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ON GENERALIZED SASAKIAN SPACE FORMS SATISFYING CERTAIN CONDITIONS ON THE CONCIRCULAR CURVATURE TENSOR

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Abstract. The object of the present paper is to study generalized Sasakian-space-forms with the conditions satisfying $\widetilde{C}(\xi, X)\widetilde{C} = 0$, $\widetilde{C}(\xi, X)R = 0$, $\widetilde{C}(\xi, X)S = 0$ and $\widetilde{C}(\xi, X)P = 0$. According these cases, generalized Sasakian-space forms have been classified.

1. Introduction

Generalized Sasakian-space-forms were introduced and studied by P. Alegre, D.E. Blair and A. Carriazo in [1]. They calculated the Riemannian curvature tensor of a generalized Sasakian-space forms and presented many examples of these manifolds in your work.

In [2], U.C. De and A. Sarkar studied the nature of a generalized Sasakian-space-form under some conditions regarding projective curvature tensor. They obtained the necessary and sufficient conditions for a generalized Sasakian-space-form satisfying $PS = 0$ and $PR = 0$.

Again A. Sarkar and U. C. De studied generalized Sasakian-space-forms with vanishing quasi-conformal curvature tensor and investigated quasi-conformal flat generalized Sasakian-space-forms, Ricci-symmetric and Ricci semisymmetric generalized Sasakian-space-forms[3].

In [5], C. Özgür and M. M. Tripathi obtained the necessary and sufficient necessary conditions for curvatures of P-Sasakian manifolds satisfying the derivations.

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Motivated by the studies of the above authors, in the present paper, we obtain necessary and sufficient conditions for a generalized Sasakain-space-form satisfying the derivation conditions $\widetilde{C}(\xi, X)\widetilde{C} = 0$, $\widetilde{C}(\xi, X)R = 0$, $\widetilde{C}(\xi, X)S = 0$ and $\widetilde{C}(\xi, X)P = 0$. Generalized Sasakain-space-forms satisfying these conditions are evaluated. I think that new results on generalized Sasakian-space-forms are also obtained.

In differential geometry, the curvature of a Riemannian manifold (M, g) plays a fundamental role as well known, the sectional curvature of a manifold determine the curvature tensor R-completely. A Riemannian manifold with constant sectional curvature c is called a real-space form and its curvature tensor is given by the equation

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}, \tag{1.1}$$

for any vector fields X, Y, Z on M. Models for these space are the Euclidean space $(c = 0)$, the sphere $(c > 0)$ and the Hyperbolic space $(c < 0)$.

A similar situation can be found in the study of complex manifolds from a Riemannian point of view. If (M, J, g) is a Kaehler manifold with constant holomorphic sectional curvature $K(X \wedge JX) = c$, then is said to be a complex space form and it is well known that its curvature tensor satisfies the equation

$$\begin{aligned} R(X, Y)Z &= \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + g(X, JZ)JY - g(Y, JZ)JX \\ &+ 2g(X, JY)JZ\}, \end{aligned} \tag{1.2}$$

for any vector fields X, Y, Z on M. These models are $\mathbb{C}^n$, $\mathbb{CP}^n$ and $\mathbb{CH}^n$ depending on $c = 0$, $c >$, and $c < 0$, respectively.

On the other hand, Sasakian-space-forms play a similar role in contact metric geometry. For such a manifold, the curvature tensor is given by

$$\begin{aligned} R(X, Y)Z &= (\frac{c+3}{4})\{g(Y, Z)X - g(X, Z)Y\} \\ &+ (\frac{c-1}{4})\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z \\ &+ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} \end{aligned} \tag{1.3}$$

for any vector fields X, Y, Z on M. These spaces can also be modeled depending on cases $c > -3$, $c = -3$ and $c < -3$.

1.1. **Preliminaries.**

In this section, we recall some definitions and basic formulas which will use later. For the more detail, on almost contact metric manifolds, we recommend the references and their references.

