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ON MEROMORPHIC FUNCTIONS THAT SHARE A SMALL FUNCTION WITH ITS DERIVATIVES

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Abstract. In this paper, we study the problem of meromorphic functions sharing a small function with its derivative and prove one theorem. The theorem improves the results of Jin-Dong Li and Guang-Xin Huang [10].

1. Introduction

Let f be a nonconstant meromorphic function defined in the whole complex plane $\mathbb{C}$. It is assumed that the reader is familiar with the notations of the Nevanlinna theory such as $T(r,f), N(r,f)$ and so on, that can be found, for instance in [1].

Let f and g be two nonconstant meromorphic functions. Let a be a finite complex number. We say that f and g share the value a CM(counting multiplicities) if $f-a$ and $g-a$ have the same zeros with the same multiplicites and we say that f and g share the value a IM(ignoring multiplicities) if we do not consider the multiplicities. When f and g share 1 IM, let z_0 be a 1-points of f of order p, a 1-points of g of order q, we denote by $N_{11}(r,\frac{1}{f-1})$ the counting function of those 1-points of f and g where $p=q=1$; and $N_E^{(2}(r,\frac{1}{f-1})$ the counting function of those 1-points of f and g where $p=q\geq 2$. $\overline{N}_L(r,\frac{1}{f-1})$ is the counting function of those 1-points of both f and g where $p>q$. In the same way, we can define $N_{11}(r,\frac{1}{g-1}), N_E^{(2}(r,\frac{1}{g-1})$ and $\overline{N}_L(r,\frac{1}{g-1})$. If f and g share 1 IM, it is easy to see that

$$\begin{aligned}\overline{N}(r,\frac{1}{f-1}) &= N_{11}(r,\frac{1}{f-1}) + \overline{N}_L(r,\frac{1}{f-1}) + \overline{N}_L(r,\frac{1}{g-1}) + N_E^{(2}(r,\frac{1}{g-1}) \\ &= \overline{N}(r,\frac{1}{g-1})\end{aligned}$$

Let f be a nonconstant meromorphic function. Let a be a finite complex number, and k be a positive integer, we denote by $N_{k)}(r,\frac{1}{f-a})(or\overline{N}_{k)}(r,\frac{1}{f-a}))$ the counting function for zeros of $f-a$ with multiplicity $\leq k$ (ignoring multiplicities), and by $N_{(k}(r,\frac{1}{f-a})(or\overline{N}_{(k}(r,\frac{1}{f-a}))$ the counting function for zeros of $f-a$ with multiplicity

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atleast k(ignoring multiplicities). Set

$$N_k(r, \frac{1}{f-a}) = \overline{N}(r, \frac{1}{f-a}) + \overline{N}_{(2}(r, \frac{1}{f-a}) + ... + \overline{N}_{(k}(r, \frac{1}{f-a})$$

$$\Theta(a,f) = 1 - \limsup_{r\longrightarrow\infty} \frac{\overline{N}(r, \frac{1}{f-a})}{T(r,f)}, \quad \delta(a,f) = 1 - \limsup_{r\longrightarrow\infty} \frac{N(r, \frac{1}{f-a})}{T(r,f)}.$$

We further define

$$\delta_k(a,f) = 1 - \limsup_{r\longrightarrow\infty} \frac{N_k(r, \frac{1}{f-a})}{T(r,f)}.$$

Clearly

$$0 \leq \delta(a,f) \leq \delta_k(a,f) \leq \delta_{k-1}(a,f)... \leq \delta_2(a,f) \leq \delta_1(a,f) = \Theta(a,f)$$

Definition 1.1(see[3]**).** Let k be a nonnegative integer or infinity. For $a \in \overline{\mathbb{C}}$ we denote by $E_k(a,f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a,f) = E_k(a,g)$, we say that f, g share the value a with weight k.

