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# A NEW NOTE ON LOCAL PROPERTY OF FACTORED FOURIER SERIES

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Abstract. The aim of this paper is to generalize a main theorem dealing with local property of Fourier series to the $|A,\theta_n|_k$ summability. Also some new and known results are obtained dealing with some basic summability methods.

## 1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums $(s_n)$, and let $(p_n)$ be a sequence of positive numbers such that

$$P_n = p_0 + ... + p_n \to \infty \quad as \quad n \to \infty. \tag{1.1}$$

The sequence-to-sequence transformation

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v \tag{1.2}$$

defines the sequence $(T_n)$ of the Riesz mean or simply the $(\bar{N},p_n)$ mean of the sequence $(s_n)$ generated by the sequence of coefficients $(p_n)$ (see [6]).
The series $\sum a_n$ is said to be summable $\mid \bar{N},p_n,\theta_n \mid_k$, $k \geq 1$, if (see [9])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid T_n - T_{n-1} \mid^k < \infty. \tag{1.3}$$

In the special case when $\theta_n = \frac{P_n}{p_n}$ and $\theta_n = n$, we obtain $\mid \bar{N},p_n \mid_k$ (see [1]) and $\mid R,p_n \mid_k$ (see [3]) summabilities, respectively. Also, if we take $\theta_n = n$ and $p_n = 1$ for all values of $n$, then we get $\mid C,1 \mid_k$ summability (see [5]).
Let $f$ be a periodic function with period $2\pi$, and Lebesgue integrable over $(-\pi,\pi)$. Without loss of generality, we may assume that the constant term of the Fourier

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series of $f$ is zero, that is

$$\int_{-\pi}^{\pi} f(t)dt = 0,$$

$$f(t) \sim \sum_{n=1}^{\infty}(a_n cosnt + b_n sinnt) = \sum_{n=1}^{\infty} C_n(t). \tag{1.4}$$

A sequence $(\lambda_n)$ is said to be convex if $\Delta^2\lambda_n \geq 0$ for every positive integer $n$, where $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$. Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \quad \bar{\Delta} a_{nv} = a_{nv} - a_{n-1}, v \quad a_{-1,0} = 0 \tag{1.5}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{\Delta}\bar{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{1.6}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{1.7}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{1.8}$$

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then $A$ defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to
$As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ... \tag{1.9}$$

Let $(\theta_n)$ be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $|A, \theta_n|_k$, $k \geq 1$, if (see [8])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty, \tag{1.10}$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s). \tag{1.11}$$

**Remark.** If we take $\theta_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then we get $| \bar{N}, p_n |_k$ summability. Also, if we take $\theta_n = n$ and $a_{nv} = \frac{p_v}{P_n}$, then we get $| R, p_n |_k$ summability.

## 2. The Known Results

Some known results have been proved dealing with local property of Fourier series (see [2], [11]). Furthermore, in [4], Bor has proved the following result.

**Theorem 2.1.** *Let $k \geq 1$ and $(p_n)$ be a sequence satisfying the conditions*

$$P_n = O(np_n) \tag{2.1}$$

$$P_n \Delta p_n = O(p_n p_{n+1}). \tag{2.2}$$

*If $(\theta_n)$ is any sequence of positive constants such that*

$$\sum_{v=1}^{m} \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \frac{1}{v}(\lambda_v)^k = O(1), \tag{2.3}$$

$$\sum_{v=1}^{m} \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \Delta\lambda_v = O(1), \tag{2.4}$$

$$\sum_{v=1}^{m} \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \frac{1}{v}(\lambda_{v+1})^k = O(1), \tag{2.5}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} = O\left(\left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \frac{1}{P_v}\right), \tag{2.6}$$

*then the summability $\mid \bar{N}, p_n, \theta_n \mid_k$ of the series $\sum_{n=1}^{\infty} C_n(t)\lambda_n P_n/np_n$ at a point can be ensured by local property, where $(\lambda_n)$ is convex sequence such that $\sum n^{-1}\lambda_n$ is convergent.*

By using the above result, Sarıgöl has obtained the following theorem (see [7]).