An odd-dimensional Riemannian manifold (M, g) is said to be an almost contact metric manifold if there exist on M a (1,1)-type field ϕ, a vector field ξ, called the structure vector field, and a 1-form η such that $\eta(\xi) = 1$,

$$\phi^2 X = -X + \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.4}$$

for any vector fields X, Y on M. In an almost contact metric manifold, we have also $\phi\xi = 0$ and $\eta o \phi = 0$.

Such a manifold is said to be a contact metric manifold if $d\eta = \Phi$, where $\Phi(X,Y) = g(X, \phi Y)$ is called the fundamental 2-form of M. If in addition, ξ is a Killing vector field, then manifold is said to be a K-contact manifold. It is well known that a contact metric manifold is a K-contact manifold if and only if $\nabla_X \xi = -\phi X$, for any vector field X on M.

On the other hand, given an almost contact metric manifold $M(\phi, \xi, \eta, g)$, we say that M is a generalized Sasakian-space-form if there exist three functions f_1, f_2, f_3 on M such that the curvature tensor R is given by

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{1.5}$$

for any vector fields X, Y, Z on M[1]. Such a manifold is denoted by $M^{2n+1}(f_1, f_2, f_3)$. This kind of manifold appears as a generalization of the well known Sasakian-space-form, which can be obtained as a particular case of generalized Sasakian space forms by taking $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$.

In a $(2n+1)$-dimensional generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$, we have the following relations[2];

$$(\bar{\nabla}_X \phi)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X], \tag{1.6}$$
$$\bar{\nabla}_X \xi = -(f_1 - f_3)\phi X, \tag{1.7}$$
$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{1.8}$$
$$R(\xi, X)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X], \tag{1.9}$$
$$\eta(R(X,Y)Z) = (f_1 - f_3)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{1.10}$$
$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \tag{1.11}$$
$$Q\xi = 2n(f_1 - f_3)\xi, \tag{1.12}$$
$$\eta(QX) = 2n(f_1 - f_3)\eta(X), \tag{1.13}$$
$$\begin{aligned} S(X,Y) &= (2nf_1 + 3f_2 - f_3)g(X,Y) \\ &- (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \end{aligned} \tag{1.14}$$
$$\tau = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, \tag{1.15}$$
$$S(X,\xi) = 2n(f_1 - f_3)\eta(X),, \tag{1.16}$$

for any vector fields X, Y on M, where R, Q, S and τ denote the Riemannian curvature tensor, Ricci operatory, Ricci tensor and scalar curvature of $M^{2n+1}(f_1, f_2, f_3)$, respectively.

Given an n-dimensional Riemannian manifold (M, g), the Concircular curvature tensor $\widetilde{C}$, the Weyl conformal curvature tensor C and projective curvature tensor

P are also, respectively, given by

$$\widetilde{C}(X,Y)Z = R(X,Y)Z - \frac{\tau}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{1.17}$$

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &- g(X,Z)QY] + \frac{\tau}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{1.18}$$

and

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \tag{1.19}$$

for any vector fields X, Y, Z on M[6].

2. Main Results on Generalized Sasakian-Space-Forms

In this section, we obtain necessary and sufficient conditions for a generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ satisfying the derivation conditions $\widetilde{C}(\xi, X)\widetilde{C} = 0$, $\widetilde{C}(\xi, X)R = 0$, $\widetilde{C}(\xi, X)S = 0$ and $\widetilde{C}(\xi, X)P = 0$.

Theorem 2.1. *A generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ satisfies the condition*

$$\widetilde{C}(\xi, X)\widetilde{C} = 0 \tag{2.1}$$

if and only if either the scalar curvature τ of $M^{2n+1}(f_1, f_2, f_3)$ is $\tau = (f_1 - f_3)2n(2n+1)$ or $M^{2n+1}(f_1, f_2, f_3)$ is a real space form with the sectional curvature $(f_1 - f_3)$.