We write f, g share (a,k) to mean that f, g share the value a with weight k; clearly if f, g share (a,k), then f, g share (a,p) for all integers p with $0 \leq p \leq k$. Also, we note that f, g share a value a IM or CM if and only if they share $(a,0)$ or (a,∞), respectively.

A meromorphic function a is said to be a small function of f where $T(r,a) = S(r,f)$, that is $T(r,a) = o(T(r,f))$ as $r \to \infty$, outside of a possible exceptional set of finite linear measure. Similarly, we can define that f and g share a small function a IM or CM or with weight k.

R.Bruck [4] first considered the uniqueess problems of an entire function sharing one value with its derivative and proved the following result.

Theorem A. Let f be a non-constant entire function satisfying $N(r, \frac{1}{f'}) = S(r,f)$. If f and f' share the value 1 CM, then $\frac{f'-1}{f-1} \equiv c$ for some nonzero constant c.

Bruck [4] further posed the following conjecture.

Conjecture 1.1. Let f be a non-constant entire function, $\rho_1(f)$ be the first iterated order of f. If $\rho_1(f)$ is not a positive integer or infinite, f and f' share the value 1 CM, then $\frac{f'-1}{f-1} \equiv c$ for some nonzero constant c.

Yang [5] proved that the conjecture is true if f is an entire function of finite order. Yu [6] considered the problem of an entire or meromorphic function sharing one small function with its derivative and proved the following two theorems.

Theorem B. Let f be a non-constant entire function and $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. If $f - a$ and $f^{(k)} - a$ share 0 CM and $\delta(0,f) > \frac{3}{4}$, then $f \equiv f^{(k)}$.

Theorem C. Let f be a non-constant non-entire meromorphic function and $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. If

(i) f and a have no common poles.

(ii) $f - a$ and $f^{(k)} - a$ share 0 CM.

(iii) $4\delta(0,f) + 2(8+k)\Theta(\infty,f) > 19 + 2k$,

then $f \equiv f^{(k)}$ where k is a positive integer.

In the same paper, Yu [6] posed the following open questions.

(i) can a CM shared be replaced by an IM share value ?

(ii) Can the condition $\delta(0,f) > \frac{3}{4}$ of theorem B be further relaxed ?

(iii) Can the condition (iii) in theorem C be further relaxed ?

(iv) Can in general the condition (i) of theorem C be dropped ?
In 2004, Liu and Gu [7] improved theorem B and obtained the following results.
Theorem D. Let f be a non-constant entire function and $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. If $f - a$ and $f^{(k)} - a$ share 0 CM and $\delta(0, f) > \frac{1}{2}$, then $f \equiv f^{(k)}$.

Lahiri and Sarkar [8] gave some affirmative answers to the first three questions imposing some restrictions on the zeros and poles of a. They obtained the following results.
Theorem E. Let f be a non-constant meromorphic function, k be a positive integer, and $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. If

(i) a has no zero (pole) which is also a zero (pole) of f or $f^{(k)}$ with the same multiplicity.

(ii) $f - a$ and $f^{(k)} - a$ share $(0, 2)$

(iii) $2\delta_{2+k}(0, f) + (4 + k)\Theta(\infty, f) > 5 + k$ then $f \equiv f^{(k)}$.

In 2005, Zhang [?] improved the above results and proved the following theorem.
Theorem F. Let f be a non-constant meromorphic function, $k(\geq 1), l(\geq 0)$ be integers. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. If
$l \geq 2$ and

$$(3 + k)\Theta(\infty, f) + 2\delta_{2+k}(0, f) > k + 4 \tag{1.1}$$

or $l = 1$ and

$$(4 + k)\Theta(\infty, f) + 3\delta_{2+k}(0, f) > k + 6 \tag{1.2}$$

or $l = 0$ and

$$(6 + 2k)\Theta(\infty, f) + 5\delta_{2+k}(0, f) > 2k + 10 \tag{1.3}$$

then $f \equiv f^{(k)}$.
In 2015, Jin-Dong Li and Guang-Xiu Huang [?] proved the following Theorem.
Theorem G. Let f be a non-constant meromorphic function, $k(\geq 1), l(\geq 0)$ be integers. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. If
$l \geq 2$ and

$$(3 + k)\Theta(\infty, f) + \delta_2(0, f) + \delta_{2+k}(0, f) > k + 4 \tag{1.4}$$