**Theorem 2.2.** *Let $k \geq 1$ and let $(p_n)$ be a sequence satisfying the conditions*

$$\Delta(P_n/np_n) = O(1/n). \tag{2.7}$$

*Let $(\lambda_n)$ be a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent. If $(\theta_n)$ is any sequence of positive constants such that*

$$\sum_{v=1}^{m} \theta_v^{k-1} \frac{P_v}{v^k p_v} \Delta\lambda_v < \infty \tag{2.8}$$

$$\sum_{v=1}^{m} \theta_v^{k-1} \left(\frac{\lambda_v}{v}\right)^k < \infty \tag{2.9}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} = O\left(\left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \frac{1}{P_v}\right), \tag{2.10}$$

*then the summability $\mid \bar{N}, p_n, \theta_n \mid_k$ of the series $\sum_{n=1}^{\infty} C_n(t)\lambda_n P_n/np_n$ at a point can be ensured by local property of $f$.*

In [10], Sulaiman has proved the following theorem covering all the results before this.

**Theorem 2.3.** *Let $k \geq 1$, and let the sequences $(p_n)$, $(\theta_n)$, $(\lambda_n)$ and $(\varphi_n)$ where $\theta_n > 0$, are all satisfying the following conditions*

$$|\lambda_{n+1}| = O(|\lambda_n|), \tag{2.11}$$

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k |\lambda_n|^k |\varphi_n|^k < \infty, \tag{2.12}$$

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |\lambda_n|^k |\Delta\varphi_n|^k < \infty, \tag{2.13}$$

$$\sum_{v=1}^{n-1} \theta_v^{1-1/k} |\varphi_v| \left(\frac{P_v}{p_v}\right)^{(1/k)-1} |\Delta\lambda_v| < \infty, \tag{2.14}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} = O\left(\left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \frac{1}{P_v}\right), \tag{2.15}$$

*then the summability $| \bar{N}, p_n, \theta_n |_k$ of the series $\sum_{n=1}^{\infty} C_n(t)\lambda_n\varphi_n$ at a point can be ensured by local property of $f$.*

## 3. The Main Result

The aim of this paper is to generalize Theorem 2.3 for $|A, \theta_n|_k$ summability factors of Fourier series in the following form.

**Theorem 3.1.** *Let $k \geq 1$ and let $A = (a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{no} = 1, \ n = 0, 1, ..., \tag{3.1}$$

$$a_{n-1,v} \geq a_{nv}, \ \text{for } n \geq v + 1, \tag{3.2}$$

$$a_{nn} = O(\frac{p_n}{P_n}), \tag{3.3}$$

$$\sum_{v=1}^{n-1} a_{vv} \hat{a}_{n,v+1} = O(a_{nn}). \tag{3.4}$$

*If the conditions (2.11)-(2.14) of Theorem 2.3 are satisfied and $(\theta_n)$ holds the following conditions,*

$$\sum_{n=v+1}^{\infty} (\theta_n a_{nn})^{k-1} \hat{a}_{n,v+1} = O\left\{(\theta_v a_{vv})^{k-1}\right\}, \tag{3.5}$$

$$\sum_{n=v+1}^{\infty} (\theta_n a_{nn})^{k-1} |\bar{\Delta} a_{nv}| = O\left\{(\theta_v a_{vv})^{k-1} a_{vv}\right\}, \tag{3.6}$$

*then the series $\sum C_n(t)\lambda_n\varphi_n$ is summable $|A, \theta_n|_k$, $k \geq 1$.*