Proof. The condition $\widetilde{C}(\xi, X)\widetilde{C} = 0$ implies that

$$\begin{aligned} (\widetilde{C}(\xi, X)\widetilde{C})(Y, Z, U) &= \widetilde{C}(\xi, X)\widetilde{C}(Y, Z)U - \widetilde{C}(\widetilde{C}(\xi, X)Y, Z)U \\ &- \widetilde{C}(Y, \widetilde{C}(\xi, X)Z)U - \widetilde{C}(Y, Z)\widetilde{C}(\xi, X)U, \end{aligned} \tag{2.2}$$

for any vector fields X, Y, Z, U on M. By virtue of (1.10) and (1.17), we reach

$$\widetilde{C}(\xi, X)Y = [f_1 - f_3 - \frac{\tau}{2n(2n+1)}][g(X,Y)\xi - \eta(Y)X] \tag{2.3}$$

and

$$\eta(\widetilde{C}(X,Y)Z) = [f_1 - f_3 - \frac{\tau}{2n(2n+1)}][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.4}$$

for any vector fields X, Y, Z on M. From (1.17) and (2.3), we can easily to see that

$$\begin{aligned} \widetilde{C}(\xi, X)\widetilde{C}(Y, Z)U &= [f_1 - f_3 - \frac{\tau}{2n(2n+1)}][g(\widetilde{C}(Y,Z)U, X)\xi \\ &- (f_1 - f_3 - \frac{\tau}{2n(2n+1)})\{g(Z,U)\eta(Y) \\ &- g(Y,U)\eta(Z)\}X], \end{aligned} \tag{2.5}$$

$$\begin{aligned} \widetilde{C}(\widetilde{C}(\xi, X)Y, Z)U &= [f_1 - f_3 - \frac{\tau}{2n(2n+1)}][(f_1 - f_3 - \frac{\tau}{2n(2n+1)})\{g(X,Y) \\ & g(Z,U)\xi - g(X,Y)\eta(U)Z\} - \eta(Y)\widetilde{C}(X,Z)U] \end{aligned} \tag{2.6}$$

and

$$\widetilde{C}(X,Y)\xi = [f_1 - f_3 - \frac{\tau}{2n(2n+1)}][\eta(Y)X - \eta(X)Y]. \tag{2.7}$$

Thus, substituting (2.3), (2.5) and (2.6) in (2.2) and after from necessary abbreviations, (2.2) takes from

$$\begin{aligned}[f_1 &- f_3 - \frac{\tau}{2n(2n+1)}][g(R(Y,Z)U,X) - (f_1 - f_3)\{g(X,Y)g(Z,U) \\ &- g(X,Z)g(Y,U)\}] = 0.\end{aligned}$$

This equation tells us that either $M^{2n+1}(f_1, f_2, f_3)$ is real space form with sectional curvature $(f_1 - f_3)$ or has the scalar curvature $\tau = (f_1 - f_3)2n(2n+1)$.

Conversely, if $M^{2n+1}(f_1, f_2, f_3)$ is either real space form with scalar curvature $(f_1 - f_3)$ or it has the scalar curvature $\tau = (f_1 - f_3)2n(2n+1)$, then we can see that (2.2) is satisfied. □

Theorem 2.2. *A generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ satisfies the condition*

$$\widetilde{C}(\xi, X)S = 0 \tag{2.8}$$

if and only if either $M^{2n+1}(f_1, f_2, f_3)$ has the scalar curvature $\tau = (f_1 - f_3)2n(2n+1)$ or is an Einstein manifold.

Proof. The condition $\widetilde{C}(\xi, X)S = 0$ implies that

$$S(\widetilde{C}(\xi, X)Y, Z) + S(Y, \widetilde{C}(\xi, X)Z) = 0, \tag{2.9}$$

for any vector fields X, Y, Z on M. Substituting (2.3) in (2.9), we obtain

$$\begin{aligned}[f_1 &- f_3 - \frac{\tau}{2n(2n+1)}][g(X,Y)S(\xi,Z) - \eta(Y)S(X,Z) + g(X,Z)S(Y,\xi) \\ &- \eta(Z)S(X,Y)] = 0.\end{aligned}$$

For $Z = \xi$, the last equation is equivalent to

$$[f_1 - f_3 - \frac{\tau}{2n(2n+1)}][S(X,Y) - (f_1 - f_3)g(X,Y)] = 0,$$

which proves our assertion. □

Theorem 2.3. *A generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ satisfies the condition*

$$\widetilde{C}(\xi, X)R = 0$$

if and only if the functions f_2 and f_3 either satisfy the condition $(2n-1)f_3 + 3f_2 = 0$ or it has the sectional curvature $(f_1 - f_3)$.

