-plane $\Re(z)>0$ with the asymptotic property given as follows:*

$$\Theta(\kappa_l;z) = \begin{cases} \displaystyle\sum_{l=0}^{\infty}\kappa_l\frac{z^l}{l!} & (|z|<R;\ 0<R<\infty;\kappa_0=1), \\ M_0 z^\omega \exp(z)\left[1+O\left(\frac{1}{z}\right)\right] & (\Re(z)\to\infty; M_0>0;\omega\in\mathbb{C}), \end{cases} \tag{1.6}$$

for some suitable constants M_0 and ω depending essentially on the sequence $\{\kappa_l\}_{l\in\mathbb{N}_0}$.

The extended gamma function $\Gamma_b^{(\kappa_l)}(z)$ and the extended beta function $\mathcal{B}_b^{(\kappa_l)}(\alpha,\beta)$ can then be expressed as (see [16, Equations (2.2) and (2.3)])

$$\Gamma_b^{(\kappa_l)}(z) = \int_0^\infty t^{z-1}\Theta\left(\kappa_l;-t-\frac{b}{t}\right)\mathrm{d}t, \tag{1.7}$$
$$(\Re(z)>0;\ \Re(b)\geq 0)$$

and

$$\mathcal{B}_b^{(\kappa_l)}(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\Theta\left(\kappa_l;-\frac{b}{t(1-t)}\right)\mathrm{d}t, \tag{1.8}$$

$$(\min\{\Re(\alpha),\Re(\beta)\}>0;\ \Re(b)\geq 0).$$

We shall also make use of the following definition of a two-parameter extension of (1.8) due to Srivastava et al. [16, p.256, Eqn. (6.1)] (see also [16, Section 6] for other related two-parameter definitions):

$$\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)\mathrm{d}t, \tag{1.9}$$

$$(\min\{\Re(\alpha),\Re(\beta)\}>0;\ \min\{\Re(b),\Re(d)\}\geq 0).$$

In this paper, we introduce some extended forms of the generalized hypergeometric functions by means of (1.9). Section 2 gives the extensions of Gauss hypergeometric functions and Section 3 treats extensions of the generalized hypergeometric functions together with some of their fundamental properties. Mellin-Barnes type integral representations are also derived by the application of the well-known Ramanujans Master Theorem. In Section 4, we consider further extensions and point out relevant connections of some of the results with known (and new) results including a useful relationship with a class of fractional calculus operators.

2. Extended Gauss Hypergeometric Functions

Using the extended beta function $\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha,\beta)$ defined by (1.9), we can easily form another series representation of the Gauss hypergeometric function

$${}_2F_1\left[\begin{matrix}\alpha_1, & \alpha_2\\ & ;z\\ & \beta_1\end{matrix}\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n}{(\beta_1)_n}\frac{z^n}{n!}, \tag{2.1}$$

$$\left(|z|<1;\alpha_1,\alpha_2\in\mathbb{C};\beta_1\in\mathbb{C}\setminus\mathbb{Z}_0^-\right).$$

Let us replace

$$\frac{(\alpha_2)_n}{(\beta_1)_n}=\frac{B(\alpha_2+n,\beta_1-\alpha_2)}{B(\alpha_2,\beta_1-\alpha_2)}\longrightarrow\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_2+n,\beta_1-\alpha_2)}{B(\alpha_2,\beta_1-\alpha_2)}$$

in (2.1), then we obtain the extended form of the Gauss hypergeometric function (2.1) in the following form:

Definition 2.1. [16] *The Extended Gauss hypergeometric function* ${}_2F_1^{(\kappa_l)}$ *is defined by*

$${}_2F_1^{(\kappa_l)}\left[\begin{matrix}\alpha_1, & \alpha_2\\ & ;z;b,d\\ & \beta_1\end{matrix}\right]=\sum_{n=0}^{\infty}(\alpha_1)_n\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_2+n,\beta_1-\alpha_2)}{B(\alpha_2,\beta_1-\alpha_2)}\frac{z^n}{n!}, \tag{2.2}$$

$$(|z|<1;\Re(\beta_1)>\Re(\alpha_2)>0;\min\{\Re(b),\Re(d)\}\geq 0).$$

The integral representation of (2.2) is contained in the following:

Theorem 2.2.

$${}_2F_1^{(\kappa_l)}\left[\begin{matrix}\alpha_1, & \alpha_2\\ & ;z;b,d\\ & \beta_1\end{matrix}\right]=\frac{1}{B(\alpha_2,\beta_1-\alpha_2)}\int_0^1 t^{\alpha_2-1}(1-t)^{\beta_1-\alpha_2-1}$$

$$\cdot(1-zt)^{-\alpha_1}\,\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)\mathrm{d}t, \tag{2.3}$$

$$(\Re(\beta_1)>\Re(\alpha_2)>0;\ \Re(b)>0,\Re(d)>0;b=d=0,|\arg(1-z)|<\pi).$$

Proof. Replacing the extended beta function $\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_2+n,\beta_1-\alpha_2)$ in (2.2) by its integral representation given by (1.9), and then interchanging the order of summation and integration (which can be justified due to the absolute convergence of the integral and the series involved), the integral representation (2.3) follows immediately after some necessary simplification. □

For $d=b$, the integral representation (2.3) corresponds to the known result [16, p. 244, Eq. (2.6)].