$l = 1$ and

$$(\frac{7}{2} + k)\Theta(\infty, f) + \frac{1}{2}\Theta(0, f) + \delta_2(0, f) + \delta_{2+k}(0, f) > k + 5 \tag{1.5}$$

or $l = 0$ and

$$(6 + 2k)\Theta(\infty, f) + 2\Theta(\infty, f) + \delta_2(0, f) + \delta_{1+k}(0, f) + \delta_{2+k}(0, f) > 2k + 10 \tag{1.6}$$

then $f \equiv f^{(k)}$.

In this paper we pay our attention to the uniqueness of more generalised form of a function namely f^m and $(f^n)^{(k)}$ sharing a small function for two arbitrary positive integer n and m.

Theorem 1.1. Let f be a non-constant meromorphic function, $k(\geq 1)$, $l(\geq 0)$ be integers. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. Suppose

that $f^m - a$ and $(f^n)^{(k)} - a$ share $(0,l)$. If
$l \geq 2$ and

$$(k+4)\Theta(\infty,f) + (k+5)\Theta(0,f) > 2k+9-m \tag{1.7}$$

$l = 1$ and

$$(k+\frac{9}{2})\Theta(\infty,f) + (k+\frac{11}{2})\Theta(0,f) > 2k+10-m \tag{1.8}$$

or $l = 0$ and

$$(2k+7)\Theta(\infty,f) + (2k+8)\Theta(0,f) > 4k+15-m \tag{1.9}$$

then $f^m \equiv (f^n)^{(k)}$.

Corollary 1.2. Let f be a non-constant meromorphic function, $m, k(\geq 1), l(\geq 0)$ be integers. Also let $a \equiv a(z)(\not\equiv 0,\infty)$ be a meromorphic small function. Suppose that $f^m - a$ and $(f^n)^{(k)} - a$ share $(0,l)$. If
$l \geq 2$ and $\Theta(0,f) > \frac{4}{5}$
or $l = 1$ and $\Theta(0,f) > \frac{9}{11}$
or $l = 0$ and $\Theta(0,f) > \frac{7}{8} - \frac{1}{8}[7\Theta(\infty,f) - 7\Theta(0,f)]$
then $f^m \equiv (f^n)^{(k)}$.

2. Lemmas

Lemma 2.1 (see [10]). Let f be a non-constant meromorphic function, k, p be two positive integers, then

$$N_p(r,\frac{1}{f^{(k)}}) \leq N_{p+k}(r,\frac{1}{f}) + k\overline{N}(r,f) + S(r,f)$$

clearly $\overline{N}(r,\frac{1}{f^{(k)}}) = N_1(r,\frac{1}{f^{(k)}})$

Lemma 2.2 (see [10]). Let

$$H = (\frac{F''}{F'} - \frac{2F'}{F-1}) - (\frac{G''}{G'} - \frac{2G'}{G-1}) \tag{2.1}$$

where F and G are two non constant meromorphic functions. If F and G share 1 IM and $H \not\equiv 0$, then

$$N_{11}(r,\frac{1}{F-1}) \leq N(r,H) + S(r,F) + S(r,G)$$

Lemma 2.3 (see [11]). Let f be a non-constant meromorphic function and let

$$R(f) = \frac{\Sigma_{k=0}^{n} a_k f^k}{\Sigma_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f with constant coefficients a_k and b_j where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r,R(f)) = dT(r,f) + S(r,f),$$

where $d = max\{n,m\}$.

