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THE PRODUCTS OF DIFFERENTIATION AND COMPOSITION OPERATORS ON BLOCH TYPE SPACES

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Abstract. Suppose that φ is an analytic self-map of the unit disk $\mathbb{D}$ and m is a nonnegative integer. A integral characterization of the boundedness and compactness of the operator $C_\varphi D^m$ on Bloch type spaces are given, where $(C_\varphi D^m f)(z) = f^{(m)}(\varphi(z))$. Moreover, an estimate of the essential norm for this operator on Bloch type spaces is also given.

1. Introduction

Let $\mathbb{D}$ be the unit disk of complex plane $\mathbb{C}$, and $H(\mathbb{D})$ the class of functions analytic in $\mathbb{D}$. For $a \in \mathbb{D}$, let σ_a denote the conformal automorphism defined by $\sigma_a = \frac{a-z}{1-\bar{a}z}$. Let $g(z,a)$ be Green's function for D with logarithmic singularity at a, *i.e.*, $g(z,a) = \log \frac{1}{|\sigma_a(z)|}$.

For $\alpha \in (0,\infty)$, recall that an $f \in H(\mathbb{D})$ is said to belong to the Bloch type space, denoted by $\mathcal{B}^\alpha$, if

$$\|f\|_\alpha = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f'(z)| < \infty.$$

With the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \|f\|_\alpha$, $\mathcal{B}^\alpha$ is a Banach space. Let $\mathcal{B}^\alpha_0$ denote the space which consists of all $f \in \mathcal{B}^\alpha$ satisfying $\lim_{|z|\to 1}(1-|z|^2)^\alpha |f'(z)| = 0$. This space is called the little Bloch type space. From [26], we see that $f \in \mathcal{B}^\alpha$ if and only if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{2\alpha-2} g^2(z,a)\, dA(z) < \infty, \tag{1.1}$$

where dA denote the normalized Lebesgue area measure on $\mathbb{D}$. $f \in \mathcal{B}^\alpha_0$ if and only if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{2\alpha-2} g^2(z,a)\, dA(z) = 0. \tag{1.2}$$

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Moreover,

$$\|f\|_{\mathcal{B}^\alpha}^2 \asymp \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^{2\alpha-2} g^2(z,a)dA(z).$$

Let φ be an analytic self-map of $\mathbb{D}$. The composition operator C_φ is defined by $C_\varphi(f) = f\circ\varphi,\ f\in H(\mathbb{D})$. Let D denote the differentiation operator, *i.e.*, $Df = f',\ f\in H(\mathbb{D})$. For a nonnegative integer m, the m-th differentiation operator is define by

$$(D^m f)(z) = f^{(m)}(z), \quad f\in H(\mathbb{D}).$$

The product of the operator D^m and the composition operators C_φ, denoted by $C_\varphi D^m$, is defined as follows.

$$(C_\varphi D^m f)(z) = f^{(m)}(\varphi(z)),\ f\in H(\mathbb{D}).$$

By Schwarz-Pick Lemma, we see that each composition operator is bounded on the Bloch space. See [9, 10, 11, 12, 13, 14, 18, 19, 21, 27] for the study of the compactness and essential norm of composition operator on the Bloch space. Product composition operator ans some other operators attracted considerable interest recently. The product of differentiation and composition operators has been studied, for example, in [3, 4, 5, 6, 7, 8, 15, 16, 17, 20, 23, 24, 25, 28, 30] and the related references therein.

In [20], Wu and Wulan characterized the boundedness and the compactness of the operator $C_\varphi D^m$ acting on the Bloch space. They proved that $C_\varphi D^m : \mathcal{B}\to\mathcal{B}$ is compact if and only if $\lim_{n\to\infty}\|C_\varphi D^m(z^n)\|_{\mathcal{B}} = 0$, as well as

$$\lim_{|a|\to 1}\left\|C_\varphi D^m\Big(\frac{a-z}{1-\overline{a}z}\Big)\right\|_{\mathcal{B}} = 0.$$