3. Extended Generalized Hypergeometric Functions

In this section, we consider the generalized hypergeometric function and extend it by using the extended beta function $\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha,\beta)$.

The generalized hypergeometric function with p numerator and q denominator parameters is defined by (see, e. g. [8, p. 27])

$$
{}_pF_q(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;z)={}_pF_q\begin{bmatrix}\alpha_1, & \cdots, & \alpha_p \\ & & ;z \\ \beta_1, & \cdots, & \beta_q\end{bmatrix}
=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k}\frac{z^k}{k!}, \tag{3.1}
$$

$$
(\alpha_l,\beta_j\in\mathbb{C},\beta_j\neq 0,-1,-2,\cdots,l=1,\cdots,p;j=1,\cdots,q)
$$

which is absolutely convergent for all values of $z\in\mathbb{C}$, if $p\le q$. When $p=q+1$, the series is absolutely convergent for $|z|<1$ and for $|z|=1$, when $\Re(\sum_{j=1}^{q}\beta_j-\sum_{l=1}^{p}\alpha_l)>0$, while it is conditionally convergent for $|z|=1$ $(z\neq 1)$, if $-1<\Re(\sum_{j=1}^{q}\beta_j-\sum_{l=1}^{p}\alpha_l)\le 0$. More details can be found in [2], [10] and [13].

In terms of the extended beta function $\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha,\beta)$ defined by (1.9), we can construct a suitable extension of (3.1). The following cases need to be considered:

(1) For $p=q+1$, the coefficients of ${}_pF_q(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;z)$ can be rewritten as

$$
(\alpha_1)_n\prod_{j=1}^{q}\frac{(\alpha_{j+1})}{(\beta_j)}=(\alpha_1)_n\prod_{j=1}^{q}\frac{B(\alpha_{j+1}+n,\beta_j-\alpha_{j+1})}{B(\alpha_{j+1},\beta_j-\alpha_{j+1})},(n\in\mathbb{N}_0).
$$

By substituting the extended beta function (1.9) for each $B(\alpha_{j+1}+n,\beta_j-\alpha_{j+1})$, we get the coefficients

$$
(\alpha_1)_n\prod_{j=1}^{q}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_{j+1}+n,\beta_j-\alpha_{j+1})}{B(\alpha_{j+1},\beta_j-\alpha_{j+1})},(n\in\mathbb{N}_0).
$$

(2) For $p=q$, the coefficients of our extension are simply

$$
\prod_{j=1}^{q}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_j+n,\beta_j-\alpha_j)}{B(\alpha_j,\beta_j-\alpha_j)},(n\in\mathbb{N}_0).
$$

(3) When $p<q$, the only reasonable construction of coefficients is

$$
\prod_{i=1}^{r}\frac{1}{(\beta_i)_n}\prod_{j=1}^{p}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_j+n,\beta_{r+j}-\alpha_j)}{B(\alpha_j,\beta_{r+j}-\alpha_j)},(n\in\mathbb{N}_0).
$$

We can now give the formal definition of our extended generalized hypergeometric function as follows:

Definition 3.1. *For suitably constrained (real or complex) parameters $\alpha_j,\ j = 1, \cdots p$; $\beta_i,\ i = 1, \cdots, q$, we define the extended generalized hypergeometric function by*

$$
{}_pF_q^{(\kappa_l)}\left(\alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; z; b, d\right) = {}_pF_q^{(\kappa_l)}\begin{bmatrix} \alpha_1, & \cdots, & \alpha_p & \\ & & & ; z; b, d \\ \beta_1, & \cdots, & \beta_q & \end{bmatrix}
$$

$$
= \begin{cases} \displaystyle\sum_{m=0}^{\infty} (\alpha_1)_m \prod_{j=1}^{q} \frac{\mathcal{B}_{b,d}^{(\kappa_l)}\left(\alpha_{j+1} + m, \beta_j - \alpha_{j+1}\right)}{B\left(\alpha_{j+1}, \beta_j - \alpha_{j+1}\right)} \frac{z^m}{m!}, \\ \qquad\qquad (|z| < 1; p = q + 1; \Re\left(\beta_j\right) > \Re\left(\alpha_{j+1}\right) > 0) \\ \displaystyle\sum_{m=0}^{\infty} \prod_{j=1}^{q} \frac{\mathcal{B}_{b,d}^{(\kappa_l)}\left(\alpha_j + m, \beta_j - \alpha_j\right)}{B\left(\alpha_j, \beta_j - \alpha_j\right)} \frac{z^m}{m!}, \\ \qquad\qquad (z \in \mathbb{C}; p = q; \Re\left(\beta_j\right) > \Re\left(\alpha_j\right) > 0) \\ \displaystyle\sum_{m=0}^{\infty} \prod_{i=1}^{r} \frac{1}{(\beta_i)_m} \prod_{j=1}^{p} \frac{\mathcal{B}_{b,d}^{(\kappa_l)}\left(\alpha_j + m, \beta_{r+j} - \alpha_j\right)}{B\left(\alpha_j, \beta_{r+j} - \alpha_j\right)} \frac{z^m}{m!}, \\ \qquad\qquad (z \in \mathbb{C}; r = q - p, p < q; \Re\left(\beta_{r+j}\right) > \Re\left(\alpha_j\right) > 0) \end{cases} \tag{3.2}
$$

The following theorem demonstrates that the form of the Euler type integral representation of ${}_pF_q^{(\kappa_l)}$ is very similar to that of the Euler type integral representation of ${}_pF_q$.

