Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 8 Issue 3(2016), Pages 49-58.

FIXED POINTS OF MULTIVALUED MAPPINGS IN DUALISTIC PARTIAL METRIC SPACES

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Abstract. We use the notion of Hausdorff metric on the family of closed bounded subsets of a dualistic partial metric space (DPMS) and establish a common fixed point theorem of a pair of multivalued mappings satisfying Mizoguchi and Takahashi's contractive conditions. Our result extends some well-known results in the literature.

1. Introduction

In 1922, Banach established the most famous fundamental fixed point theorem (the so-called the Banach contraction principle [9]) which has played an important role in various fields of applied mathematical analysis. It is known that the Banach contraction principle has been extended in many various directions by several authors (see [1]-[29]).

In the other hand, the study of metric spaces expressed the most important role to many fields both in pure and applied science such as biology, medicine, physics and computer science. Some generalizations of the notion of a metric space have been proposed by some authors, such as, rectangular metric spaces, semi metric spaces, pseudo metric spaces, probabilistic metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi metric spaces, D-metric spaces, and cone metric spaces (see [[1],[12],[15],[27],). Branciari [11] introduced the notion of a generalized metric space replacing the triangle inequality by a rectangular type inequality. He then extended Banach's contraction principle in such spaces.

In the last thirty years, the theory of multivalued functions has advanced in a variety of ways. In 1969, the systematic study of Banach-type fixed theorems of multivalued mappings started with the work of Nadler [24]. He used the concept of the Hausdorff metric to establish the multivalued contraction principle containing the Banach contraction principle as a special case. His finidings were followed by Azam et al.[8] and many others (see, e.g.,[16],[20],[28]).

2000 *Mathematics Subject Classification.* Primary 47H10; Secondary 54H25.
Key words and phrases. Common fixed point; Partial Hausdorrf metric; Quasi metric; Dualistic Partial metric; set-valued mappings.

Submitted April 14, 2016. Published August 15, 2016.

In 1994, Matthews [22] intoduced the concept of partial metric spaces and obtained various fixed point theorems. In particular, he established the precise relationship between partial metric spaces and the so-called weightable quasi-metric spaces, and proved a partial metric generalization of Banach's contraction mapping theorem. Later on, Neill in [25] introduced the concept of dualistic partial metric spaces (DPMS) by extending the range $R^+ \to R$. He developed several connections between partial metrics and the topological aspects of domain theory. In 2004, Oltra et al.,[26] established Banach fixed point theorem for complete DPMS. Recently many authors developed some fixed point theorems using complete DPMS for Banach's contraction principle and partial order respectively. For the sake of continuity of work on DPMS, we establish some common fixed point theorems of a pair of multivalued mappings satisfying Mizoguchi and Takahashi's contractive conditions in the setting of DPMS.

2. Preliminaries

Throughout this paper the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ will denote the set of real numbers, the set of nonnegative real numbers and the set of natural numbers, respectively.

Definition 2.1. [25] Let X be a nonempty set. Suppose that the mapping $D : X \times X \to \mathbb{R}$, satisfies:

(1) $x = y \Leftrightarrow D(x,x) = D(y,y) = D(x,y);$
(2) $D(x,x) \le D(x,y)$ for all $x, y \in X$;
(3) $D(x,y) = D(y,x)$ for all $x, y \in X$;
(4) $D(x,z) \le D(x,y) + D(y,z) - D(y,y)$, for all $x, y, z \in X$.

Then D is called a dualistic partial metric on X, and (X, D) is called a DPMS.

Note that if $\mathbb{R}$ is replaced by $\mathbb{R}^+$ then D is known as partial metric on X. To make a difference between partial metric and dualistic partial metric, we discuss an example. Let us define $D : X \times X \to \mathbb{R}$ by $D(x,y) = Sup\{x,y\}$. Now if $X = \mathbb{R}$, then D is dualistic partial metric but nor partial metric on X, for if $x = -1$ and $y = -3$ then $Sup\{-1, -3\} = -1 = D(x,y)$ which is not possible in partial metric. Each dualistic partial metric D on X generates a τ_0 topology $\tau(D)$ on X which has a base topology of open $D-$balls $\{B_D(x,\varepsilon) : x \in X, \varepsilon > 0\}$ and $B_D(x,\varepsilon) = \{y \in X : D(x,y) < \varepsilon + D(x,x)\}$. From this fact it fallows that a sequence $(x_n)_n$ in a DPMS converges to a point $x \in X$ if and only if $D(x,x) = \lim_{n\to\infty} D(x,x_n)$.