Liang and Zhou in [8] proved that $C_\varphi D^m : \mathcal{B}^\alpha\to\mathcal{B}^\beta$ is compact if and only if $C_\varphi D^m : \mathcal{B}^\alpha\to\mathcal{B}^\beta$ is bounded and $\lim_{n\to\infty} n^{\alpha-1}\|C_\varphi D^m(z^n)\|_{\mathcal{B}^\beta} = 0$. In [28], Zhou and Zhu showed that $C_\varphi D^m : \mathcal{B}^\alpha\to\mathcal{B}^\beta$ is compact if and only if $C_\varphi D^m : \mathcal{B}^\alpha\to\mathcal{B}^\beta$ is bounded and $\lim_{|a|\to 1^-}\left\|C_\varphi D^m\left(\frac{1-|a|^2}{(1-\overline{a}z)^\alpha}\right)\right\|_{\mathcal{B}^\beta} = 0$.

In this paper, we study the boundedness, compactness and essential norm of the operator $C_\varphi D^m$ between Bloch type spaces by using the integral characterizations which stated in (1) and (2).

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A\asymp B$ means that there is a positive constant C such that $B/C\le A\le CB$.

2. Main results and proofs

In this section, we will state and prove the main results in this paper. For this purpose we give some auxiliary results which we use in this paper. The following lemma can be proved as Proposition 3.11 in [1].

Lemma 2.1. *Let $0<\alpha,\beta<\infty$, φ be an analytic self map of $\mathbb{D}$ and m be a nonnegative integer. Then $C_\varphi D^m : \mathcal{B}^\alpha\to\mathcal{B}^\beta$ is compact if and only if for every bounded sequence $\{f_n\}$ in $\mathcal{B}$ converging to 0 uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$, $\lim_{n\to\infty}\|C_\varphi D^m f_n\|_{\mathcal{B}^\beta} = 0$.*

Similar to the proof of Proposition in [19], we have the following result.

Lemma 2.2. *Let $0 < \alpha, \beta < \infty$, φ be an analytic self map of $\mathbb{D}$ and m be a nonnegative integer. If $C_\varphi D^m : \mathcal{B}^\alpha(\mathcal{B}_0^\alpha) \to \mathcal{B}^\beta$ is compact, then for any $\epsilon > 0$ there exists a $\delta \in (0,1)$, such that for all f in $\mathbb{B}_{\mathcal{B}^\alpha}$(or $\mathbb{B}_{\mathcal{B}_0^\alpha}$), the unit ball of $\mathcal{B}^\alpha$ (or $\mathcal{B}_0^\alpha$), and $\delta < r < 1$, holds*

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f^{(m+1)}(\varphi(z))|^2 |\varphi'(z)|^2 (1-|z|^2)^{2\beta-2} g^2(z,a) dA(z) < \epsilon.$$

By modifying the proof of Theorem 4.2 of [12], we can prove the following result. We omit the details.

Lemma 2.3. *Let $0 < \alpha, \beta < \infty$, φ be an analytic self map of $\mathbb{D}$ and m be a nonnegative integer. Then $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}_0^\beta$ is compact if and only if $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}_0^\beta$ is bounded and*

$$\lim_{|a|\to 1} \sup_{\|f\|_{\mathcal{B}^\alpha}<1} \int_{\mathbb{D}} |(C_\varphi D^m f)'(z)|^2 (1-|z|^2)^{2\beta-2} g^2(z,a) dA(z) = 0.$$

The following lemma can be found, for example, in [29].

Lemma 2.4. *For $f \in H(\mathbb{D})$, $0 < \alpha < \infty$ and m be a nonnegative integer. Then $f \in \mathcal{B}^\alpha$ if and only if*

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha+m} |f^{(m+1)}(z)| < \infty.$$

$f \in \mathcal{B}_0^\alpha$ if and only if

$$\lim_{|z|\to 1} (1-|z|^2)^{\alpha+m} |f^{(m+1)}(z)| = 0.$$

Theorem 2.5. *Let $0 < \alpha, \beta < \infty$, φ be an analytic self map of $\mathbb{D}$ and m be a nonnegative integer. Then the following statements are equivalent:*

(i) $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded;

(ii) $C_\varphi D^m : \mathcal{B}_0^\alpha \to \mathcal{B}^\beta$ is bounded;

(iii)

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(m+\alpha)}} (1-|z|^2)^{2\beta-2} g^2(z,a) dA(z) < \infty. \tag{2.1}$$

