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FAREY-PELL SEQUENCE, APPROXIMATION TO IRRATIONALS AND HURWITZ'S INEQUALITY

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Abstract. The purpose of this paper is to give the notion of Farey-Pell sequence. We investigate some identities of the Farey-Pell sequence. Finally, a generalization of Farey-Pell sequence and an approximation to irrationals via Farey-Pell fractions are given.

1. Introduction

A Farey sequence of order n is the sequence of fractions $\frac{p}{q}$, where p and q are coprimes and $0 \leq p < q \leq n$, arranged in order of increasing size. This sequence has a rich history and properties. Let $\frac{a}{b}$ and $\frac{c}{d}$ be two consecutive elements of this sequence. A well-known identity of the Farey sequence is $bc - ad = 1$ which is known as "neighbor identity".

Alladi [2] defined the concept of Farey sequence of Fibonacci numbers. In this study, Alladi investigate some relations between Farey Fibonacci fractions. Then, Matyas [4] generalized the idea of Farey Fibonacci sequence and gave the sufficient conditions for the sequence $G_k = AG_{k-1} + BG_{k-2}$ so that the properties of points of symmetry hold in this more general setting. Alladi [3] gave the best approximation to irrational numbers by Farey Fibonacci fractions.

In this paper, we go a step further. We define Farey-Pell sequence using Pell numbers. Although it seems that the identities in Farey Fibonacci sequence have to hold, we need to define the concept of "center of interval", since Pell sequence does not hold the same recurrence with Fibonacci sequence. Afterwards, we give the approximation to irrational numbers via Farey-Pell fractions. Thus, it will be convenient to give the definition of Pell sequence.

For $n \geq 2$, the Pell sequence $\{P_n\}$ is defined by the following recurrence relation $P_n = 2P_{n-1} + P_{n-2}$ with the initial conditions $P_0 = 0$ and $P_1 = 1$. A few Pell numbers are $0, 1, 2, 5, 12, 29, 70, \ldots$. Its Binet formula is known as

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

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where α, β are the roots of the characteristic equation $x^2 - 2x - 1 = 0$.

2. Farey-Pell Sequence and its identities

Definition 2.1. *The Farey-Pell sequence of order P_n is the set of all possible fractions $\frac{P_i}{P_j}$, $0 \leq P_i \leq P_j \leq P_n$ arranged in ascending order of size. $0/P_{n-1}$ is the first fraction of the sequence. This sequence is denoted by $\mathcal{FP}_n$. Further, we denote the r^{th} fraction of the $\mathcal{FP}_n$ by $\mathcal{FP}_{(r)n}$.*

We give a Farey-Pell sequence of order 29 as follows,

$$\frac{0}{12}, \frac{1}{29}, \frac{2}{29}, \frac{1}{12}, \frac{2}{12}, \frac{5}{29}, \frac{1}{5}, \frac{2}{5}, \frac{12}{29}, \frac{5}{12}, \frac{1}{2}, \frac{2}{2}.$$

Definition 2.2. *We define the fraction $\mathcal{FP}_{(r)n}$ as a point of symmetry if $\mathcal{FP}_{(r-1)n}$ and $\mathcal{FP}_{(r+2)n}$ have the same denominator.*

Definition 2.3. *An $\mathcal{FP}$ interval contains the set of all $\mathcal{FP}_n$ fractions between two consecutive points of symmetry.*

Definition 2.4. *We define the center of an $\mathcal{FP}$ interval as a fraction that has the greatest denominator of all fractions in the interval.*

Definition 2.5. [2] *We define the distance between $\mathcal{FP}_{(r)k}$ and $\mathcal{FP}_{(k)n}$ as $|r - k|$.*

Definition 2.6. [1] *Weighted mediant of two fractions $\frac{a}{b}$ and $\frac{c}{d}$ is defined as*

$$\frac{2a + c}{2b + d} \text{ or } \frac{a + 2c}{b + 2d}.$$

Theorem 2.7. *$\mathcal{FP}_{(r+k+1)n}$ and $\mathcal{FP}_{(r-k)n}$ have the same denominator if and only if one of them coincides with a point of symmetry.*

Proof. In the $\mathcal{FP}_n$ sequence, the terms are arranged in the following sense. The terms in the last $\mathcal{FP}$ interval are of the form $\frac{P_{j-1}}{P_j}$. The terms in the $\mathcal{FP}$ interval previous to the last $\mathcal{FP}$ interval are of the form $\frac{P_{j-2}}{P_j}, \ldots$. The weighted mediant of two fractions $\frac{P_{i-1}}{P_{j-1}}$ and $\frac{P_{i-2}}{P_{j-2}}$, which is $\frac{P_i}{P_j}$, lies in between them. That is,

$$\text{If } \frac{P_{i-1}}{P_{j-1}} < \frac{P_{i-2}}{P_{j-2}} \text{ then } \frac{P_{i-1}}{P_{j-1}} < \frac{P_i}{P_j} < \frac{P_{i-2}}{P_{j-2}}$$
$$\text{If } \frac{P_{i-2}}{P_{j-2}} < \frac{P_{i-1}}{P_{j-1}} \text{ then } \frac{P_{i-2}}{P_{j-2}} < \frac{P_i}{P_j} < \frac{P_{i-1}}{P_{j-1}}.$$