Proof. The condition $\widetilde{C}(\xi, X)R = 0$ yields to

$$\begin{aligned}\widetilde{C}(\xi,X)R(Y,Z)U &- R(\widetilde{C}(\xi,X)Y,Z)U - R(Y,\widetilde{C}(\xi,X)Z)U \\ &- R(Y,Z)\widetilde{C}(\xi,X)U = 0,\end{aligned} \tag{2.10}$$

for any vector fields X, Y, Z, U on M. In view of (2.3), we obtain

$$\begin{aligned}\widetilde{C}(\xi,X)R(Y,Z)U &= [f_1 - f_3 - \frac{\tau}{2n(2n+1)}][g(R(Y,Z)U,X)\xi \\ &- (f_1 - f_3)\{g(Z,U)\eta(Y) - g(Y,U)\eta(Z)\}X].\end{aligned} \tag{2.11}$$

On the other hand, by direct calculations, we have

$$\begin{aligned} R(\widetilde{C}(\xi,X)Y,Z)U &= [f_1-f_3-\frac{\tau}{2n(2n+1)}][(f_1-f_3)g(X,Y)\{g(Z,U)\xi-\eta(U)Z\} \\ &- \eta(Y)R(X,Z)U] \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} R(Y,Z)\widetilde{C}(\xi,X)U &= [f_1-f_3-\frac{\tau}{2n(2n+1)}][(f_1-f_3)g(X,U)\eta(Z)Y \\ &- (f_1-f_3)g(X,U)\eta(Y)Z-\eta(U)R(Y,Z)X]. \end{aligned} \tag{2.13}$$

Substituting (2.11), (2.12) and (2.13) in (2.10), we arrive at

$$\begin{aligned} 0 &= [f_1-f_3-\frac{\tau}{2n(2n+1)}]\{g(R(Y,Z)U,X)\xi-(f_1-f_3)g(Z,U)\eta(Y)X \\ &+ (f_1-f_3)g(Y,U)\eta(Z)X-(f_1-f_3)g(X,Y)g(Z,U)\xi+(f_1-f_3)g(X,Y)\eta(U)Z \\ &+ \eta(Y)R(X,Z)U+(f_1-f_3)g(X,Z)g(Y,U)\xi-(f_1-f_3)g(X,Z)\eta(U)Y \\ &+ \eta(Z)R(Y,X)U-(f_1-f_3)g(X,U)\eta(Z)Y+(f_1-f_3)g(X,U)\eta(Y)Z \\ &+ \eta(U)R(Y,Z)X\}, \end{aligned}$$

which implies that

$$\begin{aligned} &[f_1-f_3-\frac{\tau}{2n(2n+1)}][g(R(Y,Z)U,X)-(f_1-f_3)\{g(Z,U)g(X,Y) \\ &- g(Y,U)g(X,Z)\}]=0. \end{aligned}$$

There exist two cases. Either

$$g(R(Y,Z)U,X)-(f_1-f_3)\{g(Z,U)g(X,Y)-g(Y,U)g(X,Z)\}=0,$$

which say us $M^{2n+1}(f_1,f_2,f_3)$ has the sectional curvature (f_1-f_3) or the scalar curvature $\tau=(f_1-f_3)2n(2n+1)$. By corresponding (1.15), we get $(2n-1)f_3+3f_2=0$. □

Theorem 2.4. *A generalized Sasakian-space-form $M^{2n+1}(f_1,f_2,f_3)$ satisfies the condition*

$$\widetilde{C}(\xi,X)P=0$$

if and only if $M^{2n+1}(f_1,f_2,f_3)$ has either the sectional curvature (f_1-f_3) or the functions f_2 and f_3 are linearly dependent such that $(2n-1)f_3+3f_2$=0.

