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CONSERVATIVE AND DISSIPATIVE FOR T-NORM AND T-CONORM AND RESIDUAL FUZZY CO-IMPLICATION

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Abstract. In this paper new concepts called conservative, dissipative, power stable for t-norm and t-conorm are considered. Also, residual fuzzy co-implication in dual Heyting algebra are investigated. Some examples as well as application are given as well.

1. Introduction

In fuzzy logic, the basic theory of connective like conjunction ($\wedge$) is interpreted by a triangular norm, disjunction ($\vee$) by triangular conorm, negation ($\neg$) by strong negations these important notions in fuzzy set theory is that of t-norm (T), t-conorms (S) and strong negations (N_C) that are used to define a generalized intersection, union and negation of fuzzy sets (see [3] and [4]. The notion of t-norm and t-conorm turned out to be basic tools for probabilistic metric spaces (see [8] and [10]) but also in several other parts and have found diverse applications in the theory of fuzzy sets, fuzzy decision making, in models of certain many-valued logics or in multivariate statistical analysis (see [3, , and [14]). Also, implication and co-implication functions play an important notion in fuzzy logic, approximate reasoning, fuzzy control, intuitionistic fuzzy logic and approximate reasoning of expert system (see ([1], [2], [5], [6], [7], and [15]). The conjunction and disjunction in fuzzy logic are often modeled as follows.

Definition 1.1. [8] A mapping T from $[0,1]^2$ into $[0,1]$ is a triangular norm (in short, t- norm), iff T are commutative, nondecreasing in both arguments, associative and which satisfies $T(p,1) = p$, for all $p \in [0,1]$.

Definition 1.2. [8] A mapping S from $[0,1]^2$ into $[0,1]$ is a triangular norm (in short, t- norm), iff T are commutative, nondecreasing in both arguments, associative and which satisfies $S(p,0) = p$, for all $p \in [0,1]$.

The standard examples of t-norms and dual t-conorms are stated in the following
1. Minimum t-norm, $M(p,q) = \min(p,q)$.
2. Probabilistic Product t-norm, $\Pi(p,q) = pq$.

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3. Drastic or weak t-norm, $W(p,q)=\begin{cases} p & \text{if } q=1, \\ q & \text{if } p=1, \\ 0 & \text{if } p,q\in[0,1). \end{cases}$
4. Nilpotent t-norm, $N(p,q)=\begin{cases} \min(p,q) & \text{if } p+q\geq 1, \\ 0 & \text{if } p+q<1. \end{cases}$
5. Lukasiewicz t-norm, $L(p,q)=\max(p+q-1,0)$.
6. Hamacher t-norm, $H(p,q)=\begin{cases} 0 & \text{if } p=q=0, \\ \frac{pq}{p+q-pq} & \text{otherwise.} \end{cases}$
7. Dubois-Prade t-norm, $D_\alpha(p,q)=\frac{pq}{\max(p,q,\alpha)}, \alpha\in(0,1)$.
8. Maximum t-conorm, $M(p,q)=S_M(p,q)=\max(p,q)$.
9. Probabilistic sum t-conorm, $S_\Pi(p,q)=p+q-pq$.
10. Drastic or largest t-conorm, $S_W(p,q)=\begin{cases} p & \text{if } q=0, \\ q & \text{if } p=0, \\ 1 & \text{if } p,q\in(0,1]. \end{cases}$
11. Nilpotent t-conorm, $S_N(p,q)=\begin{cases} \max(p,q) & \text{if } p+q<1, \\ 0, & \text{if } p+q\geq 1. \end{cases}$
12. Bounded Sum t-conorm, $S_L(p,q)=\min(p+q,1)$.
13. Hamacher t-conorm, $S_H(p,q)=\begin{cases} 0 & \text{if } p=q=0, \\ \frac{p+q-2pq}{1-pq} & \text{otherwise.} \end{cases}$
14. Dubois-Prade t-conorm, $S_{D_\alpha}(p,q)=1-\frac{(1-p)(1-q)}{\max(1-p,1-q,\alpha)}, \alpha\in(0,1)$.

For other family of t-norms (not needed here) we refer the reader to [11] for instance. If $T_1<T_2$ $(S_{T_1}<S_{T_2})$ and there is at least one pair $(p,q)\in[0,1]^2$ such that $T_1(p,q)<T_2(p,q)$ $(S_{T_1}(p,q)<S_{T_2}(p,q))$ then we briefly $T_1<T_2$ $(S_{T_1}<S_{T_2})$ write. With this, the above t-norms and t-conorms satisfy the next known chain of inequalities

$$W<L<\Pi<H<M<S_M<S_H<S_\Pi<S_L<S_W.$$

Two t-norms (t-conorms) are called comparable if

$$T_1\leq T_2 \text{ or } T_1\geq T_2 \ \ (S_{T_1}\leq S_{T_2} \text{ or } S_{T_1}\geq S_{T_2}),$$

holds. The above chain of inequalities shows that $W,L,\Pi,H,M,S_M,S_H,S_\Pi,S_L$, and S_W are comparable. It is not hard to see that for example Π and N are not comparable, while W,N and M comparable with $W<N<M$ [9].