Proof. $(iii) \Rightarrow (i)$. For any $f \in \mathcal{B}^\alpha$, by Lemma 2.4, we have

$$\begin{aligned}
\|C_\varphi D^m f\|_{\mathcal{B}^\beta}^2 &\asymp \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(C_\varphi D^m f)'(z)|^2 (1-|z|^2)^{2\beta-2} g^2(z,a) dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f^{(m+1)}(\varphi(z))|^2 |\varphi'(z)|^2 (1-|z|^2)^{2\beta-2} g^2(z,a) dA(z) \\
&\le \|f\|_{\mathcal{B}^\alpha}^2 \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2 (1-|z|^2)^{2\beta-2}}{(1-|\varphi(z)|^2)^{2(m+\alpha)}} g^2(z,a) dA(z) \qquad (2.2) \\
&< \infty.
\end{aligned}$$

Hence $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded.

$(i) \Rightarrow (ii)$. This implication is obvious.

$(ii) \Rightarrow (iii)$. Let $f \in \mathcal{B}^\alpha$. Set $f_r(z) = f(rz)$ for $0 < r < 1$. It is easy to check that $f_r \in \mathcal{B}_0^\alpha$ and $\|f_r\|_\alpha \le \|f\|_\alpha$. Thus, by the assumption we have

$$\|C_\varphi D^m f_r\|_\beta \le \|C_\varphi D^m\| \|f_r\|_\alpha \le \|C_\varphi D^m\|_{\mathcal{B}^\alpha\to\mathcal{B}^\beta} \|f\|_\alpha \le \|C_\varphi D^m\|_{\mathcal{B}^\alpha\to\mathcal{B}^\beta} \|f\|_{\mathcal{B}^\alpha} \tag{2.3}$$

for any $f \in \mathcal{B}^\alpha$. By [2] we know that there exists two functions $f_1,\ f_2 \in \mathcal{B}^\alpha$ such that

$$\frac{C}{(1-|z|^2)^\alpha} \le |f_1'(z)| + |f_2'(z)|, \ \ z \in \mathbb{D}.$$

By Lemma 2.4, we see that there exist $h, k \in \mathcal{B}^\alpha$ and

$$\frac{C}{(1-|z|^2)^{\alpha+m}} \le |h^{(m+1)}(z)| + |k^{(m+1)}(z)|, \ \ z \in \mathbb{D}. \tag{2.4}$$

Replacing f in (2.3) by h and k respectively, we get

$$\|C_\varphi D^m h_r\|_\beta \le \|C_\varphi D^m\|_{\mathcal{B}^\alpha \to \mathcal{B}^\beta} \|h\|_{\mathcal{B}^\alpha}, \ \|C_\varphi D^m k_r\|_\beta \le \|C_\varphi D^m\|_{\mathcal{B}^\alpha \to \mathcal{B}^\beta} \|k\|_{\mathcal{B}^\alpha}.$$

Then

$$\begin{aligned}
& \int_{\mathbb{D}} \frac{|r|^{2m+2}|\varphi'(z)|^2}{(1-|r\varphi(z)|^2)^{2(m+\alpha)}}(1-|z|^2)^{2\beta-2}g^2(z,a)dA(z) \\
\le \ \ & 2\int_{\mathbb{D}} \left(|h^{(m+1)}(r\varphi(z))|^2 + |k^{(m+1)}(r\varphi(z))|^2\right)|r|^{2m+2}|\varphi'(z)|^2(1-|z|^2)^{2\beta-2}g^2(z,a)dA(z) \\
= \ \ & 2\int_{\mathbb{D}} \left(|(h_r^{(m)} \circ \varphi)'(z)|^2 + |(k_r^{(m)} \circ \varphi)'(z)|^2\right)(1-|z|^2)^{2\beta-2}g^2(z,a)dA(z) \\
\le \ \ & 2\|C_\varphi D^m h_r\|_\beta^2 + 2\|C_\varphi D^m k_r\|_\beta^2 \\
\le \ \ & 2\|C_\varphi D^m\|^2_{\mathcal{B}^\alpha \to \mathcal{B}^\beta}(\|h\|^2_{\mathcal{B}^\alpha} + \|k\|^2_{\mathcal{B}^\alpha}) < \infty
\end{aligned}$$

for all $a \in \mathbb{D}$ and $r \in (0,1)$. This estimate and Fatou's Lemma give (2.1). □