## Proof of Theorem 3.1

*Proof.* Let $(I_n)$ denotes the A-transform of the series $\sum_{n=1}^{\infty} C_n(t)\lambda_n\varphi_n$. Then, by (1.7) and (1.8), we have

$$\bar{\Delta} I_n = \sum_{v=1}^{n} \hat{a}_{nv} a_v \lambda_v \varphi_v.$$

Applying Abel's transformation to this sum, we get that

$$\begin{aligned}\bar{\Delta}I_n &= \sum_{v=1}^{n-1} \Delta(\hat{a}_{nv}\lambda_v\varphi_v)\sum_{r=1}^{v} a_r + \hat{a}_{nn}\lambda_n\varphi_n\sum_{v=1}^{n} a_v \\ &= \sum_{v=1}^{n-1} \Delta(\hat{a}_{nv}\lambda_v\varphi_v)s_v + \hat{a}_{nn}\lambda_n\varphi_n s_n \\ &= \sum_{v=1}^{n-1} \bar{\Delta}a_{nv}\lambda_v\varphi_v s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1}\Delta\lambda_v\varphi_v s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1}\lambda_{v+1}\Delta\varphi_v s_v + a_{nn}\lambda_n s_n\varphi_n \\ &= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.\end{aligned}$$

To complete the proof of Theorem 3.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid I_{n,r} \mid^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4. \tag{3.7}$$

First, by applying Hölder's inequality with indices $k$ and $k'$, where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, we have that

$$\begin{aligned}\sum_{n=2}^{m+1} \theta_n^{k-1} \mid I_{n,1} \mid^k &= \sum_{n=2}^{m+1} \theta_n^{k-1} \left|\sum_{v=1}^{n-1} \bar{\Delta}a_{nv}\lambda_v\varphi_v s_v\right|^k \\ &\le \sum_{n=2}^{m+1} \theta_n^{k-1} \sum_{v=1}^{n-1} \left|\bar{\Delta}a_{nv}\right| |\lambda_v|^k|\varphi_v|^k|s_v|^k \times \left\{\sum_{v=1}^{n-1} \left|\bar{\Delta}a_{nv}\right|\right\}^{k-1}\end{aligned}$$

On the other hand, since by (3.1) and (3.2), we have

$$\sum_{v-1}^{n-1} |\bar{\Delta}a_{nv}| \le a_{nn} \tag{3.8}$$

Therefore, using condition (2.12), (3.6) and (3.8), we get

$$\begin{aligned}\sum_{n=2}^{m+1} \theta_n^{k-1} \mid I_{n,1} \mid^k &= O(1) \sum_{n=2}^{m+1} (\theta_n a_{nn})^{k-1} \left\{\sum_{v=1}^{n-1} |\bar{\Delta}a_{nv}||\lambda_v|^k|\varphi_v|^k\right\} \\ &= O(1) \sum_{v=1}^{m} |\lambda_v|^k|\varphi_v|^k \sum_{n=v+1}^{m+1} (\theta_n a_{nn})^{k-1}|\bar{\Delta}a_{nv}| \\ &= O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} a_{vv}|\lambda_v|^k|\varphi_v|^k \\ &= O(1) \sum_{v=1}^{m} \theta_v^{k-1} a_{vv}^k |\varphi_v|^k|\lambda_v|^k = O(1) \quad \text{as} \quad m \to \infty,\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1. Now, using Hölder's inequality and then using condition (2.14) we have that

$$\sum_{n=2}^{m+1}\theta_n^{k-1}\mid I_{n,2}\mid^k\leq\sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||\varphi_v||s_v|\right\}^k$$
$$\leq\sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|^k|\Delta\lambda_v||\varphi_v||s_v|^k\theta_v^{(1-\frac{1}{k})(1-k)}\left(\frac{P_v}{p_v}\right)^{(k-1)(1-\frac{1}{k})}\right\}$$
$$\times\left\{\sum_{v=1}^{n-1}\theta_v^{1-1/k}|\varphi_v|\left(\frac{P_v}{p_v}\right)^{(1/k)-1}|\Delta\lambda_v|\right\}^{k-1}$$
$$=O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|^{k-1}|\hat{a}_{n,v+1}||\varphi_v||\Delta\lambda_v|\theta_v^{(1-\frac{1}{k})(1-k)}\left(\frac{P_v}{p_v}\right)^{(k-1)(1-\frac{1}{k})}\right\}$$
$$=O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\varphi_v||\Delta\lambda_v|\theta_v^{(1-\frac{1}{k})(1-k)}\left(\frac{P_v}{p_v}\right)^{(k-1)(1-\frac{1}{k})}\right\}$$
$$=O(1)\sum_{v=1}^{m}|\varphi_v||\Delta\lambda_v|\theta_v^{(1-\frac{1}{k})(1-k)}\left(\frac{P_v}{p_v}\right)^{(k-1)(1-\frac{1}{k})}\sum_{n=v+1}^{m+1}(\theta_n a_{nn})^{k-1}|\hat{a}_{n,v+1}|$$