Theorem 3.2. *For the extended generalized hypergeometric function defined by* (3.2)*, we have the following integral representation:*

$$
{}_pF_q^{(\kappa_l)}\begin{bmatrix} \alpha_1, & \cdots, & \alpha_p & \\ & & & ; z; b, d \\ \beta_1, & \cdots, & \beta_q & \end{bmatrix} = \frac{\Gamma\left(\beta_q\right)}{\Gamma\left(\alpha_p\right)\Gamma\left(\beta_q - \alpha_p\right)} \int_0^1 t^{\alpha_p - 1}\left(1 - t\right)^{\beta_q - a_p - 1}
$$

$$
\cdot\, {}_{p-1}F_{q-1}^{(\kappa_l)}\begin{bmatrix} \alpha_1, & \cdots, & \alpha_{p-1} & \\ & & & ; zt; b, d \\ \beta_1, & \cdots, & \beta_{q-1} & \end{bmatrix} \Theta\left(\kappa_l; -\frac{b}{t} - \frac{d}{1-t}\right) \mathrm{d}t. \tag{3.3}
$$

$$
\left(\Re\left(\beta_q\right) > \Re\left(\alpha_p\right) > 0, \min\{\Re\left(b\right), \Re\left(d\right)\} > 0; d = b = 0, \left|\arg\left(1 - z\right)\right| < \pi\right)
$$

Proof. We need to verify that formula (3.3) holds for three different expressions of ${}_pF_q^{(\kappa_l)}(\alpha_1, \cdots, \alpha_p;\ \beta_1, \cdots, \beta_q; z; b, d)$ given in (3.2), respectively. Consider the case $p = q + 1$. In view of the representation that

$$
\frac{\mathcal{B}_{b,d}^{(\kappa_l)}\left(\alpha_{q+1} + m, \beta_q - \alpha_{q+1}\right)}{B\left(\alpha_{q+1}, \beta_q - \alpha_{q+1}\right)} = \frac{\Gamma\left(\beta_q\right)}{\Gamma\left(\alpha_{q+1}\right)\Gamma\left(\beta_q - \alpha_{q+1}\right)}
$$

$$
\cdot \int_0^1 t^{\alpha_{q+1} + m - 1}\left(1 - t\right)^{\beta_q - \alpha_{q+1} - 1} \Theta\left(\kappa_l; -\frac{b}{t} - \frac{d}{1-t}\right) \mathrm{d}t, \tag{3.4}
$$

$$
\left(m \in \mathbb{N}_0, \min\{\Re\left(b\right), \Re\left(d\right)\} > 0, \Re\left(\beta_q\right) > \Re\left(\alpha_{q+1}\right) > 0\right)
$$

we find that

$$
\begin{aligned}
&{}_{q+1}F_{q}^{(\kappa_l)}\begin{bmatrix}\alpha_1, & \cdots, & \alpha_{q+1} \\ & & ; z; b, d \\ \beta_1, & \cdots, & \beta_q\end{bmatrix} \\
&= \frac{\Gamma(\beta_q)}{\Gamma(\alpha_{q+1})\Gamma(\beta_q-\alpha_{q+1})}\int_0^1 t^{\alpha_{q+1}-1}(1-t)^{\beta_q-\alpha_{q+1}-1}\,\Theta\left(\kappa_l; -\frac{b}{t}-\frac{d}{1-t}\right) \\
&\qquad \cdot \sum_{m=0}^{\infty}(\alpha_1)_m \prod_{j=1}^{q-1}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_{j+1}+m, \beta_j-\alpha_{j+1})}{B(\alpha_{j+1}, \beta_j-\alpha_{j+1})}\frac{(zt)^m}{m!}\mathrm{d}t \\
&= \frac{\Gamma(\beta_q)}{\Gamma(\alpha_{q+1})\Gamma(\beta_q-\alpha_{q+1})}\int_0^1 t^{\alpha_{q+1}-1}(1-t)^{\beta_q-\alpha_{q+1}-1} \\
&\qquad \cdot {}_{q}F_{q-1}^{(\kappa_l)}\begin{bmatrix}\alpha_1, & \cdots, & \alpha_q \\ & & ; zt; b, d \\ \beta_1, & \cdots, & \beta_{q-1}\end{bmatrix}\Theta\left(\kappa_l; -\frac{b}{t}-\frac{d}{1-t}\right)\mathrm{d}t. \qquad (3.5)
\end{aligned}
$$

It is clear that the relation (3.3) is also valid for $p \leq q$, and this completes the proof. □