Theorem 2.6. *Let $0 < \alpha, \beta < \infty$, φ be an analytic self map of $\mathbb{D}$ and m be a nonnegative integer. Then $C_\varphi D^m : \mathcal{B}_0^\alpha \to \mathcal{B}_0^\beta$ is bounded if and only if $\varphi \in \mathcal{B}_0^\beta$ and*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(m+\alpha)}}(1-|z|^2)^{2\beta-2}g^2(z,a)dA(z) < \infty. \tag{2.5}$$

Proof. Assume that $C_\varphi D^m : \mathcal{B}_0^\alpha \to \mathcal{B}_0^\beta$ is bounded. It is clear that $C_\varphi D^m : \mathcal{B}_0^\alpha \to \mathcal{B}^\beta$ is bounded. By Theorem 2.5, (2.5) holds. Let $f(z) = z^{m+1}$. Using the boundedness of $C_\varphi D^m : \mathcal{B}_0^\alpha \to \mathcal{B}_0^\beta$ we see that $\varphi \in \mathcal{B}_0^\beta$.

Conversely, assume that $\varphi \in \mathcal{B}_0^\beta$ and (2.5) holds. By Theorem 2.5, we see that $C_\varphi D^m : \mathcal{B}_0^\alpha \to \mathcal{B}^\beta$ is bounded. To prove that $C_\varphi D^m : \mathcal{B}_0^\alpha \to \mathcal{B}_0^\beta$ is bounded, it suffices to prove that $C_\varphi D^m f \in \mathcal{B}_0^\beta$ for any $f \in \mathcal{B}_0^\alpha$. Let $f \in \mathcal{B}_0^\alpha$. For every $\epsilon > 0$, by Lemma 2.4, we can choose $\rho \in (0,1)$ such that $|f^{(m+1)}(w)|(1-|w|^2)^{\alpha+m} < \varepsilon$ for all $w \in \mathbb{D} \setminus \rho\overline{\mathbb{D}}$. Then,

$$\begin{aligned}
&\lim_{|a|\to 1} \int_{\mathbb{D}} |(C_\varphi D^m f)'(z)|^2 (1-|z|^2)^{2\beta-2} g^2(z,a) dA(z) \\
= &\lim_{|a|\to 1} \Big(\int_{|\varphi(z)|>\rho} + \int_{|\varphi(z)|\le\rho} \Big) |f^{(m+1)}(\varphi(z))|^2 |\varphi'(z)|^2 (1-|z|^2)^{2\beta-2} g^2(z,a) dA(z) \\
\le &\ \varepsilon \lim_{|a|\to 1} \int_{|\varphi(z)|>\rho} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(m+\alpha)}} (1-|z|^2)^{2\beta-2} g^2(z,a) dA(z) \\
&+ \frac{\|f\|^2_{\mathcal{B}^\alpha}}{(1-\rho^2)^{2(m+\alpha)}} \lim_{|a|\to 1} \int_{|\varphi(z)|\le\rho} |\varphi'(z)|^2 (1-|z|^2)^{2\beta-2} g^2(z,a) dA(z).
\end{aligned}$$

From the above inequality and by the assumption, we get the desired result. □

Theorem 2.7. *Let $0 < \alpha, \beta < \infty$, φ be an analytic self map of $\mathbb{D}$ and m be a nonnegative integer. Then the following statements are equivalent:*
(i) $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta_0$ is bounded;
(ii) $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta_0$ is compact;
(iii)

$$\lim_{|a|\to 1} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(m+\alpha)}} (1-|z|^2)^{2\beta-2} g^2(z,a) dA(z) = 0. \tag{2.6}$$

Proof. $(ii) \Rightarrow (i)$. It is obvious.