Proof. The condition $\widetilde{C}(\xi,X)P=0$ implies that

$$\begin{aligned} (\widetilde{C}(\xi,X)P)(Y,Z,U) &= \widetilde{C}(\xi,X)P(Y,Z)U-P(\widetilde{C}(\xi,X)Y,Z)U \\ &- P(Y,\widetilde{C}(\xi,X)Z)U-P(Y,Z)\widetilde{C}(\xi,X)=0, \end{aligned} \tag{2.14}$$

for any vector fields X,Y,Z,U on M.

In view of (1.11), we obtain from (1.19)

$$\eta(P(X,Y)Z)=0. \tag{2.15}$$

From (2.1), (2.4) and (2.14), we reach

$$\begin{aligned} \widetilde{C}(\xi,X)P(Y,Z)U &= [f_1-f_3-\frac{\tau}{2n(2n+1)}][g(R(Y,Z)U,X) \\ &- \frac{1}{2n}\{g(Z,U)S(Y,X)-g(Y,U)S(X,Z)\}]\xi \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} P(\widetilde{C}(\xi,X)Y,Z)U &= [f_1 - f_3 - \frac{\tau}{2n(2n+1)}][(f_1-f_3)g(X,Y)g(Z,U)\xi \\ &- \frac{1}{2n}g(X,Y)S(Z,U)\xi - \eta(Y)P(X,Z)U]. \end{aligned} \tag{2.17}$$

Finally, we conclude that

$$P(Y,Z)\widetilde{C}(\xi,X)U = -[f_1 - f_3 - \frac{\tau}{2n(2n+1)}]\eta(U)P(Y,Z)X. \tag{2.18}$$

So, substituting (2.16), (2.17) and (2.18) in (2.14), we can infer

$$\begin{aligned} 0 &= [f_1 - f_3 - \frac{\tau}{2n(2n+1)}][g(R(Y,Z)U,X)\xi - \frac{1}{2n}\{g(Z,U)S(Y,X) \\ &- g(Y,U)S(X,Z)\}\xi - (f_1-f_3)g(X,Y)g(Z,U)\xi + \frac{1}{2n}g(X,Y)S(Z,U)\xi \\ &+ \eta(Y)P(X,Z)U + (f_1-f_3)g(X,Z)g(Y,U)\xi + \eta(Z)P(Y,X)U \\ &- \frac{1}{2n}g(X,Z)S(Y,U)\xi + \eta(U)P(Y,Z)X]. \end{aligned}$$

Simplifying above the equation, we get

$$\begin{aligned} &[f_1 - f_3 - \frac{\tau}{2n(2n+1)}][g(R(Y,Z)U,X) + (f_1-f_3)\{g(X,Z)g(Y,U) \\ &- g(X,Y)g(Z,U)\} + \frac{1}{2n}\{g(Y,U)S(X,Z) - g(Z,U)S(Y,X) + g(X,Y)S(Z,U) \\ &- g(X,Z)S(Y,U)\}] = 0. \end{aligned} \tag{2.19}$$

Here, taking into account of (1.14), then (2.19) can be rewritten as

$$\begin{aligned} &[f_1 - f_3 - \frac{\tau}{2n(2n+1)}][R(Y,Z)U - (f_1-f_3)\{g(Z,U)Y - g(Y,U)Z\} \\ &+ \frac{1}{2n}(3f_2 + 2nf_3 - f_3)\{g(Z,U)\eta(Y)\xi - g(Y,U)\eta(Z)\xi \\ &+ \eta(Y)\eta(U)Z - \eta(Z)\eta(U)Y\}] = 0. \end{aligned} \tag{2.20}$$

Taking $Y = \xi$ in (2.20) and making use of (1.19), we obtain

$$g(Z,U)\xi - \eta(U)\eta(Z)\xi + \eta(U)Z - \eta(Z)\eta(U)\xi = 0,$$

that is,

$$g(Z,U) - \eta(Z)\eta(U) = 0.$$

So, (2.20) reduce to

$$[f_1 - f_3 - \frac{\tau}{2n(2n+1)}][R(Y,Z)U - (f_1-f_3)\{g(Z,U)Y - g(Y,U)Z\}] = 0,$$

which proves our assertion. □

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