3. Proof of the Theorem 1.2

Let $F = \frac{f^m}{a}$ and $G = \frac{(f^n)^{(k)}}{a}$. Then F and G share $(1, l)$, except the zeros and poles of $a(z)$. Let H be defined by (2.1)

Case 1. Let $H \not\equiv 0$.

By our assumptions, H have poles only at zeros of F' and G' and poles of F and G, and those 1-points of F and G whose multiplicities are distinct from the multiplicities of corresponding 1-points of G and F respectively. Thus, we deduce from (2.1) that

$$\begin{aligned} N(r,H) \le \overline{N}_{(2}(r,\frac{1}{H}) + \overline{N}_{(2}(r,\frac{1}{G}) + \overline{N}(r,H) \\ + N_0(r,\frac{1}{F'}) + N_0(r,\frac{1}{G'}) + \overline{N}_L(r,\frac{1}{F-1}) \\ + \overline{N}_L(r,\frac{1}{G-1}) \end{aligned} \tag{3.1}$$

here $N_0(r,\frac{1}{F'})$ is the counting function which only counts those points such that $F' = 0$ but $F(F-1) \neq 0$.

Because F and G share 1 IM, it is easy to see that

$$\begin{aligned} \overline{N}(r,\frac{1}{F-1}) &= N_{11}(r,\frac{1}{F-1}) + \overline{N}_L(r,\frac{1}{F-1}) + \overline{N}_L(r,\frac{1}{G-1}) + N_E^{(2}(r,\frac{1}{G-1}) \\ &= \overline{N}(r,\frac{1}{G-1}) \end{aligned} \tag{3.2}$$

By the second fundamental theorem, we see that

$$\begin{aligned} T(r,F) + T(r,G) \le \overline{N}(r,F) + \overline{N}(r,G) + \overline{N}(r,\frac{1}{F}) \\ + \overline{N}(r,\frac{1}{G}) + \overline{N}(r,\frac{1}{F-1}) + \overline{N}(r,\frac{1}{G-1}) \\ - N_0(r,\frac{1}{F'}) - N_0(r,\frac{1}{G'}) + S(r,F) + S(r,G) \end{aligned} \tag{3.3}$$

Using Lemma 2.2 and (3.1), (3.2) and (3.3) We get

$$\begin{aligned} T(r,F) + T(r,G) \le 3\overline{N}(r,F) + N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) \\ + N_{11}(r,\frac{1}{F-1}) + 2N_E^{(2}(r,\frac{1}{G-1}) \\ + 3\overline{N}_L(r,\frac{1}{F-1}) + 3\overline{N}_L(r,\frac{1}{G-1}) + S(r,F) + S(r,G) \end{aligned} \tag{3.4}$$

We discuss the following three sub cases.

Sub case 1.1. $l \geq 2$. Obviously.

$$\begin{aligned} N_{11}(r,\frac{1}{F-1}) + 2N_E^{(2}(r,\frac{1}{G-1}) + 3\overline{N}_L(r,\frac{1}{F-1}) + 3\overline{N}_L(r,\frac{1}{G-1}) \\ \le N(r,\frac{1}{G-1}) + S(r,F) \\ \le T(r,G) + S(r,F) + S(r,G) \end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5), we get

$$T(r,F) \le 3\overline{N}(r,F) + N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + S(r,F) \tag{3.6}$$

that is

$$T(r,f^m) \le 3\overline{N}(r,f^m) + N_2(r,\frac{1}{f^m}) + N_2(r,\frac{1}{(f^n)^{(k)}}) + S(r,f)$$

By Lemma 2.1 for $p=2$, we get

$$mT(r,f) \le (k+5)\overline{N}(r,\frac{1}{f}) + (k+4)\overline{N}(r,f) + S(r,f)$$

So

$$(k+4)\Theta(\infty,f) + (k+5)\Theta(0,f) \le 2k+9-m$$

which contradicts with (1.7).

