

RANK ONE SOLUTIONS OF SOME NONLINEAR OPERATOR EQUATIONS

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ABSTRACT. Let \mathcal{E} be a Banach space and A, B be bounded linear operators on \mathcal{E} . This note concerns the operator equations $XA - AX = f(X)$ and $BX - XA = \tau(X)$ and some other equations, where X is an unknown operator, f is an analytic function and τ is a linear map on $\mathcal{L}(\mathcal{E})$. Using the eigenvalues of A and A^* for mentioned operator equations, we introduce an explicit idempotent or nilpotent rank one solution.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{L}(\mathcal{E})$ denote the algebra of all bounded linear operators on a Banach space \mathcal{E} and A^* be the adjoint operator on dual space \mathcal{E}^* defined by $(A^*\phi)(x) = \phi(Ax)$ for $x \in \mathcal{E}$ and $\phi \in \mathcal{E}^*$. It is easy to show that an operator A is of rank one if and only if there exist $\phi \in \mathcal{E}^*$ and $u \in \mathcal{E}$ such that $A = \phi \otimes u$, where $(\phi \otimes u)x := \phi(x)u$ for any $x \in \mathcal{E}$. Note that for any $u, v \in \mathcal{E}$, $\phi, \psi \in \mathcal{E}^*$ and operator $B \in \mathcal{L}(\mathcal{E})$, we have

- (1) $B(\phi \otimes u) = \phi \otimes Bu$,
- (2) $(\phi \otimes u)B = B^*\phi \otimes u$,
- (3) $(\phi \otimes u)(\psi \otimes v) = \psi \otimes \phi(v)u$.

It is easy to see that the rank one operator $\phi \otimes u$ is an idempotent if and only if $\phi(u) = 1$, and it is a nilpotent if and only if $\phi(u) = 0$.

Recall that the spectrum of $B \in \mathcal{L}(\mathcal{E})$ is the set $\sigma(B)$ of all $\lambda \in \mathbb{C}$ for which the operator $\lambda I - B$ does not have an inverse in $\mathcal{L}(\mathcal{E})$. The point spectrum $\sigma_p(B)$ of A consists of all $\lambda \in \sigma(B)$ such that λ is an eigenvalue of B .

In this paper, we intend to introduce rank one solutions for some operator equations in the setting of Banach spaces. In early studies, rank one, finite rank and compact operator solutions of $AX = XA$ have been investigated; see e.g., [5] and [10]. Some other researchers focused on rank of solution of matrix equations; see e.g. [11] and [9]. This note is motivated by the following theorem, which is proved in [6].

2010 *Mathematics Subject Classification.* 47A62, 46Nxx, 15A24.

Key words and phrases. Operator equation, rank one operator, idempotent solution, nilpotent solution.

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Submitted April 30, 2016. Published May 30, 2016.

Theorem 1.1. *Let A be a nonzero bounded linear operator on a Banach space \mathcal{E} , where $\dim(\mathcal{E}) \geq 3$. If the point spectrum of either A or A^* is nonempty, then there exists a bounded linear operator B such that*

- (1) $AB + BA$ is of rank one,
- (2) $I + f(A)B$ is invertible for every function f , which is analytic in a neighborhood of the spectrum of A .

In 2005, Burde [4] solved the matrix equation $XA - AX = X^p$, with integer $p \geq 2$, which arises from Lie theory. In particular, he proved that every matrix satisfying this equation is a nilpotent matrix. In [1], the matrix equation $XA - AX = \tau(X)$, where τ is an automorphism of finite order on a field K was studied. Bourgeois [3] investigated the matrix equation $XA - AX = f(X)$, by assuming that there exists a unique number α satisfying $f(\alpha) = 0$, and proved that if $f'(\alpha) = 0$ and $f'(\alpha) \neq 0$, then the matrix equation $XA - AX = f(X)$ has a solution.

We obtain explicit rank one solutions for some operator equations such as $XA - AX = f(X)$ and $BX - XA = \tau(X)$ in the framework of Banach spaces, where f is an analytic function and τ is a linear map on $\mathcal{L}(\mathcal{E})$.

2. MAIN RESULT

We consider the following operator equation.

$$XA - AX = f(X), \quad (2.1)$$

where f is an analytic function. We aim to study Equation (2.1) under some new mild conditions.