$(i) \Rightarrow (iii)$. Assume that $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta_0$ is bounded. From the proof of Theorem 2.5, we can choose functions $g, h \in \mathcal{B}^\alpha$ such that

$$\frac{C}{(1-|z|^2)^{m+\alpha}} \le |g^{(m+1)}(z)| + |h^{(m+1)}(z)|, \;\; z \in \mathbb{D}.$$

Then we get $C_\varphi D^m g_1, C_\varphi D^m g_2 \in \mathcal{B}^\beta_0$. Therefore,

$$\begin{aligned}
&C \lim_{|a|\to 1} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(m+\alpha)}} (1-|z|^2)^{2\beta-2} g^2(z,a) dA(z) \\
\le &\ 2 \lim_{|a|\to 1} \int_{\mathbb{D}} |g^{(m+1)}(\varphi(z))|^2 |\varphi'(z)|^2 (1-|z|^2)^{2\beta-2} g^2(z,a) dA(z) \\
&+2 \lim_{|a|\to 1} \int_{\mathbb{D}} |h^{(m+1)}(\varphi(z))|^2 |\varphi'(z)|^2 (1-|z|^2)^{2\beta-2} g^2(z,a) dA(z) \\
= &\ 2 \lim_{|a|\to 1} \int_{\mathbb{D}} |C_\varphi D^m g|^2 (1-|z|^2)^{2\beta-2} g^2(z,a) dA(z) \\
&+2 \lim_{|a|\to 1} \int_{\mathbb{D}} |C_\varphi D^m h|^2 (1-|z|^2)^{2\beta-2} g^2(z,a) dA(z) \\
= &\ 0,
\end{aligned}$$

as desired.

$(iii) \Rightarrow (ii)$. Assume that (2.6) holds. By Theorem 2.5 we see that $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded. We first prove that $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta_0$ is bounded. For this purpose, we only need to prove that $C_\varphi D^m f \in \mathcal{B}_0^\beta$ for any $f \in \mathcal{B}^\alpha$. Let $f \in \mathcal{B}^\alpha$. We have

$$\begin{aligned}
&\lim_{|a|\to 1}\int_{\mathbb{D}} |(C_\varphi D^m f)'(z)|^2(1-|z|^2)^{2\beta-2}g^2(z,a)dA(z)\\
\le\ & C\|f\|^2_{\mathcal{B}^\alpha}\lim_{|a|\to 1}\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(m+\alpha)}}(1-|z|^2)^{2\beta-2}g^2(z,a)dA(z), \quad (2.7)
\end{aligned}$$

which with (2.6) imply that $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}_0^\beta$ is bounded. Moreover, we have

$$\lim_{|a|\to 1}\sup_{\|f\|_{\mathcal{B}^\alpha}\le 1}\int_{\mathbb{D}} |(C_\varphi D^m f)'(z)|^2(1-|z|^2)^{2\beta-2}g^2(z,a)dA(z) = 0. \quad (2.8)$$

From Lemma 2.4, we see that $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}_0^\beta$ is compact. □

Theorem 2.8. *Let $0 < \alpha, \beta < \infty$, φ be an analytic self map of $\mathbb{D}$ and m be a nonnegative integer. Suppose that $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded. Then*

$$\|C_\varphi D^m\|^2_{e,\mathcal{B}_0^\alpha\to\mathcal{B}^\beta} \approx \|C_\varphi D^m\|^2_{e,\mathcal{B}^\alpha\to\mathcal{B}^\beta} \approx T,$$

where

$$T := \lim_{r\to 1}\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(m+\alpha)}}(1-|z|^2)^{2\beta-2}g^2(z,a)dA(z).$$

Proof. It is clear that

$$\|C_\varphi D^m\|^2_{e,\mathcal{B}_0^\alpha\to\mathcal{B}^\beta} \le \|C_\varphi D^m\|^2_{e,\mathcal{B}^\alpha\to\mathcal{B}^\beta}.$$