Remark. A multidimensional case of the Euler type integral representation of (3.5) is given by

$$
\begin{aligned}
&{}_{q+1}F_{q}^{(\kappa_l)}\begin{bmatrix}\alpha_1, & \cdots, & \alpha_{q+1} \\ & & ; z; b, d \\ \beta_1, & \cdots, & \beta_q\end{bmatrix} = \prod_{j=1}^{q}\frac{\Gamma(\beta_j)}{\Gamma(\alpha_{j+1})\Gamma(\beta_j-\alpha_{j+1})} \\
&\qquad \cdot \int_0^1\cdots\int_0^1 \prod_{j=1}^{q}\left\{t_j^{\alpha_{j+1}}(1-t_j)^{\beta_j-\alpha_{j+1}-1}\Theta\left(\kappa_l; -\frac{b}{t_j}-\frac{d}{1-t_j}\right)\right\} \\
&\qquad\qquad \cdot (1-t_1t_2\cdots t_q z)^{-\alpha_1}\,\mathrm{d}t_1\cdots\mathrm{d}t_q,
\end{aligned}
$$

which follows from the repeated application of the functional equation (3.5). If we set $d = b = 0$, then this representation reduces to the one given in [7, p.132, Eq. (4.2)].

Theorem 3.3. *The following derivative formula holds for $p \leq q+1$.*

$$
\begin{aligned}
&\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{{}_{p}F_{q}^{(\kappa_l)}\begin{bmatrix}\alpha_1, & \cdots, & \alpha_p \\ & & ; z; b, d \\ \beta_1, & \cdots, & \beta_q\end{bmatrix}\right\} \\
&\qquad = \frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}{}_{p}F_{q}^{(\kappa_l)}\begin{bmatrix}\alpha_1+n, & \cdots, & \alpha_p+n \\ & & ; z; b, d \\ \beta_1+n, & \cdots, & \beta_q+n\end{bmatrix}(n \in \mathbb{N}_0). \qquad (3.6)
\end{aligned}
$$

Proof. Differentiating ${}_{q+1}F_q^{(\kappa_l)}$ with respect to z, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}z}\left\{{}_{q+1}F_{q}^{(\kappa_l)}\begin{bmatrix}\alpha_1, & \cdots, & \alpha_p \\ & & ; z; b, d \\ \beta_1, & \cdots, & \beta_q\end{bmatrix}\right\} \\
&\qquad = \sum_{m=1}^{\infty}(\alpha_1)_m \prod_{j=1}^{q}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_{j+1}+m, \beta_j-\alpha_{j+1})}{B(\alpha_{j+1}, \beta_j-\alpha_{j+1})}\frac{z^{m-1}}{(m-1)!}. \qquad (3.7)
\end{aligned}
$$

Replacing $m \to m+1$ in the right-hand side of (3.7), we are lead to

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{ {}_{q+1}F_q^{(\kappa_l)}\left[\begin{matrix}\alpha_1, & \cdots, & \alpha_{q+1} \\ & & ; z; b, d \\ \beta_1, & \cdots, & \beta_q\end{matrix}\right]\right\} = \alpha_1 \frac{\prod_{j=1}^{q}\alpha_{j+1}}{\prod_{j=1}^{q}\beta_j} \cdot {}_{q+1}F_q^{(\kappa_l)}\left[\begin{matrix}\alpha_1+1, & \cdots, & \alpha_{q+1}+1 \\ & & ; z; b, d \\ \beta_1+1, & \cdots, & \beta_q+1\end{matrix}\right]. \quad (3.8)$$

Recursive application of this procedure n-times gives us the general form (3.6).

Similarly, we can prove this result for the case $p \le q$. □

For $p=2$ and $q=1$, we at once get

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{ {}_2F_1^{(\kappa_l)}\left[\begin{matrix}\alpha_1, & \alpha_2 \\ & ; z; b, d \\ & \beta_1\end{matrix}\right]\right\} = \frac{(\alpha_1)_n(\alpha_2)_n}{(\beta_1)_n}\, {}_2F_1^{(\kappa_l)}\left[\begin{matrix}\alpha_1+n, & \alpha_2+n \\ & ; z; b, d \\ & \beta_1+n\end{matrix}\right]. \quad (3.9)$$

If we put $\Theta(\kappa_l; z) = {}_1F_1(\alpha; \beta; z)$ and $b=d$, then equation (3.9) reduces to

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{ F_b^{(\alpha,\beta)}\left[\begin{matrix}a, & b \\ & ; z \\ & c\end{matrix}\right]\right\} = \frac{(b)_n(a)_n}{(c)_n} F_b^{(\alpha,\beta)}\left[\begin{matrix}a+n, & b+n \\ & ; z \\ & c+n\end{matrix}\right],$$

which corresponds to the known result [12, Theorem 3.3].

Next, we derive the Mellin-Barnes type contour integral representation of the function (3.2). We need the following well-known theorem which is widely used to evaluate definite integrals and infinite series.