We use induction as the method of proof. It is true for all $\mathcal{FP}_n$ sequence up to 169. Let us consider 169 as P_{n-1}. For the $\mathcal{FP}_n$ sequence of order P_n, the following fractions need to be placed:

$$\frac{P_2}{P_n}, \frac{P_3}{P_n}, \ldots, \frac{P_i}{P_n}, \ldots, \frac{P_{n-1}}{P_n}.$$

$\frac{P_i}{P_n}$ will be exactly between $\frac{P_{i-1}}{P_{n-1}}$ and $\frac{P_{i-2}}{P_{n-2}}$. Assume that $\frac{P_{i-1}}{P_{n-1}} < \frac{P_i}{P_n} < \frac{P_{i-2}}{P_{n-2}}$. The distance of $\frac{P_i}{P_n}$ from the point of symmetry, say $\frac{1}{P_j}$, is one more than the distance $\frac{P_{i-1}}{P_n}$ from that point of symmetry. Hence it is true for 408. Continuing in this way, it can be shown for $985, 2378, \ldots$. □

Theorem 2.8. *For an $\mathcal{FP}$ interval $[\frac{1}{P_i}, \frac{1}{P_{i-1}}]$, the denominator of the fraction next to $\frac{1}{P_i}$ is P_i, and the denominator of the next fraction is P_{i+2}, then P_{i+4},... till we reach the center of the $\mathcal{FP}_n$ sequence, i.e., until P_{i+2k} does not exceed P_n. Then the denominator of the fraction after P_{i+2k} will be the maximum possible fraction not greater than P_n and not equal to any of the terms created. Thus it is either P_{i+2k+1} or P_{i+2k-1}, say P_j. After P_j, the denominator of the fractions will be $P_{j-2}, P_{j-4}, \ldots$ till we reach $\frac{1}{P_{i-1}}$.*

Proof. By induction on Theorem 2.7 gives the result. □

Theorem 2.9. *$\frac{h}{k}, \frac{h'}{k'}, \frac{h''}{k''}$ are three consecutive fractions in $\mathcal{FP}_n$ such that neither of them is center of any $\mathcal{FP}$ interval or point of symmetry. Then the following holds:*

$$\frac{h'}{k'} = \frac{h + h''}{k + k''}.$$

Proof. If $\frac{h}{k}, \frac{h'}{k'}, \frac{h''}{k''}$ are three consecutive fractions in any $\mathcal{FP}$ interval which are not center of $\mathcal{FP}$ interval and point of symmetry, then there are two possibilities.

Case 1:
Let $\frac{h}{k} = \frac{P_{i-2}}{P_{j-2}}$, $\frac{h'}{k'} = \frac{P_i}{P_j}$ and $\frac{h''}{k''} = \frac{P_{i+2}}{P_{j+2}}$. In this case, it is obvious that

$$\frac{h + h''}{k + k''} = \frac{6P_i}{6P_j} = \frac{P_i}{P_j}.$$

Case 2:
Similarly, if $\frac{h}{k} = \frac{P_{i+2}}{P_{j+2}}$, $\frac{h'}{k'} = \frac{P_i}{P_j}$ and $\frac{h''}{k''} = \frac{P_{i-2}}{P_{j-2}}$, then we have

$$\frac{h'}{k'} = \frac{h + h''}{k + k''}$$

□

When one of the three consecutive fractions is the center of an $\mathcal{FP}$ interval, we have the following result.

Theorem 2.10. *If one of the three consecutive fractions $\frac{h}{k}$, $\frac{h'}{k'}$ and $\frac{h''}{k''}$ is the center of any interval, then we have the followings:*

(1) *If $\frac{h'}{k'}$ is the center of $\mathcal{FP}$ interval, then we have either*

$$\frac{h'}{k'} = \frac{2h'' + h}{2k'' + k} \qquad or \qquad \frac{h'}{k'} = \frac{h'' + 2h}{k'' + 2k}.$$

(2) *If $\frac{h}{k}$ is the center of $\mathcal{FP}$ interval, then we have*

$$\frac{h'}{k'} = \frac{h'' + 2h}{k'' + 2k} \qquad or \qquad \frac{h'}{k'} = \frac{h'' + h}{k'' + k}.$$