Definition 1.3. [13] Let T a left-continuous t-norm. Then, the residual implication or R-implication derived form is given by

$$I_T(p,q)=\sup\{r\in[0,1]\,|T(r,p)\leq q\}, \text{ for all } p,q\in[0,1]. \qquad (R)$$

i.e. $T(r,p)\leq q\Leftrightarrow r\leq I_T(p,q)$, for all $p,q,r\in[0,1]$.

2. Main Results

In the following section we will study the relation between power stable aggregation functions and power stable t-norm and t-conorm, then introduce some new concepts for t-norm and t-conorm as conservative, dissipative.

Definition 2.1. [16] A mapping A from $[0,1]^2$ into $[0,1]$ is aggregation function, iff A are increasing in each variable, $A(0,0)=0$, and $A(1,1)=1$.

Definition 2.2. [16] An aggregation function $A:[0,1]^2\to[0,1]$ is called power stable whenever for any constant $p\in(0,\infty)$ and $p,q\in[0,1]^2$ it hold,

$$A(p^r,q^r)=(A(p,q))^r.$$

Proposition 2.1. [16] *Power stable aggregation functions are exactly those which are invariant under power transformations, i.e., aggregation function satisfying for all powers* $\varphi_r : [0,1] \to [0,1], \varphi_r(p) = p^r \in (0,\infty)$ *and all* $p,q \in [0,1]^2$ *the property*

$$A(p,q) = \varphi_r^{-1}(A(\varphi_r(p), \varphi_r(q))).$$

Definition 2.3. Let $\Phi : [0,1] \to [0,\infty]$ be a continuous strictly decreasing function such that $\Phi(1) = 0$. Let $\Phi^{(-1)}$ be the pseudo-inverse of Φ defined by

$$\Phi^{(-1)}(p) = \begin{cases} \Phi^{-1}\ (p) \text{ if } & p \in [0, \Phi(0)], \\ 0, & \text{otherwise.} \end{cases}$$

For all $p,q \in [0,1]$, we set

$$T(p,q) = \Phi^{(-1)}(\Phi(p) + \Phi(q)),$$

then T is a t-norm and Φ is called an additive generator of T.

Definition 2.4. Let $\Psi : [0,1] \to [0,\infty]$ be a continuous strictly increasing function such that $\Psi(0) = 0$. Let $\Psi^{(-1)}$ be the pseudo-inverse of Ψ defined by

$$\Psi^{(-1)}(p) = \begin{cases} \Psi^{-1}\ (p) \text{ if } & p \in [0, \Psi(1)], \\ 1, & \text{otherwise.} \end{cases}$$

For all $p,q \in [0,1]$, we set

$$S_T(p,q) = \Psi^{(-1)}(\Psi(p) + \Psi(q)),$$

then S_T is a t-conorm and Ψ is called an additive generator of S_T.

Proposition 2.2. *Let T be a t-norm, S_T be a t-conorm and $\Phi : [0,1] \to [0,\infty]$ an additive generator of T. The function $\Psi : [0,1] \to [0,\infty]$ defined by $\Psi(t) = \Phi(1-t)$ is an additive generator of S_T.*

Definition 2.5. Let T (S_T) be a t-norm (t-conorm) and $\mu : [0,1] \to [0,1]$ be a continuous strictly increasing map. If for all $p,q \in [0,1]$, we set

$$\begin{aligned} T_\mu(p,q) &= \mu^{-1}(T(\mu(p), \mu(q))), \\ S_{T_\mu}(p,q) &= \mu^{-1}(S_T(\mu(p), \mu(q))), \end{aligned}$$

then T_μ is a t-norm (S_{T_μ} is a t-conorm).

Proposition 2.3. *Let T (S_T) and R (S_R) are t-norms (t-conorms), and $\mu : [0,1] \to [0,1]$ be continuous strictly increasing function. Then*

1. *If $T_\mu = R_\mu$ then $T = R$.*
2. *If $S_{T_\mu} = S_{R_\mu}$ then $S_T = S_R$.*
3. *If $T \leq S$ then $T_\mu \leq R_\mu$.*
4. *If $S_T \leq S_R$ then $S_{T_\mu} \leq S_{R_\mu}$.*
5. *$(T_\mu)_{\mu^{-1}} = T$ and $\left(S_{T_\mu}\right)_{\mu^{-1}} = S_T$.*

Some example of continuous strictly increasing function $\mu : [0,1] \to [0,1]$ are given

1.$\mu(t) = \frac{2t}{t+1}$, 2.$\mu(t) = 1 - (1-t)^x,\ x > 0$.
3.$\mu(t) = t^x,\ x > 0$. 4.$\mu(t) = \frac{x^t - 1}{x - 1},\ x > 0,\ x \neq 0$.
5.$\mu(t) = \frac{log(1+xt^\alpha)}{log(1+x)},\ x > -1,\ \alpha > 0$.