Theorem 3.4 (**Ramanujan's Master Theorem** [1]). *Assume f admits an expansion of the form:*

$$f(x) = \sum_{k=0}^{\infty} \frac{\lambda(k)}{k!}(-x)^k \quad (\lambda(0) \neq 0).$$

Then, the Mellin transform of f is given by

$$\int_0^{\infty} x^{s-1} f(x)\,\mathrm{d}x = \Gamma(s)\,\lambda(-s).$$

By means of the Ramanujan's Master Theorem, we obtain the following Mellin-Barnes type integral representation.

Theorem 3.5. *The Mellin-Barnes type integral representation of the function* (3.2) *is given by*

$${}_pF_q^{(\kappa_l)}\left[\begin{matrix}\alpha_1, & \cdots, & \alpha_p \\ & & ; z; b, d \\ \beta_1, & \cdots, & \beta_q\end{matrix}\right]$$

$$=\begin{cases}\dfrac{1}{2\pi i}\displaystyle\int_{L_1}\prod_{j=1}^{q}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}\left(\alpha_{j+1}-s,\beta_j-\alpha_{j+1}\right)}{B\left(\alpha_{j+1},\beta_j-\alpha_{j+1}\right)}\frac{\Gamma\left(s\right)\Gamma\left(\alpha_1-s\right)}{\Gamma\left(\alpha_1\right)}\left(-z\right)^{-s}\mathrm{d}s,\\ \quad\left(p=q+1,\Re\left(\beta_j\right)>\Re\left(\alpha_{j+1}\right)>0,j=1,\cdots,q,\Re\left(\alpha_1\right)>0\right)\\ \dfrac{1}{2\pi i}\displaystyle\int_{L_2}\prod_{j=1}^{q}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}\left(\alpha_j-s,\beta_j-\alpha_j\right)}{B\left(\alpha_j,\beta_j-\alpha_j\right)}\Gamma\left(s\right)\left(-z\right)^{-s}\mathrm{d}s,\\ \quad\left(p=q,\Re\left(\beta_j\right)>\Re\left(\alpha_j\right)>0,j=1,\cdots,q\right)\\ \dfrac{1}{2\pi i}\displaystyle\int_{L_3}\prod_{j=1}^{q}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}\left(\alpha_j-s,\beta_{j+r}-\alpha_j\right)}{B\left(\alpha_j,\beta_{j+r}-\alpha_j\right)}\prod_{i=1}^{r}\frac{\Gamma\left(\beta_i\right)}{\Gamma\left(\beta_i-s\right)}\Gamma\left(s\right)\left(-z\right)^{-s}\mathrm{d}s,\\ \left(r=q-p,\ p<q,\Re\left(\beta_{r+j}\right)>\left(\alpha_j\right)>0,\Re\left(\beta_i\right)>0,i=1,\cdots,r\right)\end{cases}\tag{3.10}$$

where $L_i, i=1,2,3$ are Mellin-Barnes-type contours from $-i\infty$ to $i\infty$, with the usual indentations in order to separate one set of poles from the other set of poles of the integrand.

Proof. The result follows rather directly upon using the Ramanujan's Master Theorem and the inverse Mellin transform. □

There exists another kind of Mellin-Barnes type integral representation of the function (3.2), if we can express the integrand of the contour integral in (3.10) into a simpler form.

In what follows, we denote by

$$C_q\left(\alpha,\beta;x\right)=\prod_{j=1}^{q}\frac{\Gamma\left(x_j+\alpha_{j+1}\right)\Gamma\left(\beta_j-\alpha_{j+1}+x_j\right)}{\Gamma\left(2x_j+\beta_j\right)}\tag{3.11}$$

and

$$H_q\left(\alpha,\beta;x\right)=\frac{C_q\left(\alpha,\beta;x\right)}{C_q\left(\alpha,\beta;0\right)}.\tag{3.12}$$

Theorem 3.6. *Let $d=b$ with $\Re(b)>0$, then the Mellin-Barnes type integral representation of the second kind for the function (3.2) with $p=q+1$ is given by*

$$\begin{aligned}{}_{q+1}F_q^{(\kappa_l)}\left[\begin{matrix}\alpha_1, & \cdots, & \alpha_{q+1}\\ \beta_1, & \cdots, & \beta_q\end{matrix};z;b\right]&=\frac{1}{(2\pi i)^q}\int_{-i\infty}^{+i\infty}\cdots\int_{-i\infty}^{+i\infty}H_q\left(\alpha,\beta;s\right)\prod_{j=1}^{q}\Gamma_0^{(\kappa_l)}\left(s_j\right)\\ &\cdot{}_{q+1}F_q\left[\begin{matrix}\alpha_1, & s_1+\alpha_2, & \cdots, & s_q+\alpha_{q+1}\\ & 2s_1+\beta_1, & \cdots, & 2s_q+\beta_q\end{matrix};z\right]b^{-\sum_{j=1}^{q}s_j}\mathrm{d}s_1\cdots\mathrm{d}s_q,\end{aligned}\tag{3.13}$$

where the function $H_q(\alpha,\beta;s)$ is defined by (3.12).