(3) *If $\frac{h''}{k''}$ is the center of $\mathcal{FP}$ interval, then we have*

$$\frac{h'}{k'} = \frac{h'' + h}{k'' + k} \qquad or \qquad \frac{h'}{k'} = \frac{2h'' + h}{2k'' + k}.$$

Proof. We give the proof of first item. Let $\frac{h^{'}}{k'}$ be the center of any $\mathcal{FP}$ interval. Then two cases may be due to the ordering of the sequence $\mathcal{FP}_n$. Firstly, if $\frac{h}{k} = \frac{P_{i-2}}{P_{j-2}}$, $\frac{h^{'}}{k'} = \frac{P_i}{P_j}$ and $\frac{h^{''}}{k''} = \frac{P_{i+1}}{P_{j+1}}$, then we obtain

$$\frac{h^{'}}{k'} = \frac{2h^{''} + h}{2k'' + k}.$$

by the recurrence of Pell sequence. For the second case, that is, if $\frac{h}{k} = \frac{P_{i-1}}{P_{j-1}}$, $\frac{h^{'}}{k'} = \frac{P_i}{P_j}$ and $\frac{h^{''}}{k''} = \frac{P_{i+2}}{P_{j+2}}$, we have

$$\frac{h^{'}}{k'} = \frac{h^{''} + 2h}{k'' + 2k}.$$

The other items can be proven similarly. □

Theorem 2.11. *Let $\frac{h}{k}, \frac{1}{P_i}, \frac{2}{P_i}, \frac{h'}{k'}$ be consecutive fractions of an $\mathcal{FP}_n$ sequence, then*

$$\frac{1}{P_i} = \frac{hP_{i-2} + h'P_{i-1}}{kP_{i-2} + 2k'P_{i-1}}.$$

Proof. From Theorem 2.8, it follows that $\frac{h}{k} = \frac{5}{P_{i+2}}$ and $\frac{h'}{k'} = \frac{12}{P_{i+2}}$. Thus

$$\frac{1}{P_i} = \frac{P_{i+2}}{P_iP_{i+2}} = \frac{5P_{i-2} + 12P_{i-1}}{P_{i+2}P_{i-2} + 2P_{i+2}P_{i-1}} = \frac{hP_{i-2} + h'P_{i-1}}{kP_{i-2} + 2k'P_{i-1}},$$

and

$$\frac{2}{P_i} = \frac{2P_{i+2}}{P_iP_{i+2}} = \frac{2\left(5P_{i-2} + 12P_{i-1}\right)}{P_{i+2}P_{i-2} + 2P_{i+2}P_{i-1}} = \frac{2\left(hP_{i-2} + h'P_{i-1}\right)}{kP_{i-2} + 2k'P_{i-1}}.$$

□

Now, we present a theorem which gives a relation about three consecutive fractions depending upon including center of an $\mathcal{FP}$ interval or not.

Theorem 2.12. *Let $\frac{h}{k}, \frac{h^{'}}{k'}, \frac{h^{''}}{k''}$ be three consecutive fractions in an $\mathcal{FP}$ interval such that $\frac{1}{P_i} < \frac{h}{k} < \frac{h^{'}}{k'} < \frac{h^{''}}{k''} < \frac{1}{P_{i-1}}$ for $i \geq 2$. Then the following holds for an $\mathcal{FP}_n$ sequence:*

1. *If $\frac{h'}{k'}$ is the center of an $\mathcal{FP}$ interval, say $\frac{h'}{k'} = \frac{P_i}{P_n}$, then either $\frac{h}{k}$ or $\frac{h^{''}}{k''}$ is equal to $\frac{P_{i-1}}{P_{n-1}}$, say $\frac{h}{k} = \frac{P_{i-1}}{P_{n-1}}$, and*
$$kh^{'} - k^{'}h = P_{i-2}, \quad k'h^{''} - k^{''}h' = 2P_{i-2}.$$
2. *If neither of $\frac{h}{k}$ and $\frac{h'}{k'}$ is the center of an $\mathcal{FP}$ interval, then*
$$kh^{'} - k^{'}h = 2P_{i-2}.$$

Proof. Since the proof depends on the proof of the Theorem 2.10, we omit the proof to cut unnecessary repetition. □

Corollary 2.13. *If $\frac{h}{k}, \frac{h^{'}}{k'}, \frac{h^{''}}{k''}$ are three consecutive fractions in an $\mathcal{FP}$ interval such that $\frac{1}{P_i} < \frac{h}{k} < \frac{h^{'}}{k'} < \frac{h^{''}}{k''} < \frac{1}{P_{i-1}}$, then we have*

$$kh^{'} - k^{'}h \leq 2P_{i-2}. \tag{2.1}$$

Proof. It is clear from Theorem 2.12. □

Lemma 2.14. *If* $j_1 - i_1 = j_2 - i_2 > 0$, *then*

$$|P_{j_1}P_{i_2} - P_{j_2}P_{i_1}| = P_{|j_2-j_1|}P_{j_1-j_1} = P_{_{|j_2-j_1|}}P_{_{j_2-j_2}}.$$