Theorem 2.1. *Let \mathcal{E} be a Banach space and f be an analytic function at 0 and $f(0) = 0$. Suppose that there exist eigenvalues $\lambda \in \sigma_p(A)$ and $\mu \in \sigma_p(A^*)$. Then*

- (i) *If $f'(0) = \mu - \lambda$, then Equation (2.1) admits a nilpotent rank one solution.*
- (ii) *If $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} = \mu - \lambda$, then Equation (2.1) admits an idempotent rank one solution.*

Proof. Let $v \in \mathcal{E}$ and $\phi \in \mathcal{E}^*$ be some corresponding eigenvectors of λ and μ , respectively. Let $X = \phi \otimes v$

(i) We assume that $\phi(v) = 0$. We show that X satisfies Equation (2.1). We have

$$XA - AX = (\phi \otimes v)A - A(\phi \otimes v) = (A^* \phi \otimes v) - (\phi \otimes Av) = (\mu - \lambda)(\phi \otimes v). \quad (2.2)$$

We have $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$ for every z in a neighborhood of origin. Since 0 is extreme point of spectrum set of any compact operator, we have

$$\begin{aligned} f(\phi \otimes v) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (\phi \otimes v)^n \\ &= \frac{f(0)}{0!} I + \frac{f'(0)}{1!} (\phi \otimes v) + \frac{f''(0)}{2!} (\phi \otimes v)^2 + \dots \\ &= f'(0)(\phi \otimes v) \\ &= (\mu - \lambda)(\phi \otimes v). \end{aligned} \quad (2.3)$$

It follows from (2.2) and (2.3) that X is a nilpotent solution.

(ii) Now assume that $\phi(v) \neq 0$. Without loss of generality, we can assume that

$\phi(v) = 1$. Hence X is idempotent. As above, $XA - AX = (\mu - \lambda)(\phi \otimes v)$. We have

$$\begin{aligned} f(\phi \otimes v) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (\phi \otimes v)^n \\ &= \frac{f(0)}{0!} I + \frac{f'(0)}{1!} (\phi \otimes v) + \frac{f''(0)}{2!} (\phi \otimes v)^2 + \dots \\ &= f'(0)(\phi \otimes v) + \frac{f''(0)}{2!} (\phi \otimes v) + \dots \\ &= (\phi \otimes v) \left(\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \right) = (\mu - \lambda)(\phi \otimes v). \end{aligned}$$

Therefore, X is an idempotent solution of Equation (2.1). □

One can compare Theorem 2.1 with matrix case in [3, Theorem 2]. Note that if A is a self-adjoint operator, i.e. $A = A^*$, then λ and μ in the latter theorem must be distinct eigenvalues of A to satisfy the conditions of Theorem 2.1.

Bhatia [2] proved an integral solution of the form $X = \int_0^\infty e^{-tA} C e^{tB} dt$ for the equation $AX - XB = C$. In 2009, Garimella et al. [7] introduced an integral form operator such as B making $AB + BA$ a rank one operator. In [1], the equation $XA - AX = \tau(X)$ was solved in the setting of matrices. Moreover, in the matrix theory, it is well-known that $BX - XA = 0$ has a nonzero solution if A and B have a common eigenvalue; cf. [12]. We can introduce a rank one explicit solution by using the common eigenvalue of A^* and B in the setting of Banach spaces too. Let A and B be two nonzero bounded linear operators and X be an unknown operator on a Banach space \mathcal{E} . Suppose that λ is a common eigenvalue of A^* and B . There exist nonzero elements $u \in \mathcal{E}$ and $\phi \in \mathcal{E}^*$ such that $Bu = \lambda u$ and $A^*\phi = \lambda\phi$, respectively. Then $X = \phi \otimes u$ satisfies the equation $BX - XA = 0$, because

$$BX - XA = B(\phi \otimes u) - (\phi \otimes u)A = (\phi \otimes Bu) - (A^*\phi \otimes u) = 0.$$

Motivated by these studies, we intend to find an integral form and rank one solution for $BX - XA = \tau(X)$, where τ is a linear map.

We need the following preliminaries. We are interested in focusing on a certain family of holomorphic functions. Let $f_t(\xi) = e^{t\xi}$ be defined on a domain Ω for each $t \in \mathbb{R}$, whence $f_t(T) = e^{-tT}$ for any $T \in \mathcal{L}(\mathcal{E})$.

Properties of this exponential family are summarized in the following theorem [13].

- Theorem 2.2.** (i) $e^{-(t+s)T} = e^{-tT} e^{-sT}$;
(ii) $\lim_{t \rightarrow 0} e^{-tT} = I$;
(iii) $\frac{d}{dt} e^{-tT} = -T e^{-tT}$;
(iv) If $\sigma(T) \subset \{\zeta \in \mathbb{C} : \text{Re}(\zeta) > 0\}$, there exists a constant C_T such that $\|e^{-tT}\| \leq C_T$ for all $t \geq 0$;
(v) If $\sigma(T) \subset \{\zeta \in \mathbb{C} : \text{Re}(\zeta) > 0\}$, then $\lim_{t \rightarrow \infty} e^{-tT} = 0$.