Proof. Following [12, Theorem 2.2], we can get the Mellin transform representation of the extended beta function (1.8). Indeed, we have

$$\begin{aligned}&\mathcal{B}_b^{(\kappa_l)}\left(\alpha_{j+1}+m,\beta_j-\alpha_{j+1}\right)\\ &=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}B\left(s_j+\alpha_{j+1}+m,s_j+\beta_j-\alpha_{j+1}\right)\Gamma_0^{(\kappa_l)}\left(s_j\right)b^{-s_j}\mathrm{d}s_j\end{aligned}$$

$$= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} B\left(s_j + \alpha_{j+1}, \beta_j - \alpha_{j+1} + s_j\right) \frac{\left(s_j + \alpha_{j+1}\right)_m}{\left(2s_j + \beta_j\right)_m} \Gamma_0^{(\kappa_l)}\left(s_j\right) b^{-s_j} \mathrm{d}s_j, \quad (3.14)$$

which in view of (3.2) and some elementary simplifications give the desired result (3.13). □

Remark. In view of (3.13), we get the expansion formula:

$$ {}_{q+1}F_q^{(\kappa_l)}\left[\begin{matrix}\alpha_1, & \cdots, & \alpha_{q+1} \\ \beta_1, & \cdots, & \beta_q\end{matrix}; z\zeta; b\right]$$

$$= (1-z)^{\alpha_1} \sum_{k=0}^{\infty} \frac{(\alpha_1)_n}{k!} \left(\frac{z}{z-1}\right)^k {}_{q+1}F_q^{(\kappa_l)}\left[\begin{matrix}-k, & \alpha_2, & \cdots, & \alpha_{q+1} \\ & \beta_1, & \cdots, & \beta_q\end{matrix}; \zeta; b\right], \quad (3.15)$$

which follows on applying Theorem 3.8 and [10, Eq.16.10.2].

4. Further results

Our definition of extended generalized hypergeometric function can further be generalized to the following form:

Definition 4.1. *For suitably constrained (real or complex) parameters α_j, $j = 1, \cdots, p$; β_i, $i = 1, \cdots, q$, we define the extended generalized hypergeometric functions by*

$$ {}_pF_q^{(\kappa_l)}\left[\begin{matrix}(\alpha_1, k_1), & \cdots, & (\alpha_p, k_p) \\ \beta_1, & \cdots & \beta_q,\end{matrix}; z; b, d\right]$$

$$= \begin{cases} \sum_{m=0}^{\infty} (\alpha_1)_{k_1 m} \prod_{j=1}^{q} \frac{\mathcal{B}_{b,d}^{(\kappa_l)}\left(\alpha_{j+1} + k_{j+1}m, \beta_j - \alpha_{j+1}\right)}{B\left(\alpha_{j+1}, \beta_j - \alpha_{j+1}\right)} \frac{z^m}{m!}, \\ \qquad (|z| < 1;\ p = q+1; \Re\left(\beta_j\right) > \Re\left(\alpha_{j+1}\right) > 0) \\ \sum_{m=0}^{\infty} \prod_{j=1}^{q} \frac{\mathcal{B}_{b,d}^{(\kappa_l)}\left(\alpha_j + k_j m, \beta_j - \alpha_j\right)}{B\left(\alpha_j, \beta_j - \alpha_j\right)} \frac{z^m}{m!}, \\ \qquad (z \in \mathbb{C}; p = q; \Re\left(\beta_j\right) > \Re\left(\alpha_j\right) > 0) \\ \sum_{m=0}^{\infty} \prod_{i=1}^{r} \frac{1}{(\beta_r)_m} \prod_{j=1}^{p} \frac{\mathcal{B}_{b,d}^{(\kappa_l)}\left(\alpha_j + k_j m, \beta_{r+j} - \alpha_j\right)}{B\left(\alpha_j, \beta_{r+j} - \alpha_j\right)} \frac{z^m}{m!}, \\ \qquad (z \in \mathbb{C}; r = q-p; p < q; \Re\left(\beta_{r+j}\right) > \Re\left(\alpha_j\right) > 0) \end{cases} \quad (4.1)$$

where the new parameters $k_1 \in \{0, 1\}$, $k_j, j = 2, \cdots, p$ are non-negative integers.

Obviously, (4.1) reduces to (3.2), whenever, $k_j = 1$, $j = 1, \cdots, p$. To illustrate its advantages, we first consider the following function:

$$ {}_2F_1^{(\kappa_l)}\left[\begin{matrix}(\alpha_1, 1), & (\alpha_2, k_2) \\ & \beta_1\end{matrix}; z; b, d\right] = \sum_{n=0}^{\infty} (\alpha_1)_n \frac{\mathcal{B}_{b,d}^{(\kappa_l)}\left(\alpha_2 + k_2 n, \beta_1 - \alpha_2\right)}{B\left(\alpha_2, \beta_1 - \alpha_2\right)} \frac{z^n}{n!}. \quad (4.2)$$