Definition 2.2. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{R}^+$, satisfies:

(1) $d(x,y) = d(y,x) = 0 \Leftrightarrow x = y$, for all $x, y \in X$;
(2) $d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in X$.

The pair (X, d) is called quasi metric space.

Each quasi metric d on X generates a τ_0 topology $\tau(d)$ on X which has a base topology of open $d-$balls $\{B_d(x,\varepsilon) : x \in X, \varepsilon > 0\}$ and $B_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$.

Moreover if d is quasi metric, then $d^s(x,y) = \max\{d(x,y), d(y,x)\}$ is a metric on X.

Let us define modulus of a dualistic partial metric by

$$|D(x,y)| = \begin{cases} D(x,y) & \text{if } D(x,y) > 0; \\ -D(x,y) & \text{if } D(x,y) < 0. \end{cases}.$$

Lemma 2.3. [26] If (X,D) is a DPMS, then the function $d_p : X \times X \to \mathbb{R}^+$ defined by

$$d_p(x,y) = D(x,y) - D(x,x),$$

for all $x,\ y \in X$, is a quasi metric on X such that $\tau(D) = \tau(d_p)$. Now if d_p is quasi metric on X then $d_p^s(x,y) = \max\{d_p(x,y), d_p(y,x)\}$ is metric on X.

Lemma 2.4. [26] (i) The sequence $\{x_n\}$ in DPMS (X,D) converges to a point x if and only if $D(x,x) = \lim_{n\to\infty} D(x_n,x)$.

(ii) The sequence $\{x_n\}$ in DPMS is called cauchy sequence if $\lim_{n,m\to\infty} D(x_n,x_m)$ exists.

(iii) The DPMS is complete if and only if the metric $\left(X, d_p^s\right)$ is complete and further $\lim_{n\to\infty} d_p^s(x_n,x) = 0$ iff $D(x,x) = \lim_{n\to\infty} D(x_n,x) = \lim_{n,m\to\infty} D(x_n,x_m)$.

A subset A of X is called closed in (X,D) if it is closed with respect to $\tau(D)$. A is called bounded in (X,D) if there exists $x_0 \in X$ and $M > 0$ such that $a \in B_D(x_0,M)$ for all $a \in A$, i.e,

$D(x_0,a) < D(x_0,x_0) + M$ for all $a \in A$.

Let $CB^D(X)$ be the collection of all nonempty, closed and bounded subsets of X with respect to the dualistic partail metric D. For $A \in CB^D(X)$, we define

$D(x,A) = \inf_{y\in A} D(x,y)$.

For $A, B \in CB^D(X)$,

$\delta_D(A,B) = \sup_{a\in A} D(a,B)$,

$\delta_D(B,A) = \sup_{b\in B} D(b,A)$,

$H_D(A,B) = \max\{\delta_D(A,B), \delta_D(B,A)\}$.

Note that $D(x,A) = 0 \Longrightarrow d_p^s(x,A) = 0$, where $d_p^s(x,A) = \inf_{y\in A} d_p^s(x,y)$.

Proposition 2.5.[7] Let (X,D) be a partial metric space. For any $A,\ B,\ C \in CB^D(X)$, we have

(i) $\delta_D(A,A) = \sup\{D(a,a) : a \in A\}$;

(ii) $\delta_D(A,A) \leq \delta_D(A,B)$;

(iii) $\delta_D(A,B) = 0 \Longrightarrow A \subseteq B$;

(iv) $\delta_D(A,B) \leq \delta_D(A,C) + \delta_D(C,B) - \inf_{c\in C} D(c,c)$.