Proof. We apply Binet's formula that

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

where

$$\alpha = 1 + \sqrt{2}, \beta = 1 - \sqrt{2}.$$

Using this formula the proof can be done similar to [2] Lemma 2.1. □

Corollary 2.15. [2] *If* $\frac{P_{i_1}}{P_{j_1}}$ *and* $\frac{P_{i_2}}{P_{j_2}}$ *are in the same* $\mathcal{FP}$ *interval, then*

$$P_{j_1}P_{i_2} - P_{j_2}P_{i_1} = P_{|j_2-j_1|}P_{j_2-i_2} = P_{|j_2-j_1|}P_{j_1-i_1}.$$

$$\frac{P_{i_1}}{P_{j_1}} < \frac{P_{i_2}}{P_{j_2}}.$$

Hence

$$|P_{j_1}P_{i_2} - P_{j_2}P_{i_1}|$$

is an integral multiple of $P_{j_1-i_1}$ *or* $P_{j_2-i_2}$. *Here,* $P_{j_1-i_1}$ *(or* $P_{j_2-i_2}$*) is the term obtained by the difference in indices of the numerator and denominator of each fraction of that* $\mathcal{FP}$ *interval.*

Definition 2.16. [2] *For an* $\mathcal{FP}$ *interval* $\left[\frac{1}{P_i}, \frac{1}{P_{i-1}}\right]$ *and two fractions* $\frac{a}{b}$ *and* $\frac{a'}{b'}$ *in this* $\mathcal{FP}$ *interval, if the distance of* $\frac{a}{b}$ *from* $\frac{1}{P_i}$ *equals the distance of* $\frac{a'}{b'}$ *from* $\frac{1}{P_{i-1}}$, *then these two fractions are called conjugate fractions in this* $\mathcal{FP}$ *interval.*

Theorem 2.17. *If* $\frac{a}{b}$ *and* $\frac{a'}{b'}$ *are conjugate in an* $\mathcal{FP}$ *interval* $\left[\frac{1}{P_i}, \frac{1}{P_{i-1}}\right]$ *then*

$$ba' - b'a = P_{i-2}.$$

Proof. Let $\frac{a}{b} = \frac{P_k}{P_j}$. Then since $\frac{a}{b}$ and $\frac{a'}{b'}$ are conjugate fractions, we have $\frac{a'}{b'} = \frac{P_{k+1}}{P_{j+1}}$. Thus, we obtain

$$ba' - b'a = P_jP_{k+1} - P_{j+1}P_k = P_1P_{j-k} = P_{i-2}$$

by Corollary 2.15. □

Definition 2.18. [2] *Let* $\frac{a}{b}$ *be a fraction in the* $\mathcal{FP}$ *interval* $\left(\frac{1}{P_i}, \frac{1}{P_{i-1}}\right)$. *The couplet for* $\frac{a}{b}$ *is the ordered* $\mathcal{FP}$ *pair*

$$\left[\left(\frac{1}{P_i}, \frac{a}{b}\right), \left(\frac{a}{b}, \frac{1}{P_{i-1}}\right)\right].$$

Theorem 2.19. *Let*

$$\left[\left(\frac{1}{P_i}, \frac{a}{b}\right), \left(\frac{a}{b}, \frac{1}{P_{i-1}}\right)\right].$$

be a couplet for $\frac{a}{b}$. *Then we have*

$$aP_i - 2b = P_kP_{i-2}$$

$$b - aP_{i-1} = P_{k+1}P_{i-2}$$

where P_k *is the* k^{th} *Pell number.*

Proof. Let $\frac{a}{b}=\frac{P_{j-i+2}}{P_j}$. Then by Corollary 2.15, aP_i-2b is

$$P_{j-i+2}P_i-2P_j=P_kP_{i-2}$$

and $b-aP_{i-1}$ is

$$P_j-P_{i-1}P_{j-i+2}=P_{k+2}P_{i-2}.$$

Multiply above equation by 2 then adding the equations, we obtain

$$P_{i-2}P_{j-i+2}=P_{k+2}P_{i-2}.$$

Thus $P_{j-i+2}=P_{k+2}$ or $j-i=k$, i.e.,

$$P_{j-i+2}P_i-2P_j=P_{j-i}P_{i-2}.$$

We can establish the last equation by Lemma 2.14. □

Definition 2.20. [2] *If* $\frac{a}{b}$ *and* $\frac{a'}{b'}$ *are conjugate fractions of the* $\mathcal{FP}$ *interval* $\left(\frac{1}{P_i},\frac{1}{P_{i-1}}\right)$, *then*

$$\left[\left(\frac{1}{P_i},\frac{a}{b}\right),\left(\frac{a}{b},\frac{1}{P_{i-1}}\right)\right]\qquad and\qquad\left[\left(\frac{1}{P_i},\frac{a'}{b'}\right),\left(\frac{a'}{b'},\frac{1}{P_{i-1}}\right)\right]$$

are said to be conjugate couplets.