Its integral representation can be written as

$$
{}_2F_1^{(\kappa_l)}\left[\begin{matrix}(\alpha_1,1), & (\alpha_2,k_2) \\ & \beta_1\end{matrix};z;b,d\right]=\frac{1}{B(\alpha_2,\beta_1-\alpha_2)}
$$

$$
\int_0^1 t^{\alpha_2-1}(1-t)^{\beta_1-\alpha_2-1}\left(1-zt^{k_2}\right)^{-\alpha_1}\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)\mathrm{d}t. \quad (4.3)
$$

$$
(\Re(\beta_1)>\Re(\alpha_2)>0;\min\{\Re(b),\Re(d)\}>0;b=d=0,|\arg(1-z)|<\pi)
$$

As an example, let us compute the case for $k_2=2$ and $z=1$. We have

$$
\begin{aligned}
&\frac{1}{B(\alpha_2,\beta_1-\alpha_2)}\int_0^1 t^{\alpha_2-1}(1-t)^{\beta_1-\alpha_2-1}\left(1-t^2\right)^{-\alpha_1}\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)\mathrm{d}t\\
&=\int_0^1\frac{t^{\alpha_2-1}(1-t)^{\beta_1-\alpha_2-1}}{B(\alpha_2,\beta_1-\alpha_2)}(1-t)^{-\alpha_1}(1+t)^{-\alpha_1}\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)\mathrm{d}t\\
&=\sum_{k=0}^{\infty}\binom{-\alpha_1}{k}\int_0^1\frac{t^{\alpha_2+k-1}(1-t)^{\beta_1-\alpha_2-\alpha_1-1}}{B(\alpha_2,\beta_1-\alpha_2)}\Theta\left(\kappa_l;-\frac{b}{t}-\frac{d}{1-t}\right)\mathrm{d}t\\
&=\frac{1}{B(\alpha_2,\beta_1-\alpha_2)}\sum_{k=0}^{\infty}\frac{(-1)^k(\alpha_1)_k}{k!}\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_2+k,\beta_1-\alpha_2-\alpha_1)\\
&=\frac{\Gamma(\beta_1)}{\Gamma(\alpha_2)\Gamma(\beta_1-\alpha_2)}\frac{\Gamma(\beta_1-\alpha_2-\alpha_1)\Gamma(\alpha_2)}{\Gamma(\beta_1-\alpha_1)}\\
&\qquad\cdot\sum_{k=0}^{\infty}(\alpha_1)_k\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_2+k,\beta_1-\alpha_2-\alpha_1)}{B(\alpha_2,\beta_1-\alpha_2-\alpha_1)}\frac{(-1)^k}{k!}\\
&=\frac{\Gamma(\beta_1)\Gamma(\beta_1-\alpha_2-\alpha_1)}{\Gamma(\beta_1-\alpha_2)\Gamma(\beta_1-\alpha_1)}{}_2F_1^{(\kappa_l)}\left[\begin{matrix}\alpha_1, & \alpha_2\\ & \beta_1-\alpha_1\end{matrix};-1;b,d\right].
\end{aligned}
$$

We thus obtain the following:

Theorem 4.2. *If* $\Re(\beta_1)>\Re(\alpha_2)>0$ *and* $\Re(\beta_1-\alpha_2-\alpha_1)>0$, *then*

$$
{}_2F_1^{(\kappa_l)}\left[\begin{matrix}(\alpha_1,1), & (\alpha_2,2)\\ & \beta_1\end{matrix};1;b,d\right]=\frac{\Gamma(\beta_1)\Gamma(\beta_1-\alpha_2-\alpha_1)}{\Gamma(\beta_1-\alpha_2)\Gamma(\beta_1-\alpha_1)}
$$

$$
\cdot\,{}_2F_1^{(\kappa_l)}\left[\begin{matrix}\alpha_1, & \alpha_2\\ & \beta_1-\alpha_1\end{matrix};-1;b,d\right]. \quad (4.4)
$$

Remark. Theorem 4.2 is in fact a generalization of the following well-known result [6, p.117, Theorem 2.3]:

$$
{}_3F_2\left[\begin{matrix}a, & \frac{b}{2}, & \frac{b+1}{2}\\ & \frac{c}{2}, & \frac{c+1}{2}\end{matrix};1\right]=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)}{}_2F_1\left[\begin{matrix}a, & b\\ & c-a\end{matrix};-1\right]. \quad (4.5)
$$

It also shows some intrinsic relationship between the extended hypergeometric function and the usual generalized hypergeometric function.

Theorem 4.3. *For the extended generalized hypergeometric function* (4.1), *we have the following formula holds true:*

$$
{}_pF_q^{(\kappa_l)}\left[\begin{matrix}(\alpha_1,k_1), & \cdots, & (\alpha_p,k_p) \\ & & & ;cz^{k_p};b,d \\ \beta_1, & \cdots, & \beta_q\end{matrix}\right] = z^{1-\beta_q}\int_0^z \frac{t^{\alpha_p-1}(z-t)^{\beta_q-a_p-1}}{B(\alpha_p,\beta_q-\alpha_p)}
$$
$$
\cdot\, {}_{p-1}F_{q-1}^{(\kappa_l)}\left[\begin{matrix}(\alpha_1,k_1), & \cdots, & (\alpha_{p-1},k_{p-1}) \\ & & & ;ct^{k_p};b,d \\ \beta_1, & \cdots, & \beta_{q-1}\end{matrix}\right]\Theta\left(\kappa_l;-\frac{bz}{t}-\frac{dz}{z-t}\right)\mathrm{d}t. \tag{4.6}
$$