Take $\mu(t) = t^x$ $(x > 0)$ then $\mu^{-1}(t) = t^{1/x}$, we get

$$L_\mu(p,q) = \mu^{-1}(max(p^x + q^x - 1, 0)) = (max(p^x + q^x - 1, 0))^{1/x}.$$

Take $\mu(t) = 1 - (1-t)^x$ $(x > 0)$ then $\mu^{-1}(t) = 1 - (1-t)^{1/x}$, we get

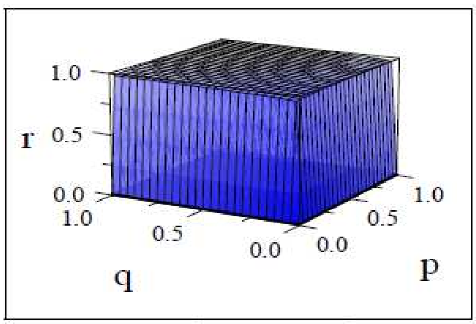
$$\Pi_\mu(p,q) = 1 - \left((1-p)^x + (1-q)^x - (1-p)^x(1-q)^x\right)^{1/x}.$$

But the most interesting applications when $\mu(t) = t^x$ for some $t > 0$. We then have the next related result.

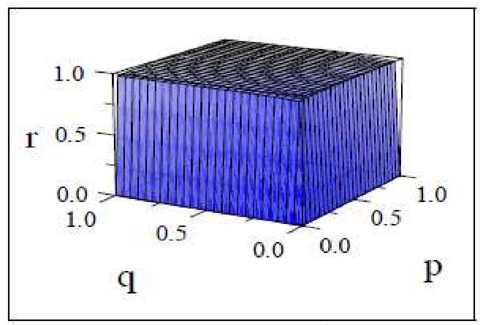
Definition 2.6. Let T (S_T) b e a t-norm (t-conorm) for any constant $x \in (0, \infty)$ and all $p, q \in [0,1]$. T is called T-power stable if holds $T(p^x, q^x) = (T(p,q))^x$. S_T is called S_T-power stable if holds,

$$S_T(p^x, q^x) = (S_T(p,q))^x.$$

Probabilistic product t-norm is T-power stable, for any constant $x \in (0, \infty)$ and all $p, q \in [0,1]$, then $\Pi(p^x, q^x) = p^x q^x = (pq)^x = (\Pi(p,q))^x$, and the following groups illustrate that.

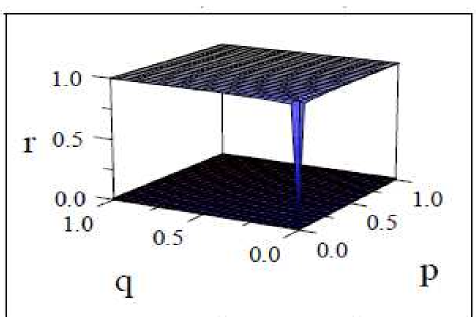


$\Pi\left(p^{\frac{1}{100}}, q^{\frac{1}{100}}\right)$

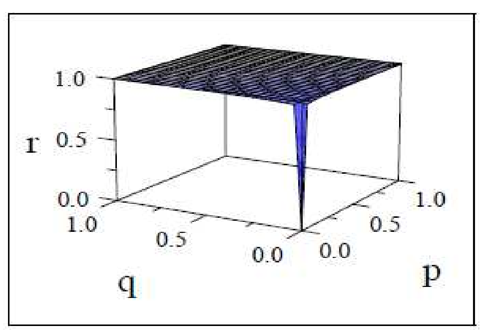


$(\Pi(p,q))^{\frac{1}{100}}$

Let T be a given power stable t-norm where it doesn't necessary that S_T-power stable t-conorm. Probabilistic product t-norm is T-power stable but probabilistic product t-conorm is not S_T-power stable and the following groups illustrate that.

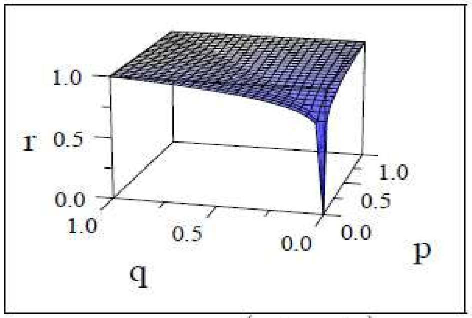


$S_\Pi\left(p^{\frac{1}{100}}, q^{\frac{1}{100}}\right)$

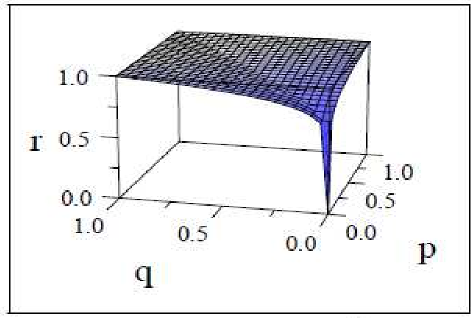


$(S_\Pi(p,q))^{\frac{1}{100}}$

Maximum t-conorm is S_T-power stable, for any constant $x \in (0, \infty)$ and all $p, q \in [0,1]$, then $S_M(p^x, q^x) = \max(p^x, q^x) = (\max(p,q))^x = (S_M(p,q))^x$, and the following groups illustrate that.



$\max\left(p^{\frac{1}{10}}, q^{\frac{1}{10}}\right)$



$(\max(p,q))^{\frac{1}{10}}$

Definition 2.7. Let T be a given power stable t-norm. We say that T is closed if the following limits

$$T_0(p,q) = lim_{x\to 0} T_x(p,q) \text{ and } T_\infty(p,q) = lim_{x\to\infty} T_x(p,q),$$

where $T_x(p,q) = (T(p^x, q^x))^{\frac{1}{x}}$, exist for all $p, q \in [0,1]$.