We consider the following equation

$$BX - XA = \tau(X). \tag{2.4}$$

Theorem 2.3. *Let \mathcal{E} be a Banach space. If $\sigma(A) \subset \{z \in \mathbb{C} | \operatorname{Re}(z) < 0\}$ and $\sigma(B) \subset \{z \in \mathbb{C} | \operatorname{Re}(z) > 0\}$, then there exists a rank one solution for Equation 2.4, where $\tau : \mathcal{L}(\mathcal{E}) \rightarrow \mathcal{L}(\mathcal{E})$ is a linear map on which $\tau(e^{-sB}) = -A$ for any $s > 0$.*

Proof. We adapt the following operators defined in [7]. Let $S_v : L^2(0, \infty) \rightarrow \mathcal{E}$ and $R_\phi : \mathcal{E} \rightarrow L^2(0, \infty)$ be defined by

$$S_v(u) = \int_0^\infty u(s)e^{-sB}v ds, \quad u \in L^2(0, \infty),$$

$$(R_\phi w)(s) = \phi(e^{sA}w), \quad w \in \mathcal{E}, \quad s \in (0, \infty),$$

where $v \neq 0$ is an arbitrary element of \mathcal{E} . Now we show that $X := S_v R_\phi$ satisfies Equation (2.4). Note that X has the following integral representation.

$$X(w) = \int_0^\infty \phi(e^{sA}w)e^{-sB}v ds, \quad w \in \mathcal{E}.$$

We have

$$\begin{aligned} (BX - XA)w &= \int_0^\infty (\phi(e^{sA}w)e^{-sB}Bv - \phi(e^{sA}Aw)e^{-sB}v) ds \\ &= \int_0^\infty \frac{d}{ds}(-\phi(e^{sA}w)e^{-sB}v) ds \\ &= -\phi(e^{sA}w)e^{-sB}v|_0^\infty \\ &= \phi(w)v. \end{aligned}$$

and

$$\begin{aligned} \tau(X)w &= \tau\left(\int_0^\infty \phi(e^{sA}w)e^{-sB}v ds\right)w \\ &= \tau\left(\int_0^\infty \phi(e^{sA}w)e^{-sB}v ds\right)w \\ &= \left(\int_0^\infty \phi(e^{sA})\tau(e^{-sB}v) ds\right)w \\ &= \left(\int_0^\infty -\phi(e^{sA}w)A ds\right)v \\ &= (-\phi(e^{sA}w)|_0^\infty)v \\ &= \phi(w)v. \end{aligned}$$

Therefore X is a rank one solution of (2.4). \square

In the following, we consider different equations in the setting of Banach spaces. First, consider the equation $XAX = AX$. Holbrook et al. [8] verified that every solution of the equation $XAX = AX$ in a Hilbert space is an idempotent precisely when every restriction of A to an invariant subspace has a dense range. Next proposition introduce a rank one idempotent solution for this equation.

Proposition 2.4. *If the point spectrum of A is nonempty, then $XAX = AX$ has a nonzero idempotent rank one solution.*

Proof. Suppose that λ is an eigenvalue of A , so there exists a unit eigenvector $u \in S$ such that $Au = \lambda u$. By the Hahn–Banach Theorem there exists a bounded

linear functional ϕ on \mathcal{E} such that $\phi(u) = \|u\| = 1$. Then $X = \phi \otimes u$ satisfies the $XAX = AX$, because

$$XAX = (\phi \otimes u)A(\phi \otimes u) = (\phi \otimes u)(\phi \otimes Au) = \lambda(\phi \otimes \phi(u).u) = (\phi \otimes Au) = AX.$$

Thus $X = \phi \otimes u$ is a rank one idempotent solution. \square

We recall that a bounded linear operator $A \neq 0$ on \mathcal{E} is said to have an outer inverse if there is a bounded linear operator B on \mathcal{E} such that

$$BAB = B.$$

Proposition 2.5. *If A has an eigenvalue, then A has a nonzero rank one idempotent outer inverse.*

Proof. Let λ be an eigenvalue of A and v be the corresponding eigenvector. By the Hahn–Banach Theorem there exists $\phi \in \mathcal{E}^*$ such that $\phi(v) = 1$. Therefore, $B = \lambda^{-1}(\phi \otimes v)$ is an idempotent outer inverse for A , since

$$BAB = \lambda^{-2}(\phi \otimes v)A(\phi \otimes v) = \lambda^{-1}(\phi \otimes \phi(v).v) = \lambda^{-1}(\phi \otimes v) = B.$$

\square

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