Proposition 2.6.[7] Let (X,D) be a partial metric space. For any $A,\ B,\ C \in CB^D(X)$, we have

(i) $H_D(A,A) \leq H_D(A,B)$;

(ii) $H_D(A,B) \leq H_D(B,A)$;

(iii) $H_D(A,B) \leq H_D(A,C) + H_D(C,B) - \inf_{c\in C} D(c,c)$.

Remark 2.7.[7] Let (X,D) be a partial metric space and A be any nonempty set in (X,D), then $a \in \overline{A}$ if and only if

$D(a,A) = D(a,a)$,

where $\overline{A}$ denotes the clouser of A with respect to partial metric D. Note that A is closed in (X,D) if and only if $\overline{A} = A$.

Lemma.2.8. Let A and B be nonempty, closed and bounded subsets of a DPMS (X,D) and $0 < h \in \mathbb{R}$. Then for every $a \in A$, there exists $b \in B$ such that $D(a,b) \leq H_D(A,B) + h$.

Proof. We argue by contradiction. Suppose there exist $h>0$, such that for any $b\in B$ we have

$$D\left(a,b\right) > H_D\left(A,B\right)+h.$$

Then,

$$D\left(a,B\right)=\inf\left\{D\left(a,b\right),\ b\in B\right\}\geq H_D\left(A,B\right)+h\geq \delta_D\left(A,B\right)+h,$$

which is a contradiction. Hence, there exists $b\in B$ such that $D\left(a,b\right)\leq H_D\left(A,B\right)+h$.

Definition 2.9. [13] A function $\varphi:[0,+\infty)\longrightarrow[0,1)$ is said to be $MT-$function if it satisfies Mizoguchi and Takahashi's conditions (i.e., lim $\sup_{r\to t^+}\varphi\left(r\right)<1$ for all $t\in[0,+\infty)$).

Proposition 2.10. [13] Let $\varphi:[0,+\infty)\longrightarrow[0,1)$ be a function. Then the following statements are equivalent.

1. φ is an $MT-$function.
2. For each $t\in[0,\infty)$, there exists $r_t^{(1)}\in[0,1)$ and $\varepsilon_t^{(1)}>0$ such that $\varphi\left(s\right)\leq r_t^{(1)}$ for all $s\in\left(t,t+\varepsilon_t^{(1)}\right)$.
3. For each $t\in[0,\infty)$, there exists $r_t^{(2)}\in[0,1)$ and $\varepsilon_t^{(2)}>0$ such that $\varphi\left(s\right)\leq r_t^{(2)}$ for all $s\in\left(t,t+\varepsilon_t^{(2)}\right)$.
4. For each $t\in[0,\infty)$, there exists $r_t^{(3)}\in[0,1)$ and $\varepsilon_t^{(3)}>0$ such that $\varphi\left(s\right)\leq r_t^{(3)}$ for all $s\in\left(t,t+\varepsilon_t^{(3)}\right)$.
5. For each $t\in[0,\infty)$, there exists $r_t^{(4)}\in[0,1)$ and $\varepsilon_t^{(4)}>0$ such that $\varphi\left(s\right)\leq r_t^{(4)}$ for all $s\in\left(t,t+\varepsilon_t^{(4)}\right)$.
6. For any nonincreasing sequence $\left\{x_n\right\}_{n\in N}$ in $[0,\infty)$, we have $0\leq\sup_{n\in N}\varphi\left(x_n\right)<1$.
7. φ is a function of contractive factor [14], that is, for any strictly decreasing sequence $\left\{x_n\right\}_{n\in N}$ in $[0,\infty)$, we have $0\leq\sup_{n\in N}\varphi\left(x_n\right)<1$.

3. The Results

Mizoguchi and Takahashi proved the following theorem on complete metric spaces in [23].

Theorem. 3.1. Let (X,d) be a complete metric space and let the mapping $S:X\to CB(X)$ be a multivalued map and $\varphi:[0,+\infty)\longrightarrow[0,1)$ be an $MT-$function. Assume that

$$H(Sx,Sy)\leq\varphi\left(d\left(x,y\right)\right)d\left(x,y\right); \tag{3.1}$$

for all x, $y\in X$, Then S has a fixed point in X.