The elements $\hat{a}_{nv}\geq 0$ for each $v,n$. In fact, it is easily seen from the positiveness of the matrix, (3.1) and (3.2), that $\hat{a}_{00}=1$,

$$\hat{a}_{nv}=\bar{a}_{n0}-\bar{a}_{v-1,0}+\sum_{i=0}^{v-1}(a_{n-1,i}-a_{ni})$$
$$=\sum_{i=0}^{v-1}(a_{n-1,i}-a_{ni})\geq 0\quad\text{for}\quad 1\leq v\leq n. \tag{3.9}$$

So, using the conditions (2.14) and (3.5), we get

$$\sum_{n=2}^{m+1}\theta_n^{k-1}\mid I_{n,2}\mid^k=O(1)\sum_{v=1}^{m}|\varphi_v||\Delta\lambda_v|\theta_v^{(1-\frac{1}{k})(1-k)}\left(\frac{P_v}{p_v}\right)^{(k-1)(1-\frac{1}{k})}(\theta_v a_{vv})^{k-1}$$
$$=O(1)\sum_{v=1}^{m}\theta_v^{1-1/k}|\varphi_v|\left(\frac{P_v}{p_v}\right)^{(\frac{1}{k})-1}|\Delta\lambda_v|=O(1)\quad\text{as}\quad m\to\infty,$$

by virtue of the hypotheses of Theorem 3.1. Furthermore, using the conditions (2.11), (2.13), (3.4)-(3.5), and (3.9), we have that

$$\sum_{n=2}^{m+1}\theta_n^{k-1}|I_{n,3}|^k\leq\sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\varphi_v||\lambda_{v+1}||s_v|\right\}^k$$
$$\leq\sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\left(\frac{P_v}{p_v}\right)^{k-1}|\Delta\varphi_v|^k|\lambda_{v+1}|^k|s_v|^k\right\}\times\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\frac{p_v}{P_v}\right\}^{k-1}$$
$$=O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^{k-1}|\hat{a}_{n,v+1}||\Delta\varphi_v|^k|\lambda_v|^k$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1}|\Delta\varphi_v|^k|\lambda_v|^k\sum_{n=v+1}^{m+1}(\theta_n a_{nn})^{k-1}|\hat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1}(\theta_v a_{vv})^{k-1}|\Delta\varphi_v|^k|\lambda_v|^k\\
&= O(1)\sum_{v=1}^{m}\theta_v^{k-1}|\Delta\varphi_v|^k|\lambda_v|^k = O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1. Finally, using the conditions (2.12) and (3.3), we have that

$$
\sum_{n=1}^{m}\theta_n^{k-1}|I_{n,4}|^k \le \sum_{n=1}^{m}\theta_n^{k-1}a_{nn}^k|\lambda_n|^k|s_n|^k|\varphi_n|^k = O(1)\sum_{n=1}^{m}\theta_n^{k-1}a_{nn}^k|\lambda_n|^k|\varphi_n|^k = O(1) \quad \text{as} \quad m\to\infty,
$$

by virtue of hypotheses of the Theorem 3.1. Since the behaviour of the Fourier series concerns the convergence for a particular value of $x$ depends on the behaviour on the function in the immediate neighborhood of this point only, this justifies (1.4) and valid. This completes the proof of Theorem 3.1. □

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