Proposition 2.4. *Let T be a T-power stable t-norm. Then the following assertions are met for all $p, q \in [0,1]$.*

(1) $(T_x)_y(p,q) = T_{xy}(p,q) = (T_y)_x(p,q)$. In particular, $(T_x)_{1/x}(p,q) = T_1(p,q) = T(p,q)$ for $x > 0$.
(2) If T and S be two T-power stable t-norms such that $T_x(p,q) = S_x(p,q)$ for some $x > 0$ then $T(p,q) = S(p,q)$.
(3) $T_x(p,q) = T_y(p,q)$ does not ensure $x = y$.

Definition 2.8. Let S_T be a given power stable t-conorm. We say that S_T is closed if the following limits

$$S_{T_0}(p,q) = lim_{x\to 0} S_{T_x}(p,q) \text{ and } S_{T_\infty}(p,q) = lim_{x\to\infty} S_{T_x}(p,q),$$

where $S_{T_x}(p,q) = (S_T(p^x, q^x))^{\frac{1}{x}}$, exist for all $p, q \in [0,1]$.

Definition 2.9. Let T (S_T) be a closed t-norm (t-conorm).
i. T (S_T) is called to be conservative if

$$\begin{aligned} T_0(p,q) &= T_\infty(p,q) = T_x(p,q) \\ S_{T_0}(p,q) &= S_{T_\infty}(p,q) = S_{T_x}(p,q) \end{aligned}$$

for all $p, q \in [0,1]$.
ii. We say that T (S_T) is dissipative if there exist two conservative t-norm (t-conorm) U (S_U) and V (S_V) that

$$\begin{aligned} T_0(p,q) &= U(p,q) \text{ and } T_\infty(p,q) = V(p,q), \\ S_{T_0}(p,q) &= S_U(p,q) \text{ and } S_{T_\infty}(p,q) = S_V(p,q), \end{aligned}$$

for all $p, q \in [0,1]$. In this case we say that T is (U,V)-dissipative and S_T is (S_U, S_V)- dissipative.

Proposition 2.5. *i. Every conservative t-norm T (S_T) is (T,T)- dissipative $((S_T, S_T)$- dissipative).*
ii. Let T (S_T) be a closed t-norms (t-conorm), if T (S_T) is (U,V)- dissipative $((S_U, S_V)$- dissipative) then T_x (S_{T_x}) is also (U,V)-dissipative $((S_U, S_V)$- dissipative) for each $r > 0$, T_x (S_{T_x}) conservative whenever $T(S_T)$ is conservative.

Example 2.1. M and S_M are conservative.

Proof. It easy to see that $M_x(p,q) = (M(p^x,q^x))^{\frac{1}{x}}$ and $S_{M_x}(p,q) = (S_M(p^x,q^x))^{\frac{1}{x}}$ for all $p,q \in [0,1]$ and $x > 0$. Then

$$\begin{aligned} M(p,q) &= M_0(p,q) = M_\infty(p,q). \\ S_M(p,q) &= S_{M_0}(p,q) = S_{M_\infty}(p,q). \end{aligned}$$

□

Example 2.2. Π is Conservative but S_Π is not Conservative.

Proof. It easy to see that that $\Pi_x(p,q) = (\Pi(p^x,q^x))^{\frac{1}{x}}$ for all $p,q \in [0,1]$ and $x > 0$. Then $\Pi(p,q) = \Pi_0(p,q) = \Pi_\infty(p,q)$. But $S_\Pi(p,q) \neq S_{\Pi_0}(p,q) \neq S_{\Pi_\infty}(p,q)$,. □

Example 2.3. The t-norm L is (Π, W)-dissipative.

Proof. For all $p,q \in [0,1]$ and $x > 0$, L is given by

$$L_x(p,q) = \begin{cases} (\max(p^x+q^x-1,0))^{\frac{1}{x}} & \text{if } p^x+q^x \geq 1, \\ 0 & \text{if } p^x+q^x \leq 1. \end{cases}$$

Assume that $p,q \in (0,1]$. For x enough small we have

$$p^x = \exp(x\ln p) = 1 + x(\ln p) + xo(1), o(1) \to 0 \text{ as } x \to 0,$$

With similar expansion for x^p. We then obtain

$$p^x + q^x - 1 = 1 + x\ln(pq) + xo(1).$$

Since $p^x + q^x > 1$ for all $p,q \in (0,1]$ and x enough small, we then have

$$ln(p^x + q^x - 1) = x\ln(pq) + xo(1),$$

For which we deduce

$$(p^x + q^x - 1)^{\frac{1}{x}} = \exp((1/x)\ln(p^x + q^x - 1)) = pq\exp(o(1)).$$

It follows that

$$L_x(p,q) = (p^x + q^x - 1)^{\frac{1}{x}} = pq\exp(o(1)),$$

and so

$$\lim_{x\to 0} L_x(p,q) = pq = \Pi(p,q),$$

for all $p,q \in (0,1]$. This, with $L_x(p,0) = 0$ and $L_x(0,q) = 0$ for all $p,q \in [0,1]$, yields the desired result.