We use the notion of Hausdorff metric on the family of closed bounded subsets of a dualistic partial metric space and establish a common fixed point theorem of a pair of multivalued mappings satisfying MT-function. Following is our main result.

Theorem. 3.2. Let (X,D) be a complete DPMS. $S,T:X\to CB^D(X)$ be multivalued mappings and $\varphi:[0,+\infty)\longrightarrow[0,1)$ be an $MT-$function. Assume that

$$H_D(Sx,Ty)\leq\varphi\left(D\left(x,y\right)\right)D\left(x,y\right); \tag{3.2}$$

for all x, $y\in X$, then there exists $z\in X$ such that $z\in Sz$ and $z\in Tz$.

Proof: Let $x_0 \in X$ and $x_1 \in Sx_0$. If $D(x_0, x_1) = 0$, then $x_0 = x_1$ and

$$H_D(Sx_0, Tx_1) \leq \varphi(D(x_0, x_1)) D(x_0, x_1) = 0.$$

Thus, $Sx_0 = Tx_1$, which implies that
$x_1 = x_0 \in Sx_0 = Tx_1 = Tx_0$, and we finished. Assume that $D(x_0, x_1) > 0$. By Lemma 2.8, we can take $x_2 \in Tx_1$ such that

$$|D(x_1, x_2)| \leq \frac{H_D(Sx_0, Tx_1) + |D(x_0, x_1)|}{2}. \tag{3.3}$$

If $D(x_1, x_2) = 0$, then $x_1 = x_2$ and

$$H_D(Tx_1, Sx_2) \leq \varphi(D(x_1, x_2)) D(x_1, x_2) = 0,$$

then $Tx_1 = Sx_2$. That is $x_2 = x_1 \in Tx_1 = Sx_2 = Sx_2$ and we finished.
Assume that $D(x_1, x_2) > 0$. Again By Lemma 2.8, we can take $x_3 \in Sx_2$ such that

$$|D(x_2, x_3)| \leq \frac{H_D(Tx_1, Sx_2) + |D(x_1, x_2)|}{2}. \tag{3.4}$$

By repeating this process, we can construct a sequence x_n of points in X and a sequence A_n of elements in $CB^D(X)$ such that

$$x_{j+1} \in A_j = \begin{cases} Sx_j, & j = 2k, k \geq 0 \\ Tx_j, & j = 2k+1, \ k \geq 0 \end{cases}, \tag{3.5}$$

and

$$|D(x_j, x_{j+1})| \leq \frac{H_D(A_{j-1}, A_j) + |D(x_{j-1}, x_j)|}{2}, \tag{3.6}$$

with $j \geq 0$, along with the assumption that $D(x_j, x_{j+1}) > 0$ for each $j \geq 0$. Now for $j = 2k+1$, we have

$$\begin{aligned} |D(x_j, x_{j+1})| &\leq \frac{H_D(A_{j-1}, A_j) + |D(x_{j-1}, x_j)|}{2}, \\ &\leq \frac{H_D(Sx_{2k}, Tx_{2k+1}) + |D(x_{2k}, x_{2k+1})|}{2}, \\ &\leq \frac{\varphi(D(x_{2k}, x_{2k+1}))(D(x_{2k}, x_{2k+1}) + |D(x_{2k}, x_{2k+1})|}{2}, \\ &\leq \left(\frac{\varphi(D(x_{j-1}, x_j)) + 1}{2}\right) |D(x_{j-1}, x_j)|, \\ &\leq D(x_{j-1}, x_j). \end{aligned}$$

Similarly for $j = 2k+2$, we obtain

$$\begin{aligned} |D(x_j, x_{j+1})| &\leq \frac{H_D(Tx_{2k+1}, Sx_{2k+2}) + |D(x_{j-1}, x_j)|}{2}, \\ &\leq \left(\frac{\varphi(D(x_{j-1}, x_j)) + 1}{2}\right) |D(x_{j-1}, x_j)|, \\ &\leq D(x_{j-1}, x_j). \end{aligned}$$