Next we prove that

$$\|C_\varphi D^m\|^2_{e,\mathcal{B}_0^\alpha\to\mathcal{B}^\beta} \gtrsim T.$$

Let $\{r_i\} \subset (1/2, 1)$ such that $r_i \to 1$ as $i \to \infty$. Define

$$f_{i,j,\theta}(z) = \frac{1}{r_i}\sum_{k=1}^{\infty} \frac{2^{k(m+\alpha)}}{(2^k+2^{n_2})(2^k+2^j-1)\cdots(2^k+2^j-m)}(r_i e^{i\theta})^{2^k} z^{2^k+2^j}$$

for $i, j \in \mathbb{N}$ such that $2^j - m \ge 0$ and $\theta \in [0, 2\pi)$. Since

$$\lim_{k\to\infty} 2^{k(1-\alpha)}\left|\frac{2^{k(m+\alpha)}}{(2^k+2^j)(2^k+2^j-1)\cdots(2^k+2^j-m)}(r_i e^{i\theta})^{2^k}\right| = 0,$$

the function $f_{i,j,\theta} \in \mathcal{B}_0^\alpha$ by Theorem 1 of [22]. Moreover,

$$\sup_{k\in\mathbb{N}} 2^{k(1-\alpha)}\left|\frac{2^{k(m+\alpha)}}{(2^k+2^j)(2^k+2^j-1)\cdots(2^k+2^j-m)}(r_i e^{i\theta})^{2^k}\right| \le 1.$$

Hence there exists a positive constant M such that $\|f_{i,j,\theta}\|_{\mathcal{B}^\alpha} \le M$ for all $i, j \in \mathbb{N}$ such that $2^j - m \ge 0$ and $\theta \in [0, 2\pi)$. Moreover, $f_{i,j,\theta}$ tends to zero uniformly on compact subsets of $\mathbb{D}$ for every i and θ as $j \to \infty$, and therefore $f_{i,j,\theta}$ tends to zero weakly as $j \to \infty$. It follows that for any compact operator $J : \mathcal{B}_0^\alpha \to \mathcal{B}^\beta$,

$$\begin{aligned}
\|C_\varphi D^m - J\|_{\mathcal{B}_0^\alpha\to\mathcal{B}^\beta} &\gtrsim \limsup_{j\to\infty}\sup_{i,\theta}\|(C_\varphi D^m - J)(f_{i,j,\theta})\|_{\mathcal{B}^\beta}\\
&\ge \limsup_{j\to\infty}\sup_{i,\theta}\|C_\varphi D^m(f_{i,j,\theta})\|_{\mathcal{B}^\beta} - \limsup_{j\to\infty}\sup_{i,\theta}\|J(f_{i,j,\theta})\|_{\mathcal{B}^\beta}\\
&= \limsup_{j\to\infty}\sup_{i,\theta}\|C_\varphi D^m(f_{i,j,\theta})\|_{\mathcal{B}^\beta}.
\end{aligned}$$

Therefore,

$$\begin{aligned}&\|C_\varphi D^m\|^2_{e,\mathcal{B}_0^\alpha\to\mathcal{B}^\beta} = \inf_J \|C_\varphi D^m - J\|^2_{\mathcal{B}_0^\alpha\to\mathcal{B}^\beta}\\ \gtrsim\ & \limsup_{j\to\infty}\sup_{i,\theta}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |f^{(m+1)}_{i,j,\theta}(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^{2\beta-2}g^2(z,a)dA(z).\end{aligned}$$

Given $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that

$$\|C_\varphi D^m\|^2_{e,\mathcal{B}_0^\alpha\to\mathcal{B}^\beta} + \varepsilon \gtrsim \int_{\mathbb{D}} |f^{(m+1)}_{i,j,\theta}(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^{2\beta-2}g^2(z,a)dA(z)$$

for all a, θ and i when $j \ge N$. Let $a \in \mathbb{D}$ be fixed. Integrating with respect to θ, using Fubini's theorem and Parseval's formula, we obtain

$$\begin{aligned}&2\pi(\|C_\varphi D^m\|^2_{e,\mathcal{B}_0^\alpha\to\mathcal{B}^\beta} + \varepsilon)\\ \gtrsim\ & \int_{\mathbb{D}}\int_0^{2\pi} |f^{(m+1)}_{i,j,\theta}(\varphi(z))|^2 d\theta|\varphi'(z)|^2(1-|z|^2)^{2\beta-2}g^2(z,a)dA(z)\\ =\ & \int_{\mathbb{D}} |\varphi(z)|^{2(2^j-m)}\int_0^{2\pi}\Big|\sum_{k=1}^{\infty} 2^{k(m+\alpha)}e^{2^k i\theta}(r_i\varphi(z))^{2^k-1}\Big|^2 d\theta\\ & \times|\varphi'(z)|^2(1-|z|^2)^{2\beta-2}g^2(z,a)dA(z)\\ =\ & \int_{\mathbb{D}} |\varphi(z)|^{2^{j+1}-2m}\Big(\sum_{k=1}^{\infty} 2^{2k(m+\alpha)}|r_i\varphi(z)|^{2(2^k-1)}\Big)|\varphi'(z)|^2(1-|z|^2)^{2\beta-2}g^2(z,a)dA(z).\end{aligned}$$

