Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 6 Issue 1 (2014), Pages 34-37

AN ESTIMATE OF THE DOUBLE GAMMA FUNCTION

(COMMUNICATED BY FATON MEROVCI)

CRISTINEL MORTICI AND SORINEL DUMITRESCU

Abstract. The object of the present paper is to establish some bounds for the double gamma function.

1. Introduction

The double gamma function G, or the G-function satisfies

$$\ln G(x+1) = \left(-\frac{1}{2} + \ln\sqrt{2\pi}\right)x - \frac{\gamma+1}{2}x^2 + S(x) \tag{1.1}$$

for $x > 0$ where

$$S(x) = \sum_{k=1}^{\infty}\left[k\ln\left(1+\frac{x}{k}\right) - x + \frac{x^2}{2k}\right]. \tag{1.2}$$

See, e.g., [5]. The G-function is closely related to the Euler gamma function

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-tx}dt, \quad x > 0,$$

since $G(1) = 1$ and $G(x+1) = \Gamma(x)G(x)$, for $x > 0$ and $G(n+2) = 1!2!\cdots n!$, for all positive integers n. The double gamma function is also called the Barnes G-function since it was introduced by Barnes [1–3].

Batir [4, Theorem 2.2] estimated $S(x)$ from (1.2) via some convexity arguments and obtained some double inequalities for the G-function.

The aim of this note is to give a different method for estimating $S(x)$ and consequently to establish the error estimate made in the approximation formula

$$\ln G(x+1) \approx \left(-\frac{1}{2} + \ln\sqrt{2\pi}\right)x - \frac{\gamma+1}{2}x^2 + S_n(x),$$

where

$$S_n(x) = \sum_{k=1}^{n}\left[k\ln\left(1+\frac{x}{k}\right) - x + \frac{x^2}{2k}\right].$$

Precisely, we give the following

2010 *Mathematics Subject Classification.* 26D15, 33B15, 26D07.
Key words and phrases. Gamma function, Double gamma function, Approximations, Inequalities.

Submitted January 10, 2011. Published February 14, 2014.

Theorem 1.1. *Let*

$$\varepsilon_n(x) = \ln G(x+1) - \left\{ \left(-\frac{1}{2} + \ln\sqrt{2\pi} \right) x - \frac{\gamma+1}{2} x^2 + S_n(x) \right\}.$$

Then for every $x > \sqrt[3]{3}$, there exists a positive integer $n(x)$ such that

$$\frac{x^3}{3n + \frac{x^{12}(3x+4)}{216}} \le \varepsilon_n(x) \le \frac{x^3}{3n}, \quad n \ge n(x).$$

(the right-hand side inequality holds for all $x > 0$ and integers $n \ge 1$).

2. The Proofs

We first give the following

Lemma 2.1. *For every $x > \sqrt[3]{3}$, there exists a positive integer $n(x)$ such that for all $n \ge n(x)$, it holds*

$$\begin{aligned} & \frac{x^3}{3n + \frac{x^{12}(3x+4)}{216}} - \frac{x^3}{3(n+1) + \frac{x^{12}(3x+4)}{216}} \\ < \; & (n+1)\ln\left(1 + \frac{x}{n+1}\right) - x + \frac{x^2}{2(n+1)} \\ < \; & \frac{x^3}{3n} - \frac{x^3}{3(n+1)}. \end{aligned} \tag{2.1}$$

(the right-hand side inequality holds for all $x > 0$ and integers $n \ge 1$).

Proof. Let

$$f(t) = (t+1)\ln\left(1 + \frac{x}{t+1}\right) - x + \frac{x^2}{2(t+1)} - \left(\frac{x^3}{3t} - \frac{x^3}{3(t+1)} \right),$$

with

$$\begin{aligned} f''(t) &= -\frac{x^3\left(10t + 4x + 16tx + 20t^2 + 12t^3 + 2x^2 + 6tx^2 + 24t^2x + 9t^3x + 6t^2x^2 + 2\right)}{3t^3(t+1)^3(t+x+1)^2} \\ &< 0. \end{aligned}$$

Now f is strictly concave, with $f(\infty) = 0$, so $f(t) < 0$, for all $t > 0$. This completely justifies the right-hand side inequality (2.1).

Let

$$g(t) = (t+1)\ln\left(1 + \frac{x}{t+1}\right) - x + \frac{x^2}{2(t+1)} - \left(\frac{x^3}{3t + \frac{x^{12}(3x+4)}{216}} - \frac{x^3}{3(t+1) + \frac{x^{12}(3x+4)}{216}} \right),$$

with

$$g''(t) = \frac{x^3 P(t)}{(t+x+1)^2 (t+1)^3 (648t + 4x^{12} + 3x^{13})^3 (648t + 4x^{12} + 3x^{13} + 648)^3},$$

where $P(t) = \sum_{k=0}^{6} a_k(x) t^k$, having the leading coefficient

$$a_6(x) = 914039610015744\,(3x+4)\left(x^3+3\right)\left(x^3-3\right)\left(x^6+9\right).$$

For $x > \sqrt[3]{3}$, we have $a_6(x) > 0$, so we can find a positive integer $n(x)$ such that $P(t) > 0$, for all $t \ge n(x)$.