Sub case 1.2. $l=1$. It is easy to see that

$$\begin{aligned} N_{11}(r,\frac{1}{F-1}) + 2N_E^{(2}(r,\frac{1}{G-1}) + 2\overline{N}_L(r,\frac{1}{F-1}) + 3\overline{N}_L(r,\frac{1}{G-1}) & \\ \le N(r,\frac{1}{G-1}) + S(r,F) & \\ \le T(r,G) + S(r,F) + S(r,G) & \end{aligned} \tag{3.7}$$

$$\begin{aligned} \overline{N}_L(r,\frac{1}{F-1}) &\le \frac{1}{2}N(r,\frac{F}{F'}) \\ &\le \frac{1}{2}N(r,\frac{F'}{F}) + S(r,F) \\ &\le \frac{1}{2}[\overline{N}(r,\frac{1}{F}) + \overline{N}(r,F)] + S(r,F). \end{aligned} \tag{3.8}$$

Combining (3.4) and (3.7) and (3.8), we get

$$T(r,F) \le N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + \frac{7}{2}\bar{N}(r,F) + \frac{1}{2}\bar{N}(r,\frac{1}{F}) + S(r,F) \tag{3.9}$$

that is

$$mT(r,f) \le N_2(r,\frac{1}{f^m}) + N_2(r,\frac{1}{(f^n)^{(k)}}) + \frac{7}{2}\bar{N}(r,f^m) + \frac{1}{2}\bar{N}(r,\frac{1}{f^m}) + S(r,f).$$

By Lemma 2.1 for $p=2$, we get

$$mT(r,f) \le (k+\frac{9}{2})\overline{N}(r,f) + (k+\frac{11}{2})\overline{N}(r,\frac{1}{f}) + S(r,f)$$

So

$$(k+\frac{9}{2})\Theta(\infty,f) + (k+\frac{11}{2})\Theta(0,f) \le 2k+10-m$$

which contradicts with (1.8).

Sub case 1.3. $l=0$. It is easy to see that

$$\begin{aligned} N_{11}(r,\frac{1}{F-1}) + 2N_E^{(2}(r,\frac{1}{G-1}) + \overline{N}_L(r,\frac{1}{F-1}) + 2\overline{N}_L(r,\frac{1}{G-1}) & \\ \le N(r,\frac{1}{G-1}) + S(r,F) & \\ \le T(r,G) + S(r,F) + S(r,F) & \end{aligned} \tag{3.10}$$

$$\begin{aligned}\overline{N}_L(r,\frac{1}{F-1}) &\leq N(r,\frac{1}{F-1}) - \overline{N}(r,\frac{1}{F-1}\\ &\leq N(r,\frac{F}{F'}) \leq N(r,\frac{F'}{F}) + S(r,F)\\ &\leq \overline{N}(r,\frac{1}{F}) + \overline{N}(r,F) + S(r,F).\end{aligned} \tag{3.11}$$

Similarly, we have

$$\begin{aligned}\overline{N}_L(r,\frac{1}{G-1}) &\leq \overline{N}(r,\frac{1}{G}) + \overline{N}(r,G) + S(r,F)\\ &\leq N_1(r,\frac{1}{G}) + \overline{N}(r,F) + S(r,G).\end{aligned} \tag{3.12}$$

Combining (3.4) and (3.10) – (3.12), we get

$$\begin{aligned}T(r,F) \leq N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + 2\overline{N}(r,\frac{1}{F})\\ + 6\overline{N}(r,F) + N_1(r,\frac{1}{G}) + S(r,F)\end{aligned} \tag{3.13}$$

that is

$$\begin{aligned}mT(r,f) \leq N_2(r,\frac{1}{f^m}) + N_2(r,\frac{1}{(f^n)^{(k)}}) + 2\overline{N}(r,\frac{1}{f^m})\\ + 6\overline{N}(r,\frac{1}{f^m}) + N_1(r,\frac{1}{(f^n)^{(k)}}) + S(r,f).\end{aligned}$$

By Lemma 2.1 for $p = 2$ and for $p = 1$ respectively, we get

$$mT(r,f) \leq (2k+8)\overline{N}(r,\frac{1}{f}) + (2k+7)\overline{N}(r,f).$$

So

$$(2k+7)\Theta(\infty,f) + (2k+8)\Theta(0,f) \leq 4k+15-m$$

which contradicts with (1.9).