It fallows that the sequence $\{D(x_n, x_{n+1})\}$ is decreasing and converges to a nonnegative real number $t \geq 0$. Define a function $\psi : [0, \infty) \longrightarrow [0, 1)$ as fallows:

$$\psi(\zeta) = \frac{\varphi(\zeta) + 1}{2}.$$

Then

$$\limsup_{\zeta\to t^{+}} \psi(\zeta) < 1.$$

Using Proposition 2.10, for $t \geq 0$, we can find $\delta(t) > 0$, $\lambda_t < 1$, such that $t \leq r \leq \delta(t) + t$ implies $\psi(r) < \lambda_t$ and there exists a natural number N such that $t \leq D(x_n, x_{n+1}) \leq \delta(t) + t$, when ever $n > N$. Hence

$$\psi(D(x_n, x_{n+1})) < \lambda_t, \text{ whenever } n > N.$$

Then for $n = 1, 2, 3, \cdots$

$$\begin{aligned} |D(x_n, x_{n+1})| &\leq \left(\frac{\varphi(D(x_{n-1}, x_n)) + 1}{2}\right) |D(x_{n-1}, x_n)|, \\ &\leq \psi(D(x_{n-1}, x_n)) |D(x_{n-1}, x_n)|, \\ &\leq \max\left\{\max_{n=1}^{N} \psi(D(x_{n-1}, x_n)), \lambda_t\right\} |D(x_{n-1}, x_n)|, \\ &\leq \left[\max\left\{\max_{n=1}^{N} \psi(D(x_{n-1}, x_n)), \lambda_t\right\}\right]^2 |D(x_{n-2}, x_{n-1})|, \\ &\leq \left[\max\left\{\max_{n=1}^{N} \psi(D(x_{n-1}, x_n)), \lambda_t\right\}\right]^n |D(x_0, x_1)|. \end{aligned}$$

Put $\max\left\{\max_{n=1}^{N} \psi(D(x_{n-1}, x_n)), \lambda_t\right\} = \Phi$, then $\Phi < 1$,

$$|D(x_n, x_{n+1})| \leq \Phi^n |D(x_0, x_1)|. \tag{3.7}$$

Also we can deduce from the contraction that

$$|D(x_n, x_n)| \leq 2\Phi^{n-1} |D(x_0, x_1)|. \tag{3.8}$$

To prove that $\{x_n\}$ is a cauchy sequence in (X, D), we will prove that $\{x_n\}$ is a cauchy sequence in (X, d_p^s). Since

$$d_p(x, y) = D(x, y) - D(x, x).$$

Therefore

$$\begin{aligned} d_p(x_n, x_{n+1}) &= D(x_n, x_{n+1}) - D(x_n, x_n), \\ d_p(x_n, x_{n+1}) + D(x_n, x_n) &= D(x_n, x_{n+1}), \\ &\leq |D(x_n, x_{n+1})|. \end{aligned}$$

By (3.7), we have

$$\begin{aligned} d_p(x_n, x_{n+1}) + D(x_n, x_n) &\leq \Phi^n |D(x_0, x_1)|, \\ d_p(x_n, x_{n+1}) &\leq \Phi^n |D(x_0, x_1)| - D(x_n, x_n), \\ &\leq \Phi^n |D(x_0, x_1)| + |D(x_n, x_n)|. \end{aligned}$$

By using (3.8), we have

$$d_p(x_n, x_{n+1}) \leq \Phi^n |D(x_0, x_1)| + 2\Phi^{n-1} |D(x_0, x_1)|.$$

This implies that

$$d_p(x_n, x_{n+1}) \leq \Phi^n (3 - 2\varphi) |D(x_0, x_1)|, \tag{3.9}$$

and

$$d_p(x_{n+1}, x_{n+2}) \leq \Phi^{n+1} (3 - 2\varphi) |D(x_0, x_1)|. \tag{3.10}$$

Continuing in the same way, we have

$$d_p(x_{n+\gamma-1}, x_{n+\gamma}) \leq \Phi^{n+\gamma-1} (3 - 2\varphi) |D(x_0, x_1)|. \tag{3.11}$$