Now $g''(t) > 0$, for all $t \ge n(x)$, so g is strictly convex on $[n(x), \infty)$. But $g(\infty) = 0$, so $g(t) > 0$, for all $t \ge n(x)$ and the left-hand side of (2.1) follows. □

Proof of Theorem 1. Inequality (2.1) can be written as

$$\frac{x^3}{3n+\frac{x^{12}(3x+4)}{216}}-\frac{x^3}{3(n+1)+\frac{x^{12}(3x+4)}{216}}<S_{n+1}(x)-S_n(x)<\frac{x^3}{3n}-\frac{x^3}{3(n+1)}.$$

By adding these telescoping inequalities from $n\geq n(x)$ to $n+p-1$, we deduce

$$\frac{x^3}{3n+\frac{x^{12}(3x+4)}{216}}-\frac{x^3}{3(n+p)+\frac{x^{12}(3x+4)}{216}}<S_{n+p}(x)-S_n(x)<\frac{x^3}{3n}-\frac{x^3}{3(n+p)},$$

then taking the limit as $p\to\infty$, we get

$$\frac{x^3}{3n+\frac{x^{12}(3x+4)}{216}}\leq S(x)-S_n(x)\leq\frac{x^3}{3n}.$$

Now the conclusion follows since $\varepsilon_n(x)=S(x)-S_n(x)$. □

3. A power series proof

In this concluding section we give an alternative proof of (2.1). In fact, we show how increasingly better estimates of

$$\phi_x(n)=(n+1)\ln\left(1+\frac{x}{n+1}\right)-x+\frac{x^2}{2(n+1)}$$

can be obtained by truncation of the associated power series. As before, we assume in this section that x is arbitrary, but fixed positive number. By standard computations, or better by using a computer software for symbolic computations such as Maple, we deduce that

$$\begin{aligned}\phi_x(n) &= \frac{1}{3n^2}x^3-\frac{1}{12n^3}x^3(3x+8)+\frac{1}{20n^4}x^3\left(15x+4x^2+20\right)\\ &\quad -\frac{1}{30n^5}x^3\left(45x+24x^2+5x^3+40\right)+\frac{1}{42n^6}x^3\left(105x+84x^2+35x^3+6x^4+70\right)\\ &\quad +O\left(\frac{1}{n^7}\right).\end{aligned}$$

Evidently,

$$\lim_{n\to\infty}n^3\left(\phi_x(n)-\frac{1}{3n^2}x^3\right)=-\frac{1}{12}x^3(3x+8)<0,$$

so there is a positive integer $m=m(x)$ such that

$$\phi_x(n)<\frac{1}{3n^2}x^3,$$

for every $n\geq m$. By similar arguments, we can state the following inequality

$$\begin{aligned}&\frac{1}{3n^2}x^3-\frac{1}{12n^3}x^3(3x+8)+\frac{1}{20n^4}x^3\left(15x+4x^2+20\right)-\frac{1}{30n^5}x^3\left(45x+24x^2+5x^3+40\right)\\ <\ &\phi_x(n)\\ <\ &\frac{1}{3n^2}x^3-\frac{1}{12n^3}x^3(3x+8)+\frac{1}{20n^4}x^3\left(15x+4x^2+20\right)\end{aligned}$$

for values of n greater than an initial value n_0, which is a stronger inequality than (2.1). For the lower term, we have

$$\left(\frac{x^3}{3n+\frac{x^{12}(3x+4)}{216}}-\frac{x^3}{3(n+1)+\frac{x^{12}(3x+4)}{216}}\right)$$
$$-\left(\frac{1}{3n^2}x^3-\frac{1}{12n^3}x^3(3x+8)+\frac{1}{20n^4}x^3\left(15x+4x^2+20\right)-\frac{1}{30n^5}x^3\left(45x+24x^2+5x^3+40\right)\right)$$
$$=-\frac{x^3A(x)}{60n^5\left(648n+4x^{12}+3x^{13}\right)\left(648n+4x^{12}+3x^{13}+648\right)}<0,$$

where $A(x)=\left(77\,760x^{13}+103\,680x^{12}-6298\,560x-8398\,080\right)n^4+\cdots$ is a fourth degree polynomial in n, with positive leading coefficient when $x\geq 2$.

For the upper term in (2.1), we have

$$\left(\frac{1}{3n^2}x^3-\frac{1}{12n^3}x^3(3x+8)+\frac{1}{20n^4}x^3\left(15x+4x^2+20\right)\right)$$
$$-\left(\frac{x^3}{3n}-\frac{x^3}{3(n+1)}\right)$$
$$=\quad-\frac{x^3B(x)}{60n^4(n+1)}<0,$$

where $B(x)=(15x+20)n^2+\left(-12x^2-30x-20\right)n-\left(12x^2+45x+60\right)$.

Our assertion is now completely proved.

Acknowledgement. The work of the first author was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0087. Some computations in this paper were performed using Maple software.

References

[1] Barnes E. W., *The theory of G-function*. Quart. J. Math. 31(1899), 264-314.

[2] Barnes E. W., *On the theory of multiple gamma function*. Trans. Cambridge Philos. Soc. 19(1904), 374-439.

[3] Barnes E. W., *Genesis of the double gamma function*. Proc. London Math. Soc. 31(1900), 358-381.

[4] Batir N., *Inequalities for the double gamma function*. J. Math. Anal. Appl. 351 (2009) 182–185.

[5] Ferreira C. and Lopez J. L., *An asymptotic expansion of the double gamma function*. J. Approx. Theory 111(2001), 298-314.

Valahia University of Târgovişte,, Department of Mathematics, Bd. Unirii 18, 130082 Târgovişte/ROMANIA

E-mail address: cristinel.mortici@hotmail.com

Ph. D. Student, University Politehnica of Bucharest,, Splaiul Independenţei 313, Bucharest/ROMANIA

E-mail address: sorineldumitrescu@yahoo.com