Now, if For all p is enough large then $p^x + q^x < 1$ for all $p,q \in (0,1)$ and so $L_x(p,q) = L_x(0,q) = 0$ and $L_x(p,1) = p$, $L_x(1,q) = q$, for all $p,q \in [0,1]$, yields

$$\lim_{x\to 0} L_x(x,y) = W(p,q),$$

for all $p,q \in [0,1]$. The proof is then completed. □

Example 2.4. The t-conorm S_N is (S_M, S_W)–dissipative.

Proof. For all $p,q \in [0,1]$ and $x > 0$, we have

$$S_{N_x}(p,q) = \max(p,q) \text{ if } p^x + q^x < 1, \; S_{N_x}(1,q) = 1, \text{ else}.$$

It easy to see that $S_{N_x}(p,1) = S_{N_x}(1,q) = 1$, for all $p,q \in [0,1]$. Since N_x a t-conorm then $S_{N_x}(p,0) = p$ and $S_{N_x}(0,q) = q$ for all $p,q \in [0,1]$. Now, if $p,q \in (0,1)$ and x

is enough small, we have $p^x + q^x < 1$ and so $S_{N_x}(p,q) = max(p,q)$. Summarizing, we then obtain

$$S_{N_0}(p,q) = \lim_{x\to 0} S_{N_x}(p,q) = max(p,q) = S_M(p,q),$$

for all $p, q \in (0,1)$.

Now, if x is enough large then $p^x + q^x \geq 1$ for all $p, q \in (0,1)$ and so $S_{N_x}(p,q) = 1$. It follows that

$$S_{N_\infty}(p,q) = \lim_{x\to 0} S_{N_p}(p,q) = 1,$$

for all $p, q \in (0,1)$. Summarizing, we have shown that

$$S_{N_\infty}(p,q) = S_W(p,q),$$

for all $p, q \in [0,1]$, so completes the proof. □

Theorem 2.6. *The t-norm H is (Π, M)-dissipative.*

Proof. We have, for all $p, q \in (0,1]$ and $x > 0$,

$$H_x(p,q) = \frac{pq}{\left(p^x + q^x - p^x q^x\right)^{1/x}}.$$

We first show that $H_0 = \Pi$. For all $p, q \in (0,1]$ and x enough small we can write

$$p^x = \exp\left(x \ln\, p\right) = 1 + x \ln\, p + \frac{1}{2}x^2(\ln\, p)^2 + x^2 o(1),$$

with similar expansions for q^x and $(pq)^x$. After all computation and reduction we obtain

$$p^x + q^x - p^x q^x = 1 + x^2(\ln\, p)(\ln\, q) + x^2 o(1)$$

and so

$$\ln\left(p^x + q^x - p^x q^x\right) = x^2(\ln\, p)(\ln\, q) + x^2 o(1).$$

It follows that

$$\left(p^x + q^x - p^x q^x\right)^{1/x} = \exp\Big((1/x)\ln\left(p^x + q^x - p^x q^x\right)\Big) = \exp\Big(x(\ln\, p)(\ln\, q) + x\, o(1)\Big),$$

from which we deduce that $(p^x + q^x - p^x q^x)^{1/x}$ tends to 1 when $x \downarrow 0$. This, with $H_x(0,q) = H_x(p,0) = 0$, yields $H_0(p,q) = pq := \Pi(p,q)$ for all $p, q \in [0,1]$.

Now, we will prove that $H_\infty = M$. For $p \in \{0,1\}$ or $q \in \{0,1\}$, the desired result is obvious. For $p = q$, it is easy to see that $H_x(p,p) = p$. Assume that $p, q \in (0,1)$ with $q < p$. We then write

$$H_x(p,q) = \frac{pq}{\left(p^x + q^x - p^x q^x\right)^{1/x}} = \frac{q}{\Big(1 + (q/p)^x - q^x\Big)^{1/x}}.$$

Clearly, $q^x \to 0$ and $(q/p)^x \to 0$ when $x \uparrow \infty$. It follows that $H_x(p,q) \to q = \min(p,q)$ when $x \uparrow \infty$. By symmetry, we have $H_x(p,q) \to p = \min(p,q)$ if $p < q$. The desired result is obtained and the proof is completed. □

Corollary 2.7. *Let T be a t-norm such that $H \leq T$. Then $T_\infty = M$.*

Proof. If $H \leq T$ then $H_\infty = M \leq T_\infty \leq M$. So $T_\infty = M$. □

Theorem 2.8. *The t-norm D is (M, Π)-dissipative for every $\alpha \in (0,1)$.*

Proof. It is easy to see that

$$D_x(p,q) = \frac{pq}{\max\left(p,q,\alpha^{1/x}\right)},$$

for all $p,q \in [0,1]$ and $\alpha \in (0,1)$. Obviously, $\alpha^{1/x} \to 0$ when $x \downarrow 0$ and $\alpha^{1/x} \to 1$ when $x \uparrow \infty$. The desired result follows after a simple manipulation. □

Corollary 2.9. *Let T be a t-norm such that $D \leq T$ for some $\alpha \in (0,1)$. Then $T_0 = M$.*

Proof. $D \leq T$ implies $D_0 \leq T_0$ and so $M \leq T_0 \leq M$ i.e. $T_0 = M$. □

3. Residual fuzzy co-implication

The following properties are generalization of fuzzy implication and fuzzy co implication from classical logic.