Case 2. Let $H \equiv 0$.

on integration we get from (2.1)

$$\frac{1}{F-1} \equiv \frac{C}{G-1} + D, \tag{3.14}$$

where C, D are constants and $C \neq 0$. we will prove that $D = 0$.

Sub case 2.1. Suppose $D \neq 0$. If z_0 be a pole of f with multiplicity p such that $a(z_0) \neq 0, \infty$, then it is a pole of G with multiplicity $np + k$ respectively. This contradicts (3.14). It follows that $N(r,f) = S(r,f)$ and hence $\Theta(\infty,f) = 1$. Also it is clear that $\overline{N}(r,f) = \overline{N}(r,G) = S(r,f)$. From (1.7)-(1.9) we know respectively

$$(k+5)\Theta(0,f) > k+5-m \tag{3.15}$$

$$(k+\frac{11}{2})\Theta(0,f) > k+\frac{11}{2}-m \tag{3.16}$$

and

$$(2k+8)\Theta(0,f) > 2k+8-m \tag{3.17}$$

Since $D \neq 0$, from (3.14) we get

$$\overline{N}\left(r,\frac{1}{F-(1+\frac{1}{D})}\right) = \overline{N}(r,G) = S(r,f)$$

Suppose $D \neq -1$.
Using the second fundamental theorem for F we get

$$\begin{aligned} T(r, F) &\leq \overline{N}(r, F) + \overline{N}(r, \frac{1}{F}) + \overline{N}\left(r, \frac{1}{F - (1 + \frac{1}{D})}\right) \\ &\leq \overline{N}(r, \frac{1}{F}) + S(r, f) \\ i.e., & \\ mT(r, F) &\leq \overline{N}(r, \frac{1}{F}) + S(r, f) \\ &\leq mT(r, f) + S(r, f). \end{aligned}$$

So, we have $mT(r, f) = \overline{N}(r, \frac{1}{f})$ and so $\Theta(0, f) = 1 - m$. Which contradicts (3.15)−(3.17).
If $D = -1$, then

$$\frac{F}{F - 1} \equiv C \frac{1}{G - 1} \tag{3.18}$$

and from which we know $\overline{N}(r, \frac{1}{F}) = \overline{N}(r, G) = S(r, f)$ and hence, $\overline{N}(r, \frac{1}{F}) = S(r, f)$.
If $C \neq -1$,
we know from (3.18) that

$$\overline{N}\left(r, \frac{1}{G - (1 + C)}\right) = \overline{N}(r, F) = S(r, f).$$

So from Lemma 2.1 and the Second fundamental theorem we get

$$\begin{aligned} T(r, (f^n)^{(k)}) &\leq \overline{N}(r, G) + \overline{N}(r, \frac{1}{G}) + \overline{N}\left(r, \frac{1}{G - (1 + C)}\right) + S(r, f) \\ &\leq \overline{N}\left(r, \frac{1}{(f^n)^{(k)}}\right) + S(r, f) \\ mT(r, f) &\leq (k + 1)\overline{N}(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f), \end{aligned}$$

which is absurd. So $C = -1$ and we get from (3.18) that $FG \equiv 1$, which implies $\left[\frac{(f^n)^{(k)}}{f^n}\right] = \frac{a^2}{f^{n+m}}$.
In view of the first fundamental theorem, we get from above

$$(n + m)T(r, f) \leq k[\overline{N}(r, f) + \overline{N}(r, \frac{1}{f})] + S(r, f) = S(r, f),$$

which is impossible.
Sub case 2.2. $D = 0$ and so from (3.14) we get

$$G - 1 \equiv C(F - 1).$$

If $C \neq 1$, then

$$\begin{aligned} & G \equiv C(F - 1 + \frac{1}{C}) \\ \textit{and} \quad & \overline{N}(r, \frac{1}{G}) = \overline{N}\left(r, \frac{1}{F - (1 - \frac{1}{C})}\right). \end{aligned}$$