Now using the triangular inequality and equations (3.10)-(3.11), we have

$$\begin{aligned} d_p(x_n, x_{n+\gamma}) &\leq d_p(x_n, x_{n+1}) + d_p(x_{n+1}, x_{n+2}) + \cdots + d_p(x_{n+\gamma-1}, x_{n+\gamma}), \\ &\leq \Phi^n (3-2\varphi) |D(x_0, x_1)| + \Phi^{n+1}(3-2\varphi) \ |D(x_0, x_1)| + \cdots + \\ &\quad \lambda^{n+\gamma-1}(3-2\varphi)|D(x_0, x_1)|, \\ &\leq \frac{\Phi^n}{1-\Phi}(3-2\varphi)|D(x_0, x_1)|. \end{aligned}$$

Similarly, we can conclude that

$$d_p(x_{n+\gamma}, x_n) \leq \frac{\Phi^n}{1-\Phi}(3-2\varphi)|D(x_0, x_1)|.$$

Now taking limit as $n \to \infty$ of last two inequalities, we obtain that

$$\lim_{n\to\infty} d_p(x_n, x_{n+\gamma}) = 0 = \lim_{n\to\infty} d_p(x_{n+\gamma}, x_n).$$

This implies

$$\lim_{n\to\infty} d_p^s(x_n, x_{n+\gamma}) = 0.$$

This implies that $\{x_n\}$ is a Cauchy sequence in (X, d_p^s). Since (X, d_p^s) is complete metric space, there exist $z \in X$ such that $x_n \longrightarrow z$ as $n \to \infty$. i.e,

$$\lim_{n\to\infty} d_p^s(x_n, z) = 0.$$

Now from Lemma 2.4, we have $\lim_{n\to\infty} d_p^s(x_n, z) = 0$ if and only if

$$D(z,z) = \lim_{n\to\infty} D(x_n, z) = \lim_{n,m\to\infty} D(x_n, x_m).$$

Since

$$\begin{aligned} \lim_{n,m\to\infty} d_p(x_n, x_m) &= 0, \\ \lim_{n,m\to\infty} [D(x_n, x_m) - D(x_n, x_n)] &= 0, \\ \lim_{n,m\to\infty} D(x_n, x_m) &= \lim_{n,m\to\infty} D(x_n, x_n). \end{aligned}$$

But (3.8) implies that

$$\lim_{n,m\to\infty} D(x_n, x_n) = 0.$$

It fallows directly that

$$\lim_{n,m\to\infty} D(x_n, x_m) = 0.$$

This implies that

$$D(z,z) = \lim_{n\to\infty} D(x_n, z) = \lim_{n\to\infty} D(x_n, x_n) = 0. \tag{3.12}$$

Now, by (3.12), we have

$$\begin{aligned} d_p(z, Tz) &= D(z, Tz) - D(z,z), \\ &= D(z, Tz). \end{aligned} \tag{3.13}$$

So

$$D(z, Tz) \geq 0.$$

Now from ($P_{2.6}$) and (3.2), we get

$$\begin{aligned} D(Sz,z) &\leq D(Sz,x_{2n+2})+D(x_{2n+2},z)-D(x_{2n+2},x_{2n+2}), \\ &\leq D(x_{2n+2},Sz)+D(x_{2n+2},z)+|D(x_{2n+2},x_{2n+2})|, \\ &\leq \sup_{u\in Tx_{2n+1}} D(u,Sz)+D(x_{2n+2},z)+|D(x_{2n+2},x_{2n+2})|, \\ &\leq \delta_D(Tx_{2n+1},Sz)+D(x_{2n+2},z)+|D(x_{2n+2},x_{2n+2})|, \\ &\leq H_D(Tx_{2n+1},Sz)+D(x_{2n+2},z)+|D(x_{2n+2},x_{2n+2})|, \\ &\leq \varphi(D(x_{2n+1},z))\,D(x_{2n+1},z)+D(x_{2n+2},z)+|D(x_{2n+2},x_{2n+2})|, \\ &\leq D(x_{2n+1},z)+D(x_{2n+2},z)+|D(x_{2n+2},x_{2n+2})|. \end{aligned}$$

Taking limit as $n \longrightarrow \infty$, we get

$$D(Sz,z)=0. \tag{3.14}$$

Thus from (3.12) and (3.14), we get
$D(z,z)=D(Sz,z)$
Thus by remark 2.7, we get that $z\in Sz$. It fallows similarly that $z\in Tz$. This completes the proof of the theorem.