Definition 3.1. [12] A mapping $I : [0,1] \times [0,1] \to [0,1]$ is a fuzzy implication if, for all $p,q,r \in [0,1]$, the following conditions are satisfied:

$I1 : I(1,1) = I(0,1) = I(0,0) = 1$ and $I(1,0) = 0$.
$I2 : I(p,q) \geq I(r,q)$ if $p \leq r$.
$I3 : I(p,q) \leq I(p,r)$ if $q \leq r$.
The set of all fuzzy implications is denoted by $\mathcal{F}I$.

Definition 3.2. [14] A mapping $J : [0,1] \times [0,1] \to [0,1]$ is a fuzzy implication if, for all $p,q,r \in [0,1]$, the following conditions are satisfied:

$J1 : J(1,1) = J(1,0) = J(0,0) = 0$ and $J(0,1) = 1$.
$J2 : J(p,q) \geq J(r,q)$ if $p \leq r$.
$J3 : J(p,q) \leq I(p,r)$ if $q \leq r$.
The set of all fuzzy co-implications is denoted by $Co - \mathcal{F}I$.
From last definition $J(1,q) = J(p,0) = 0$ and $J(p,p) = 0$, for all $p,q \in [0,1]$.

Definition 3.3. [13] A fuzzy implication I and fuzzy co-implication J are satisfy the following most important properties, for all $p,q,r \in [0,1]$

$I(1,q) = q,$	(NP)	$J(0,q) = q,$	(Co-NP)
$I(p,I(q,r)) = I(q,I(p,r)),$	(EP)	$J(p,J(q,r)) = J(q,J(p,r)),$	(Co-EP)
$I(p,p) = 1,$	(IP)	$I(p,p) = 0,$	(Co-IP)
$I(p,q) = 1 \Leftrightarrow p \leq q,$	(OP)	$J(p,q) = 0 \Leftrightarrow p \geq q.$	(Co-OP)

Heyting algebra logic is the system on Heyting algebras and Brouweriaun algebras. Heyting algebra $\langle L, \wedge, \vee, \Longrightarrow, 0, 1\rangle$ is lattice with the bottom 0, the top 1, and the binary operation called implication $\Longrightarrow$ such that, for all $p,q,r \in L$, $p \Longrightarrow q$ is the relative pseudocomplement of a with respect to r [13]. That is to say

$$p \wedge r \leq q \Leftrightarrow p \Longrightarrow q, \text{ for all } p,q,r \in L.$$

In other words, the set of all $p \in L$ such that $p \wedge r \leq q$ contains the greatest element, denoted by $p \Longrightarrow q$. Precisely

$$p \Longrightarrow q = \sup\left\{r \in L | p \wedge r \leq q\right\}.$$

The dual of Heyting algebra is called Brouwerian algebra $\left\langle L, \wedge, \vee, \overset{*}{\Longrightarrow}, 0, 1 \right\rangle$ is a lattice with 0 and 1, and the binary operation called co-implication $\overset{*}{\Longrightarrow}$ in dual Heyting algebra. Satisfying for all $p, q, r \in L$,

$$p \vee r \geq q \Leftrightarrow p \overset{*}{\Longrightarrow} q.$$

The set of all r in L such that $p \vee r \geq q$ contains the smallest element, denoted by $p \overset{*}{\Longrightarrow} q$. Precisely

$$p \overset{*}{\Longrightarrow} q = \inf \{ r \in L | p \vee r \geq q \}.$$

Definition 3.4. Let S is the t-conorm of right continuous T. Then, the residual co-implication (R^*-coimplication) derived from S, is

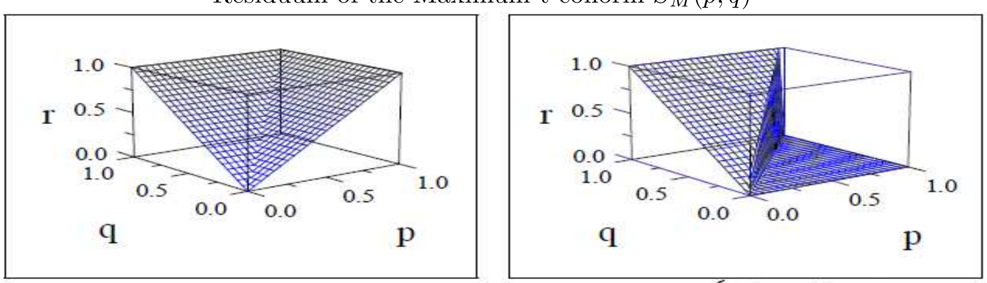
$$J_S(p, q) = \inf \{ r \in [0, 1] \, | S(r, p) \geq q \}, \text{ for all } p, q \in [0, 1]. \qquad (R^*)$$

R^*-co-implication come from residuted lattices based on residuation property (R^*P) that can be written as

$$S(r, p) \geq q \Leftrightarrow r \geq J_S(p, q), \text{ for all } p, q, r \in [0, 1]. \qquad (R^*P)$$

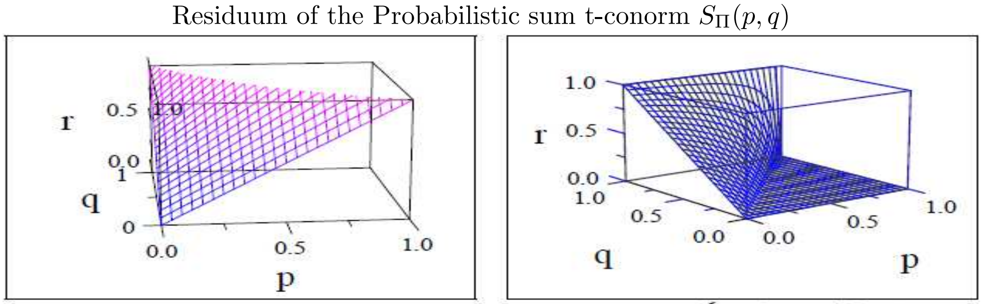
The $J_S(p, q)$ operation is called residual co-implication of the t-conorm S. Applying the above concepts to the standard t-norms we obtain the following interesting results.