From the formula (3.8) in [9], we have that

$$\sum_{k=1}^{\infty} 2^{2k(m+\alpha)}|r_i\varphi(z)|^{2(2^k-1)} \gtrsim \frac{1}{(1-|r_i\varphi(z)|^2)^{2(m+\alpha)}}$$

for all $z \in \mathbb{D}$ with $|\varphi(z)| > 1/2$. Thus by Fatou's Lemma, we get

$$\begin{aligned}&2\pi(\|C_\varphi D^m\|^2_{e,\mathcal{B}_0^\alpha\to\mathcal{B}^\beta} + \varepsilon)\\ \gtrsim\ & \liminf_{i\to\infty}\int_{\mathbb{D}} |\varphi(z)|^{2^{j+1}-2m}\frac{|\varphi'(z)|^2}{(1-|r_i\varphi(z)|^2)^{2(m+\alpha)}}(1-|z|^2)^{2\beta-2}g^2(z,a)dA(z)\\ \gtrsim\ & \int_{\mathbb{D}} |\varphi(z)|^{2^{j+1}-2m}\frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(m+\alpha)}}(1-|z|^2)^{2\beta-2}g^2(z,a)dA(z)\\ \gtrsim\ & \int_{\mathbb{D}} |\varphi(z)|^{2^{j+1}}\frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(m+\alpha)}}(1-|z|^2)^{2\beta-2}g^2(z,a)dA(z).\end{aligned}$$

Since $a \in \mathbb{D}$ was arbitrary, we obtain that

$$\begin{aligned}&2\pi(\|C_\varphi D^m\|^2_{e,\mathcal{B}_0^\alpha\to\mathcal{B}^\beta} + \varepsilon)\\ \gtrsim\ & \frac{1}{e}\limsup_{j\to\infty}\sup_{a\in\mathbb{D}}\int_{\{|\varphi(z)|>1-2^{-(j+1)}\}} \frac{|\varphi'(z)|^2(1-|z|^2)^{2\beta-2}g^2(z,a)}{(1-|\varphi(z)|^2)^{2(m+\alpha)}}dA(z)\\ =\ & \frac{1}{e}\limsup_{r\to1}\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r} \frac{|\varphi'(z)|^2(1-|z|^2)^{2\beta-2}g^2(z,a)}{(1-|\varphi(z)|^2)^{2(m+\alpha)}}dA(z),\end{aligned}$$

for all $\varepsilon > 0$. Therefore $\|C_\varphi D^m\|^2_{e,\mathcal{B}_0^\alpha\to\mathcal{B}^\beta} \gtrsim T$.

Finally, we prove that

$$\|C_\varphi D^m\|^2_{e,\mathcal{B}^\alpha\to\mathcal{B}^\beta} \lesssim T.$$

For $j \in \mathbb{N}$, define $(K_j f)(z) = (K_{\psi_j} f)(z) = f(\frac{j}{j+1}z)$, where $\psi_j(z) = \frac{jz}{j+1}$. Since the operator K_j is compact on $\mathcal{B}^\alpha$ for all $j \in \mathbb{N}$ and $C_\varphi D^n : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded, we get