Example 3.3. Let $X=\mathbb{R}$ and $D(x,y)=\frac{1}{4}|x-y|+\frac{1}{2}\max\{x,y\}$, for all $x,y\in X$. Note that if d_p is quasi metric on X, then $d_p^s(x,y)=\max\{d_p(x,y),d_p(y,x)\}$ is metric on X. Hence, $d_p^s(x,y)=|x-y|$ and so (X,d_p^s) is a complete metric space. Also define mappings $S,T:X\longrightarrow CB^D(X)$ by
$Sx=\overline{B\left(0,\frac{x}{4}\right)},\quad Ty=\overline{B\left(0,\frac{x}{3}\right)}$.
Then
$H_D\left(\overline{B\left(0,\frac{x}{4}\right)},\overline{B\left(0,\frac{x}{3}\right)}\right)=\max\left[\frac{x}{4},\frac{x}{3}\right]$ and

$$\begin{aligned} H_D(Sx,Ty) &= \max\left[\frac{x}{4},\frac{x}{3}\right] \\ &\leq \frac{1}{12}\max\{x,y\}\leq kD(x,y). \end{aligned}$$

Therefore, for $\varphi(t)=\frac{1}{12}$, all the conditions of theorem 3.2 are satisfied. Also it is clear that for all $x\in X$, the set Sx and Tx are bounded and closed with respect to the topology $\tau(D)=\tau(d_p)$. Hence, we can show that (3.2) holds for all $x,y\in X$. i.e.,

$$H_D(Sx,Ty)=H_D\left(0,\left[\frac{y}{4},\frac{y}{3}\right]\right)=\frac{y}{4}.$$

Now we deduce the result for single-valued self-mappings from Theorem 3.2.

Theorem 3.4. Let (X,d) be a complete DPMS. $S,T:X\to X$ be two self mappings and $\varphi:[0,+\infty)\longrightarrow[0,1)$ be an $MT-$function. Assume that

$$D(Sx,Ty)\leq\varphi(D(x,y))\,D(x,y);$$

for all x, $y\in X$, then S and T have a common fixed point.

Corollary 3.5. Let (X,d) be a complete DPMS. $S,T:X\to CB^D(X)$ be multivalued mappings satisfying the following condition

$$H_D(Sx,Ty)\leq kD(x,y);$$

for all x, $y\in X$, and $k\in[0,1)$, then S and T have a common fixed point.

Conflict of Interests

The authors declare that they have no competing interests.

Acknowledgement

Both authors are heartily thankful to referee for giving valuable suggestions to this paper.

References

[1] M. Abbas, B. Ali, C. Vetro, A Suzuki type fixed point theorem for a generalized multivalued mapping on partial Hausdorff metric space, Topology and its Appl, 160 (2013) 553–563.

[2] J. Ahmad, A. Azam, S. Saejung, Common fixed point results for contractive mappings in complex valued metric spaces, Fixed Point Theory and Applications, 2014(1):67.

[3] J. Ahmad, A. Azam, M. Arshad, Fixed points of multivalued mappings in partial metric spaces, Fixed Point Theory and Applications 2013, 2013:316.

[4] J. Ahmad, C Di. Bari, Y.J. Cho, M. Arshad, Some fixed point results for multi-valued mappings in partial metric spaces, Fixed Point Theory and Applications 2013 (1), 1-13.

[5] M. Arshad, A. Azam, M. Abbas, A. Shoaib, Fixed point results of dominated mappings on a closed ball in ordered partial metric spaces without continuity, U.P.B. Sci. Bull., Series A, Vol. 76, Iss.2. 2014.