By the second fundamental theorem and Lemma 2.1 for $p = 1$ and Lemma 2.3 we have

$$\begin{aligned} mT(r,f) + S(r,f) &= T(r,F) \\ &\le \overline{N}(r,F) + \overline{N}(r,\frac{1}{F}) + \left(r, \frac{1}{F-(1-\frac{1}{C})}\right) + S(r,G) \\ &\le \overline{N}(r,f^m) + \overline{N}(r,\frac{1}{f^m}) + \overline{N}\left(r, \frac{1}{(f^n)^{(k)}}\right) + S(r,f) \\ &\le \overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) + (k+1)\overline{N}(r,\frac{1}{f}) + k\overline{N}(r,f) + S(r,f) \\ &\le (k+2)\overline{N}(r,\frac{1}{f}) + (k+1)\overline{N}(r,f) + S(r,f). \end{aligned}$$

Hence

$$(k+1)\Theta(\infty,f) + (k+2)\Theta(0,f) \le 2k+3-m.$$

So, it follows that

$$\begin{aligned} (k+4)\Theta(\infty,f) + (k+5)\Theta(0,f) &\le 3\Theta(\infty,f) + (k+1)\Theta(\infty,f) \\ &\quad + (k+3)\Theta(0,f) + 2\Theta(0,f) \\ &\le 2k+9-m \end{aligned}$$

$$(k+\frac{9}{2})\Theta(\infty,f) + (k+\frac{11}{2})\Theta(0,f) \le 2k+10-m,$$

and

$$(2k+7)\Theta(\infty,f) + (2k+8)\Theta(0,f) \le 4k+15-m.$$

This contradicts $(1.7)-(1.9)$. Hence $C = 1$ and so $F \equiv G$, that is $f^m \equiv (f^n)^{(k)}$. This completes the proof of the theorem.

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References

[1] W. K. Hayman, *Meromorphic functions,* Clarendon Press, Oxford (1964).

[2] L. Yang, *Distribution Theory,* Springer Verlag, Berlin (1993).

[3] I.Lahiri, *Weighted sharing and uniqueness of meromorphic function,* Nagoya Math. J., **161**, 193-206(2001).

[4] R.Bruck, *On entire functions which share one value CM with their first derivative,* Result. Math. **30**, 21-24(1996).

[5] L.Z.Yang, *Solution of a differential equation and its applications,* Kodai. Math. J. **22,** 458-464(1999).

[6] K.W.Yu, *On entire and meromorphic functions that share small functions with their derivatives,* J. Inequal. Pure Appl. Math. **4(1)**(2003) Art.21(Online:http://jipam.vu.edu.au/).

[7] L.P.Liu and Y.X.Gu, *Uniqueness of meromorphic functions that share one small function with their derivatives,* Kodai. Math. J. **27,** 272-279(2004).

[8] I.Lahiri and A.Sarkar, *Uniqueness of meromorphic function and its derivative,* J. Inequal. Pure Appl. Math. **5(1)**(2004) Art.20 (Online:http://jipam.vu.edu.au/).

[9] Q.C. Zhang, *Meromorphic function that shares one small function with their derivatives,* J. Inequal. Pure. Appl. Math. **6(4)**(2005) Art.116 (Online:http://jipam.vu.edu.au/).

[10] Jin-Dong Li and Guang-Xin Huang, *On meromorphic functions that share one small function with their derivatives,* Palestine Journal of Mathematics, Vol. **4(1)** (2015), 91-96.

[11] A.Z. Mohon'ko, *On the Nevanlinna charecteristics of some meromorphic functions,* Theory of Functions, Funct. Anal. Appl. **14** (1971), 83-87.

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