Theorem 2.21. *Let*

$$\left[\left(\frac{1}{P_i},\frac{a}{b}\right),\left(\frac{a}{b},\frac{1}{P_{i-1}}\right)\right]\qquad and\qquad\left[\left(\frac{1}{P_i},\frac{a'}{b'}\right),\left(\frac{a'}{b'},\frac{1}{P_{i-1}}\right)\right]$$

be conjugate couplets. If

$$aP_i-2b=P_kP_{i-2}\qquad and\qquad b-aP_{i-1}=P_{k+1}P_{i-2}$$

then

$$a'P_i-2b'=P_{k+1}P_{i-2}\qquad and\qquad b'-a'P_{i-1}=P_{k+2}P_{i-2}.$$

Proof. In Theorem 2.19, $j-i$ is the difference in the indices of P_j and $P_{i.}$ If $\frac{a}{b}=\frac{P_{j-i+2}}{P_j}$, then $k=j-i$. Since $\frac{a'}{b'}$ is conjugate with $\frac{a}{b}$, we have $\frac{a'}{b'}=\frac{P_{j-i+3}}{P_{j+1}}$. Thus, in the equation for $\frac{a'}{b'}$,

$$P_ia'-2b'=P_mP_{i-2},$$

the index of the constant factor P_m will be $m=j-i+1=k+1$. That is, $a'P_i-2b'=P_{k+1}P_{i-2}$. Therefore $b'-a'P_{i-1}=P_{k+2}P_{i-2}$, by Theorem 2.19. □

Theorem 2.22. *If* $\mathcal{FP}_{(r)n}$ *is a point of symmetry then* $r\in\{2,4,7,11,16,22,...\}$ *that is, the sequence of distance between two consecutive points of symmetry will be* $2,3,4,5,6,....$

Proof. We have to show that if there are n terms in an $\mathcal{FP}$ interval then there are $(n+1)$ terms in the next. Let there be k terms of the form P_i/P_j. Clearly there are $k+1$ terms of the form P_{i+1}/P_j and these fractions are in next to the $\mathcal{FP}$ interval in which the k terms of the form P_i/P_j lie. Thus the sequence is an arithmetic progression with common difference of 1. Also, the second term is always $1/P_n$. Hence we have the result. Here $j-i$ is assumed constant. □

2.1. **Generalized $\mathcal{FP}_n$ Sequence.** We defined the $\mathcal{FP}_n$ sequence in the interval $[0,1]$. We now define it in the interval $[0,\infty)$.

Definition 2.23. *The $\mathcal{FP}_n$ sequence of order P_n is the set of all fractions $\frac{P_i}{P_j}$, $j \leq n$, arranged in ascending order of magnitude $i,j \geq 0$. If $i \leq j$ then the $\mathcal{FP}_n$ sequence is in the interval $[0,1]$.*

Definition 2.24. *Suppose that $\mathcal{FP}_{(r)n} > 1$. If $\mathcal{FP}_{(r-2)n}$ and $\mathcal{FP}_{(r+1)n}$ have the same numerator then $\mathcal{FP}_{(r)n}$ is a point of symmetry.*

Definition 2.25. *A fraction with denominator P_n is called a center.*

Now, we give some results for $\mathcal{FP}_n$ sequence in the interval $[0,\infty)$. Their proofs can be done similarly to the proofs of the $\mathcal{FP}_n$ sequence in the interval $[0,1]$.

Theorem 2.26. *If $\mathcal{FP}_{(r)n} > 1$ is a point of symmetry, then $\mathcal{FP}_{(r+k)n}$ and $\mathcal{FP}_{(r-k-1)n}$ have the same numerator until one of them coincides with a center.*

Theorem 2.27. *A point of symmetry is the fraction which has either numerator or denominator 1.*

Theorem 2.28. *For fractions greater than 1, any $\mathcal{FP}$ interval is given by $[P_{n-1}, P_n]$.*

3. Approximation to irrationals with Farey-Pell fractions

In this section, we present the approximation of irrationals with Farey-Pell fractions.