[6] M. Arshad, S. U. Khan, J. Ahmad, Fixed point results for F-contractions involving some new rational expressions, JP Journal of Fixed Point Theory and Applications Volume 11, Number 1, 2016, Pages 79-97.

[7] H. Aydi, M. Abbas, C. Vetro, Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces. Topol. Appl. 159, 3234-3242 (2012).

[8] A. Azam, M. Arshad, Fixed points of sequence of locally contractive multivalued maps, Comput. Math. Appl. 57, 96-100 (2009).

[9] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations int egrales, Fund. Math., 3 (1922), 133-181.

[10] I. Beg, A. Azam, Fixed points of asymptotically regular multivalued mappings, J. Aust. Math. Soc. A 53, 313-326 (1992).

[11] A. Branciari, A fixed point theorem of Banach-Caccippoli type on a class of generalized metric spaces, Public. Math. Debrecen 57 31–37 (2000).

[12] B. C. Dhage, Generalized metric spaces with fixed point. Bull. Calcutta Math. Soc. 84, 329–336 (1992).

[13] W. S. Du, On coincidence point and fixed point theorems for nonlinear multivalued maps, Topol. Appl. 159, 49-56 (2012).

[14] W. S. Du, Coupled fixed point theorems for nonlinear contractions satisfied Mizoguchi-Takahashi's conditions in quasi-ordered metric spaces. Fixed Point Theory Appl. 2010, Article ID 876372 (2010).

[15] L. G. Haung, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings. J. Math. Anal. Appl. 332, 1468–1476 (2007).

[16] N. Hussain, A. Amin-Harandi, Y. J. Cho, Approximate endpoints for set-valued contractions in metric spaces, Fixed Point Theory Appl. 2010, Article ID 614867 (2010).

[17] N. Hussain, J.R. Roshan, V. Parvaneh, A. Latif, A unification of G-metric, Partial metric and b-metric spaces, Abstr. Appl. Anal,Volume 2014 (2014), Article ID 180698.

[18] N. Hussain, Z. Kadelburg, S. Radenovic, F. Al-Solamy, Comparision functions and fixed point results in partial metric spaces, Abstr. Appl. Anal, Volume 2012 (2012), Article ID 605781,

[19] S. U. Khan, A. Bano. Common fixed point theorems in Cone metric spaces using W-distance, Int. J. of Math. Anal, Vol. 7, 2013, no. 14, 657-663

[20] M. Kikkawa, T. Suzuki, Three fixed point theorems for generalized contractions with constants in complete metric spaces, Nonlinear Anal., Theory Methods Appl. 69, 2942-2949 (2008).

[21] MA. Kutbi, J. Ahmad, N. Hussain, M. Arshad, Common fixed point results for mappings with rational expressions, Abstract and Applied Analysis Volume 2013 (2013), Article ID 549518, 11 pages.

[22] S. G. Matthews, Partial metric topology Proc. 8th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci.,vol. 728, 1994, pp. 183–197.

[23] N. Mizoguchi, W. Takahashi, Fixed point theorems for multivalued mappings on complete metric spaces, J. Math. Anal. Appl. 141, 177-188 (1989).

[24] S. Nadler, Multi-valued contraction mappings, Pac. J. Math. 20, 475-488 (1969).

[25] SJ. O'Neill, Partial metrics, valuations and domain theory Proc. 11th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci, vol. 806, 1996, pp. 304–315.
[26] S. Oltra, O. Valero, Banach fixed point theorems for partial metric spaces, Rend. Ist. Mat. Univ, Trieste 36(2004), 17-26.
[27] S. Radenovic, B. E. Rhoades, Fixed point theorem for two non-self mappings in cone metric spaces. Comput. Math. Appl. 57 1701–1707 (2009).
[28] T. Suzuki, Mizoguchi-Takahashi's fixed point theorems is a real generalization of Nadler's, J. Math. Anal. Appl. 340, 752-755 (2008).
[29] O. Valero, On Banach fixed point theorems for partial metric spaces, Applied General Topology, Vol.6, No.2, 2005.

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