Residuum of the Maximum t-conorm $S_M(p, q)$



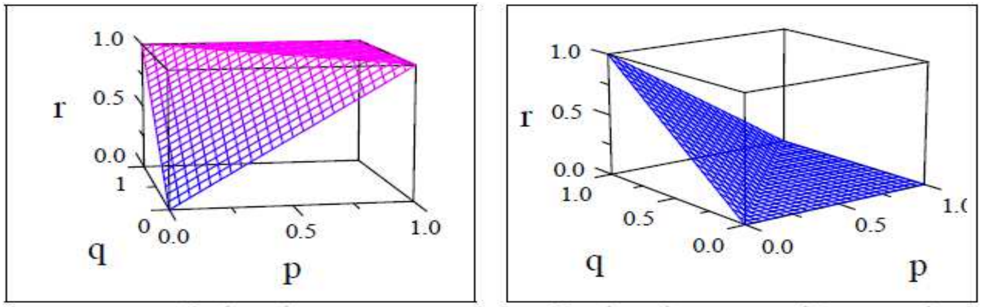
$S_M(p, q)$ $\qquad$ $J_{S_M}(p, q) = \begin{cases} 0 & \text{if } p \geq q \\ q & \text{otherwise} \end{cases}$

Residuum of the Probabilistic sum t-conorm $S_\Pi(p, q)$



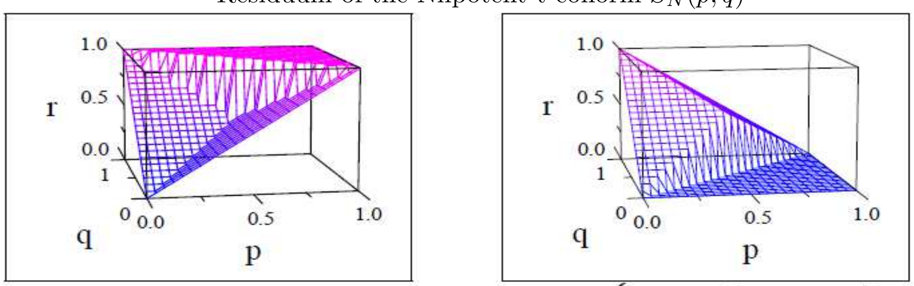
$S_\Pi(p, q)$ $\qquad$ $J_{S_\Pi}(p, q) = \begin{cases} 0 & \text{if } p \geq q \\ \frac{q-p}{1-p} & \text{otherwise} \end{cases}$

Residuum of the Bounded Sum t-conorm $S_L(p, q)$



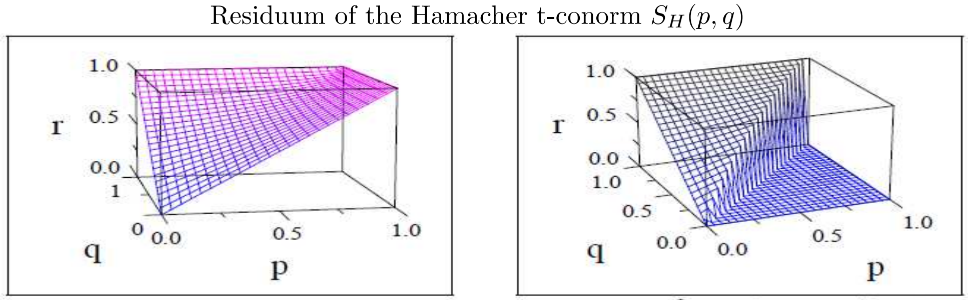
$S_L(p, q)$ $\quad$ $J_{S_L}(p, q) = \max(0, q - p)$

Residuum of the Nilpotent t-conorm $S_N(p, q)$



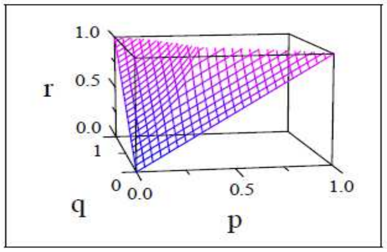
$S_N(p, q)$ $\quad$ $J_{S_N}(x, y) = \begin{cases} 0 & \text{if } p \geq q \\ \min(1 - p, q) & \text{otherwise} \end{cases}$

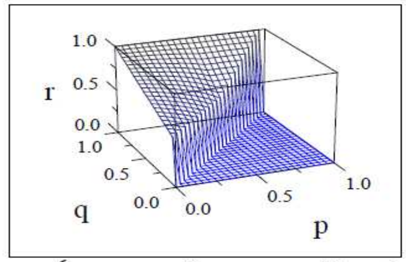
Residuum of the Hamacher t-conorm $S_H(p, q)$