Definition 3.1. *For any $\mathcal{FP}_n$, we form a new ordered set $\mathcal{FP}_{n,1}$ consisting all rationals in $\mathcal{FP}_n$ together with mediants[1] of consecutive rationals in $\mathcal{FP}_n$ and define $\mathcal{FP}_{n,r+1}$ as all rationals in $\mathcal{FP}_{n,r}$ with their mediants of consecutive rationals. We define*

$$\mathcal{PP}_n = \bigcup_{r=1}^{\infty} \mathcal{FP}_{n,r}.$$

Proposition 3.2. *$\mathcal{PP}_n$ is dense in the interval $[0,\infty)$. Thus we can approximate every irrational τ by a sequence of rationals $\frac{a}{b}$ in $\mathcal{PP}_n$.*

Let's take the irrational $\tau > 0$. For $\tau < 0$, we can approximate by $-\frac{a}{b}$, where $\frac{a}{b}$ in $\mathcal{PP}_n$.

Before giving further, we give a lemma which helps us to prove the main theorem for this section.

Lemma 3.3. *There does not exist integers x and y such that the inequalities*

$$\frac{1}{xy} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{x^2} + \frac{1}{y^2}\right)$$

and

$$\frac{1}{x\left(x^2+y^2\right)} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{x^2} + \frac{1}{x^2+y^2}\right)$$

simultaneously hold.

For the proof, one can see [5] (Lemma 1.4, pp. 5).

In the following theorems, we give a best approximation for some irrational numbers.

[1] If $a/b < c/d$, then $(a+c)/(b+d)$ is the mediant fraction of these fractions.

Theorem 3.4. *Suppose that τ is an irrational number between two consecutive fractions $\frac{a}{b}$ and $\frac{a'}{b'}$, where one of the fractions is the center of an $\mathcal{FP}$ interval, say $\frac{P_m}{P_n}$, and the other one is $\frac{P_{m-1}}{P_{n-1}}$, in the sequence $\mathcal{FP}_n$. Then for every integer $n \geq 2$, there exist infinitely many fractions $\frac{h}{k} \in \mathcal{PP}_n$ such that*

$$\left|\tau - \frac{h}{k}\right| < \frac{P_{i-2}}{\sqrt{5}k^2}. \tag{3.1}$$

Moreover, $\sqrt{5}$ is the best possible constant, i.e., if $\sqrt{5}$ is replaced by a bigger constant the assertion fails.

Proof. Assume that $\tau < 1$ and let τ is between two consecutive points of symmetry, say $\frac{1}{P_i}$ and $\frac{1}{P_{i-1}}$ and between two consecutive fractions $\frac{a}{b}$ and $\frac{a'}{b'}$, where one of the fractions is the center of an $\mathcal{FP}$ interval, say $\frac{P_m}{P_n}$, and the other one is $\frac{P_{m-1}}{P_{n-1}}$ in $\mathcal{FP}_n$. For each positive integers $n \geq 2$, there are successive fractions $\frac{x}{y}, \frac{u}{v}$ in $\mathcal{FP}_{n,r}$ so that $\tau \in \left[\frac{x}{y}, \frac{u}{v}\right]$. It is clear that

$$\frac{x}{y} < \frac{x+u}{y+v} < \frac{u}{v} \quad \text{and} \quad \frac{x+u}{y+v} \in \mathcal{FP}_{n,r+1}.$$

Then there are two cases, namely we have

$$\text{either } \frac{x}{y} < \frac{x+u}{y+v} < \tau < \frac{u}{v} \text{ or } \frac{x}{y} < \tau < \frac{x+u}{y+v} < \frac{u}{v} .$$

Assume that

$$\frac{x}{y} < \frac{x+u}{y+v} < \tau < \frac{u}{v}$$

and none of these fractions satisfy the inequality (3.1). Then

$$\tau - \frac{x}{y} \geq \frac{P_{i-2}}{\sqrt{5}y^2}; \quad \tau - \frac{x+u}{y+v} \geq \frac{P_{i-2}}{\sqrt{5}\left(y+v\right)^2}; \qquad \frac{u}{v} - \tau \geq \frac{P_{i-2}}{\sqrt{5}v^2}.$$

If we combine these three inequalities, we obtain

$$\frac{u}{v} - \frac{x}{y} \geq \frac{P_{i-2}}{\sqrt{5}}\left(\frac{1}{v^2} + \frac{1}{y^2}\right) \text{ and } \frac{u}{v} - \frac{x+u}{y+v} \geq \frac{P_{i-2}}{\sqrt{5}}\left(\frac{1}{v^2} + \frac{1}{y^2+v^2}\right)$$

From the inequality (2.1), we have

$$\frac{1}{vy} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{v^2} + \frac{1}{y^2}\right) \text{ and } \frac{1}{v\left(y+v\right)} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{v^2} + \frac{1}{y^2+v^2}\right)$$

which is a contradiction according to the Lemma 3.3. Therefore, one of the fractions, say $\frac{h}{k}$, satisfies the inequality

$$\left|\tau - \frac{h}{k}\right| < \frac{P_{i-2}}{\sqrt{5}k^2}.$$

The proof can be done similarly for the case $\frac{x}{y} < \tau < \frac{x+u}{y+v} < \frac{u}{v}$.