Proof. Denoting the right-hand side of (4.6) by $A(z)$, then (as indicated before), we just need to prove the case for $p=q+1$. Let $t=zv$ in $A(z)$, we have then

$$
\begin{aligned}
A(z) &= \frac{\Gamma(\beta_q)\,z^{\beta_q-1}}{\Gamma(\alpha_p)\Gamma(\beta_q-\alpha_p)}\int_0^1 v^{\alpha_{q+1}-1}(1-v)^{\beta_q-a_{q+1}-1} \\
&\quad\cdot\, {}_qF_{q-1}^{(\kappa_l)}\left[\begin{matrix}(\alpha_1,k_1), & \cdots, & (\alpha_q,k_q) \\ & & & ;c(zv)^{k_{q+1}};b,d \\ \beta_1, & \cdots, & \beta_{q-1}\end{matrix}\right]\Theta\left(\kappa_l;-\frac{b}{v}-\frac{d}{1-v}\right)\mathrm{d}v \\
&= \frac{\Gamma(\beta_q)\,z^{\beta_q-1}}{\Gamma(\alpha_{q+1})\Gamma(\beta_q-\alpha_{q+1})}\sum_{m=0}^{\infty}(\alpha_1)_{k_1m}\prod_{j=1}^{q-1}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_{j+1}+k_{j+1}m,\beta_j-\alpha_{j+1})}{B(\alpha_{j+1},\beta_j-\alpha_{j+1})} \\
&\quad\cdot\frac{\left(cz^{k_{q+1}}\right)^m}{m!}\int_0^1 v^{\alpha_{q+1}+k_{q+1}m-1}(1-v)^{\beta_q-a_{q+1}-1}\,\Theta\left(\kappa_l;-\frac{b}{v}-\frac{d}{1-v}\right)\mathrm{d}v \\
&= z^{\beta_q-1}\sum_{m=0}^{\infty}(\alpha_1)_{k_1m}\prod_{j=1}^{q}\frac{\mathcal{B}_{b,d}^{(\kappa_l)}(\alpha_{j+1}+k_{j+1}m,\beta_j-\alpha_{j+1})}{B(\alpha_{j+1},\beta_j-\alpha_{j+1})}\frac{\left(cz^{k_{q+1}}\right)^m}{m!} \\
&= z^{\beta_q-1}\,{}_{q+1}F_q^{(\kappa_l)}\left[\begin{matrix}(\alpha_1,k_1), & \cdots, & (\alpha_q,k_q), & (\alpha_{q+1},k_{q+1}) \\ & & & & ;cz^{k_{q+1}};b,d \\ \beta_1, & \cdots, & \beta_{q-1}, & \beta_q\end{matrix}\right].
\end{aligned}
$$

□

Theorem 4.4 has the significance and advantages of implementing the modified definition (4.1) of the extended generalized hypergeometric function. It not only enables us to formulate the classical results of hypergeometric functions in a new way, but also provide some new important interpretations. We consider here one of such new interpretations.

Srivastava et al. in [16] defined the following extended fractional derivative operator:

$$
\mathcal{D}_{z,(\kappa_l)}^{\mu,b,d}\{f(z)\} = \begin{cases} \dfrac{1}{\Gamma(-\mu)}\displaystyle\int_0^z (z-t)^{-\mu-1}\,\Theta\left(\kappa_l;-\frac{bz}{t}-\frac{dz}{z-t}\right)f(t)\,\mathrm{d}t, \\ \qquad\qquad (\Re(\mu)<0), \\ \dfrac{\mathrm{d}^m}{\mathrm{d}z^m}\left\{\mathcal{D}_{z,(\kappa_l)}^{\mu-m,b,d}\{f(z)\}\right\},\ (m-1\le\Re(\mu)\le m\,(m\in\mathbb{N})). \end{cases} \tag{4.7}
$$

The path of integration in (4.7) is a line in the complex $t-$plane from $t=0$ to $t=z$.

By using the operator (4.7), we can rewrite (4.6) as

$$
{}_pF_q^{(\kappa_l)}\left[\begin{matrix}(\alpha_1,k_1), & \cdots, & (\alpha_p,k_p)\\ \beta_1, & \cdots, & \beta_q\end{matrix}; cz^{k_p}; b,d\right] = \frac{\Gamma(\beta_q)}{\Gamma(\alpha_p)} z^{1-\beta_q}
$$

$$
\cdot\, \mathcal{D}_{z,(\kappa_l)}^{-(\beta_q-\alpha_p),b,d}\left\{ t^{\alpha_p-1}\,{}_{p-1}F_{q-1}^{(\kappa_l)}\left[\begin{matrix}(\alpha_1,k_1), & \cdots, & (\alpha_{p-1},k_{p-1})\\ \beta_1, & \cdots, & \beta_{q-1}\end{matrix}; ct^{k_p}; b,d\right]\right\}. \tag{4.8}
$$

Acknowledgments. The authors wish to thank the referee for useful suggestions.

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