$S_H(p, q)$ $\quad$ $J_{S_H}(p, q) = \begin{cases} 0 & \text{if } p \geq q \\ \frac{p+q-2pq}{1-pq} & \text{otherwise} \end{cases}$

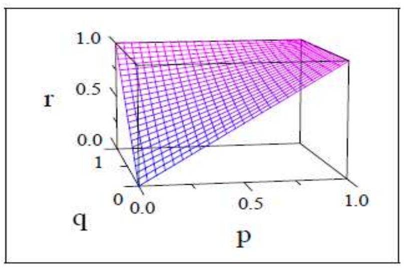
Residuum of the Dubois-Prade t-conorm $S_D(p,q)$

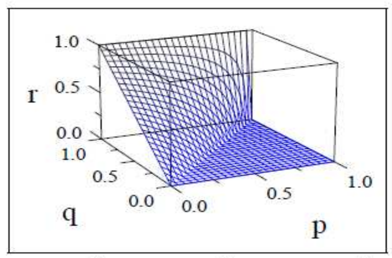




$S_{D_{0.5}}(p,q)$ $\qquad J_{S_{D_{0.5}}}(p,q)=\begin{cases}0 & \text{if } p\geq q\\ \max(q,\frac{\alpha q-\alpha}{1-p}+1) & \text{if } p<q\end{cases}$

Residuum of the Hamacher's parametric t-conorm $S_{T_\alpha}(p,q)$





$S_{T_{0.5}}(p,q)$ $\qquad J_{S_{T_{0.5}}}(p,q)=\begin{cases}0 & \text{if } p\geq q\\ \frac{0.5q-p+(1-0.5)q}{1-(2-0.5)p+(1-0.5)p} & \text{if } p<q\end{cases}$

In the following we introduce some properties for residual co-implication.

Theorem 3.1. *For a right continuous t-conorm S then $J_S \in Co-FI$*

Proof. We have to show that J_1, J_2 and J_3 in definition of fuzzy co-implication are satisfied for all $p,q,r\in[0,1]$.

$J_1: J_S(1,1)=J_S(1,0)=J_S(0,0), J_S(0,1)=1.$

$J_2: p\leq r \Longrightarrow \{t\in[0,1]|S(t,p)\geq q\}\subseteq\{t\in[0,1]|S(t,r)\geq q\}$
$\Longrightarrow \inf\{t\in[0,1]|S(t,p)\geq q\}\geq\inf\{t\in[0,1]|S(t,r)\geq q\}$
$\Longrightarrow J_S(p,q)\geq J_S(r,q).$

$J_3: q\leq r \Longrightarrow \{t\in[0,1]|S(t,p)\geq q\}\supseteq\{t\in[0,1]|S(t,p)\geq r\}$
$\Longrightarrow \inf\{t\in[0,1]|S(t,p)\geq q\}\leq\inf\{t\in[0,1]|S(t,p)\geq r\}$
$\Longrightarrow J_S(p,q)\leq J_S(p,r).$ □

Theorem 3.2. *A co-implications J_S satisfy (Co-NP) and (Co-IP).*

Proof. For any S t-conorm and for all $p,q,r\in[0,1]$ we get $J_S(0,q)=\inf\{r\in[0,1]|S(r,0)\geq q\}=\inf\{r\in[0,1]|r\geq q\}=q$.

Also, $J_S(p,p)=\inf\{r\in[0,1]|S(r,p)\geq p\}=0.$

□

Theorem 3.3. *If S is a right continuous, then J_S satisfy (Co-EP) and (Co-OP).*

Proof. For any right continuous t-conorm S and for all $p, q, r \in [0,1]$ and by using R^* condition we have

$$\begin{aligned} J_S\left(p, J_S\left(q, r\right)\right) &= \inf\left\{t \in [0,1] | S(t,p) \geq J_S\left(q,r\right)\right\} = \inf\left\{t \in [0,1] | S\left(S(t,p),q\right) \geq r\right\} \\ &= \inf\left\{t \in [0,1] | S\left(t, S(p,q)\right) \geq r\right\} = \inf\left\{t \in [0,1] | S\left(t, S(q,p)\right) \geq r\right\} \\ &= \inf\left\{t \in [0,1] | S\left(S(t,q),p\right) \geq r\right\} = \inf\left\{t \in [0,1] | S\left(t,q\right)\right) \geq J_S\left(p,r\right)\right\} \\ &= J_S\left(q, J_S\left(p,r\right)\right). \end{aligned}$$

Now, we would like to prove that $J_S\left(p,q\right) = 0 \Leftrightarrow p \geq q$. If $p \geq q$ then $S\left(p,0\right) = p \geq q$, so $J_S\left(p,q\right) = 0$. Conversely, if $J_S\left(p,q\right) = 0$ then because of R^* condition we get $S\left(p,0\right) \geq q$, i.e., $p \geq q$. □

4. Conclusion

The definition of power stable t-norm and t-conorm are introduced then the new concepts of dissipative and conservative for t-norm and t-conorm are studied with examples. Also, there are four usual models of fuzzy implications *(S,N)*, residual, QL-operation and D-operations implication. In this paper we introduced residual co-implication. Now, an interesting natural question arises that to find *(T,N)*, Co-QL-operation and Co-D-operations

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