Using the process above, we can prove the theorem for $\tau > 1$. Thus for any irrational number τ which is between two consecutive fractions $\frac{a}{b}$ and $\frac{a'}{b'}$, where one of the fractions is the center of an $\mathcal{FP}$ interval, say $\frac{P_m}{P_n}$, and the other one is $\frac{P_{m-1}}{P_{n-1}}$, and integer $n \geq 2$, there exists infinitely many fractions $\frac{h}{k} \in \mathcal{PP}_n$ such that (3.1) holds.

Now, we will investigate that $\sqrt{5}$ is the best possible constant. Let $\tau = 1+\sqrt{2}$. Since

$$2 < \frac{70}{29} < \tau < \frac{169}{70} < 5,$$

we have

$$\left|\tau - \frac{h}{k}\right| < \frac{P_{i-2}}{\sqrt{5}k^2}.$$

Here $P_{i-2} = 1$, and so we obtain that

$$\left|\tau - \frac{h}{k}\right| < \frac{1}{\sqrt{5}k^2}$$

for infinitely many $\frac{h}{k} \in \mathcal{PP}_n$. Thus from Hurwitz Theorem [5], we can't enlarge $\sqrt{5}$. □

We have seen in Theorem 3.4 that $\sqrt{5}$ is the best possible constant over the interval $(0,\infty)$. Now, the question arises that "Is it the best possible constant for every interval (P_{i-1}, P_i) and $\left(\frac{1}{P_i}, \frac{1}{P_{i-1}}\right)$ for $i = 3, 4, ..?$". The following theorems give the answer of this question.

Theorem 3.5. *Suppose that τ is an irrational number between two consecutive fractions $\frac{a}{b}$ and $\frac{a'}{b'}$ in the $\mathcal{FP}$ interval $[P_{i-1}, P_i]$, where one of the fractions is the center of this $\mathcal{FP}$ interval, say $\frac{P_m}{P_n}$, and the other one is $\frac{P_{m-1}}{P_{n-1}}$. Then for every integer $n \geq 2$, there exist infinitely many fractions $\frac{h}{k} \in \mathcal{PP}_n$ such that*

$$\left|\tau - \frac{h}{k}\right| < \frac{P_{i-2}}{\sqrt{5}k^2}.$$

Further, $\sqrt{5}$ is the best possible constant for the assertion.

Proof. The existence follows by Theorem 3.4. Let's show that the assertion fails when we replace $\sqrt{5}$ by a bigger constant. To prove this, it is enough to show for $n = 2$.

Take the interval $[P_{i-1}, P_i]$. By assumption $i \geq 3$. Since there is no fraction between $\frac{P_i}{2}$ and $\frac{P_i}{1}$ in the $\mathcal{FP}$ interval $[P_{i-1}, P_i]$, τ can not be between $\frac{P_i}{2}$ and $\frac{P_i}{1}$. So we consider the interval $\left[\frac{P_{i-1}}{1}, \frac{P_i}{2}\right]$. Let

$$X = \left[\frac{P_{i-1}}{1}, \frac{P_i}{2}\right] \qquad \text{and} \qquad X_1 = \left\{\frac{P_{i-1}}{1}, \frac{P_{i-1}+P_i}{3}, \frac{P_i}{2}\right\}.$$

We form X_{r+1} by taking all fractions and the mediants of consecutive fractions in X_r. Let $Y = \bigcup_{r=1}^{\infty} X_r$.

Now, let

$$X' = \left[\frac{2}{1}, \frac{5}{2}\right] \qquad \text{and} \qquad X'_1 = \left\{\frac{2}{1}, \frac{7}{3}, \frac{5}{2}\right\}.$$

Form X'_{r+1} by taking all fractions and the mediants of consecutive fractions in X'_r and let $Y' = \bigcup_{r=1}^{\infty} X'_r$. Then we have a one to one correspondence between X_r and X'_r as follows:

$$\frac{a\cdot 2 + b\cdot 5}{a\cdot 1 + b\cdot 2} \to \frac{aP_{i-1} + bP_i}{a\cdot 1 + b\cdot 2}.$$

Thus the distance between the consecutive rationals in X_r is P_{i-2} times the distance between consecutive rationals in X'_r. For $\alpha_0 = 1+\sqrt{2}$ and $\alpha_1 = \sqrt{2}-1$, let $P_{i-1} + P_{i-2}\alpha_1 = \alpha_0^{i-2} = \alpha'$.