$$\begin{aligned}
&\|C_\varphi D^m\|^2_{e,\mathcal{B}^\alpha\to\mathcal{B}^\beta} \le \|C_\varphi D^m - C_\varphi D^m K_j\|^2_{\mathcal{B}^\alpha\to\mathcal{B}^\beta} = \|C_\varphi D^m (I-K_j)\|^2_{\mathcal{B}^\alpha\to\mathcal{B}^\beta}\\
\approx\ & \sup_{\|f\|_{\mathcal{B}^\alpha}\le 1} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|(f-K_j f)^{(m+1)}(\varphi(z))\right|^2 |\varphi'(z)|^2 (1-|z|^2)^{2\beta-2} g^2(z,a)\, dA(z)\\
\le\ & \sup_{\|f\|_{\mathcal{B}^\alpha}\le 1} \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|\le r} \left|(f-K_j f)^{(m+1)}(\varphi(z))\right|^2 |\varphi'(z)|^2 (1-|z|^2)^{2\beta-2} g^2(z,a)\, dA(z)\\
&+ \sup_{\|f\|_{\mathcal{B}^\alpha}\le 1} \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|> r} \left|(f-K_j f)^{(m+1)}(\varphi(z))\right|^2 |\varphi'(z)|^2 (1-|z|^2)^{2\beta-2} g^2(z,a)\, dA(z)\\
=\ & J_1 + J_2
\end{aligned}$$

for all $r \in (0,1)$ and $j \in \mathbb{N}$, where $I(f) = f$ and

$$J_1 = \sup_{\|f\|_{\mathcal{B}^\alpha}\le 1} \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|\le r} \left|(f-K_j f)^{(m+1)}(\varphi(z))\right|^2 |\varphi'(z)|^2 (1-|z|^2)^{2\beta-2} g^2(z,a)\, dA(z)$$

and

$$J_2 = \sup_{\|f\|_{\mathcal{B}^\alpha}\le 1} \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|> r} \left|(f-K_j f)^{(m+1)}(\varphi(z))\right|^2 |\varphi'(z)|^2 (1-|z|^2)^{2\beta-2} g^2(z,a)\, dA(z).$$

Since $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded, we see that $\varphi \in \mathcal{B}^\beta$. Since $f - f\circ\psi_j$ and its derivative tend to zero uniformly in a compact subset of $\mathbb{D}$ as $j \to \infty$, it follows that

$$J_1 \le \|\varphi\|_{\mathcal{B}^\beta} \limsup_{j\to\infty} \sup_{\|f\|_{\mathcal{B}^\alpha}\le 1} \sup_{|\varphi(z)|\le r} |(f-K_j f)^{(m+1)}(\varphi(z))|^2 = 0.$$

On the other hand, Since

$$\|f - K_j f\|_{\mathcal{B}^\alpha} \le \|f\|_{\mathcal{B}^\alpha} + \|f\circ\psi_j\|_{\mathcal{B}^\alpha} \le 2\|f\|_{\mathcal{B}^\alpha} \le 2, \tag{2.9}$$

by Lemma 2.4 we get

$$J_2 \le \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(m+\alpha)}} (1-|z|^2)^{2\beta-2} g^2(z,a)\, dA(z).$$

Consequently,

$$\begin{aligned}
\|C_\varphi D^m\|^2_{e,\mathcal{B}^\alpha\to\mathcal{B}^\beta} &\le \limsup_{j\to\infty} \|C_\varphi D^m - C_\varphi D^m K_j\|^2_{\mathcal{B}^\alpha\to\mathcal{B}^\beta}\\
&\le \limsup_{j\to\infty} J_1 + \limsup_{j\to\infty} J_2\\
&\lesssim \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(m+\alpha)}} (1-|z|^2)^{2\beta-2} g^2(z,a)\, dA(z)
\end{aligned}$$

for all $r \in (0,1)$. Thus $\|C_\varphi D^m\|^2_{e,\mathcal{B}^\alpha\to\mathcal{B}^\beta} \lesssim T$. The proof is completed. □

From the last Theorem, we get the following result.

Corollary 2.9. *Let $0 < \alpha, \beta < \infty$, φ be an analytic self map of $\mathbb{D}$ and m be a nonnegative integer such that $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded. Then the following statements are equivalent:*

(i) $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact;

(ii) $C_\varphi D^m : \mathcal{B}^\alpha_0 \to \mathcal{B}^\beta$ is compact;

(iii) $\varphi \in \mathcal{B}^\beta$ *and*

$$\lim_{r\to 1}\sup_{a\in\mathbb{D}} \int\limits_{|\varphi(z)|>r} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(m+\alpha)}}(1-|z|^2)^{2\beta-2}g^2(z,a)dA(z)=0. \qquad (2.10)$$

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