Then α' is an irrational number between two consecutive fractions in X_r. Assume that there exists infinitely many $\frac{p_r}{q_r} \in X$ with

$$\left|\tau - \frac{p_r}{q_r}\right| < \frac{P_{i-2}}{\beta q_r^2}, \text{ where } \beta > \sqrt{5}, r = 1, 2, ...,$$

then there exists infinitely many $\frac{p'_r}{q_r} \in Y'$ such that

$$\left|\tau - \frac{p'_r}{q_r}\right| < \frac{1}{\beta q_r^2}, \text{ where } \beta > \sqrt{5}, r = 1, 2, ...$$

But this contradicts with Hurwitz's theorem in [5], pp. 6. Hence if we replace $\sqrt{5}$ by a larger constant the theorem fails. □

Theorem 3.6. *Suppose that τ is an irrational number between two consecutive fractions $\frac{a}{b}$ and $\frac{a'}{b'}$ in the $\mathcal{FP}$ interval $\left[\frac{1}{P_i}, \frac{1}{P_{i-1}}\right]$, where one of the fractions is the center of this $\mathcal{FP}$ interval, say $\frac{P_m}{P_n}$, and the other one is $\frac{P_{m-1}}{P_{n-1}}$. Then for every integer $n \geq 2$, there exist infinitely many fractions $\frac{h}{k} \in \mathcal{PP}_n$ such that*

$$\left|\tau - \frac{h}{k}\right| < \frac{P_{i-2}}{\sqrt{5}k^2}.$$

Proof. Consider the interval $\left[\frac{1}{P_i}, \frac{1}{P_{i-1}}\right]$. Let τ be an irrational number between two consecutive fractions $\frac{a}{b}$ and $\frac{a'}{b'}$, where one of the fractions is the center of the $\mathcal{FP}$ interval $\left[\frac{1}{P_i}, \frac{1}{P_{i-1}}\right]$, say $\frac{P_m}{P_n}$, and the other one is $\frac{P_{m-1}}{P_{n-1}}$ in this interval. Then we have seen that there exists infinitely many rationals $\frac{h}{k} \in \mathcal{PP}_n$ such that

$$\left|\tau - \frac{h}{k}\right| < \frac{P_{i-2}}{\sqrt{5}k^2}.$$

Let

$$X'' = \left[\frac{2}{P_i}, \frac{1}{P_{i-1}}\right] \qquad \text{and} \qquad X''_1 = \left\{\frac{2}{P_i}, \frac{3}{P_i + P_{i-1}}, \frac{1}{P_{i-1}}\right\}.$$

We form X''_{r+1} by taking all fractions with mediants of consecutive fractions in X''_r. Let $Y'' = \bigcup_{r=1}^{\infty} X''_r$. Then there exists a one to one correspondence between Y and Y'', i.e., $\frac{a}{b} \to \frac{b}{a}$.

Let $\alpha'' = \frac{1}{\alpha'}$ and suppose that there exists infinitely many $\frac{p_r}{q_r} \in \mathcal{PP}_n$ such that

$$\left|\alpha'' - \frac{p_r}{q_r}\right| < \frac{P_{i-2}}{\beta q_r^2}.$$

If

$$\alpha'' = \frac{p_r}{q_r} + \frac{\delta P_{i-2}}{q_r^2}$$

then $|\delta| < 1/\beta$ and this implies that for infinitely many $\frac{q_r}{p_r} \in Y$

$$\left|\alpha' - \frac{q_r}{p_r}\right| < \frac{P_{i-2}}{\beta\left(p_r + \frac{\delta P_{i-2}}{q_r}\right)(p_r)}.$$

For any $\gamma > \delta$ with $\sqrt{5} < \gamma < \beta$, we obtain that for infinitely many $\frac{q_r}{p_r} \in Y$

$$\left| \alpha' - \frac{q_r}{p_r} \right| < \frac{P_{i-2}}{\gamma p_r^2},$$

which contradicts with Theorem 3.5. This proves Theorem 3.6. □

Finally, we give an approximation for some irrational numbers τ.

Theorem 3.7. *Suppose that* τ *is an irrational number in the interval* $\left[\frac{P_{i-1}}{1}, \frac{P_i}{2}\right]$ *or* $\left[\frac{2}{P_i}, \frac{1}{P_{i-1}}\right]$. *Then for every integer* $n \geq 2$, *there exist infinitely many fractions* $\frac{h}{k} \in \mathcal{PP}_n$ *such that*

$$\left| \tau - \frac{h}{k} \right| < \frac{2P_{i-2}}{\sqrt{5}k^2}.$$

Proof. The proof can be done similar to the proof of Theorem 3.4. □

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