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SYMMETRY AND PERIODIC-CHAOS IN 3-D SINUSOID DISCRETE MAPS

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Abstract. In this paper, we investigate the effectiveness of the sinusoid map in a new simple 3-D discrete map, that realizes a new physical phenomenon, called symmetry and periodic-chaos. This phenomenon is justified by numerical investigation, and the new simple 3-D discrete maps produce several chaotic attractors obtained by the quasi-periodic route to chaos.

1. Introduction

It is well-known in the theoretical research the sinusoid map play an important role in mathematics due to the properties of this map, sinusoid map or sinusoidal map is generally the sine map which is related to the oscillations, can describe many oscillating phenomena. This map is very commonly used in pure and applied mathematics [1], [2], in addition to mathematics, sinusoid map occur in other fields of study such as science, and physics and engineering [3], [5]. This map also occur in nature, many processes in nature display repeating patterns described by sinusoid map as seen in ocean waves, sound waves, light waves and many other fields. Some authors have described chaotic map with sinusoid map [4], [6], [7],[8], [9]. In recent years, many documents have described 3-D chaotic maps such as with quadratic inverse [10], [11], [12], [13], [14]. Doubtless, the study of 3-D discrete map such as with sinusoid map is interesting contribution to the development of the theory of dynamical systems. This short paper investigate the effect of the sinusoid map in a 3-D discrete maps, the proposed 3-D discrete map (1) is defined with two sinusoid nonlinearities, topologically different from any other know 3-D maps, this paper introduces and justifies numerically a new physical phenomenon, shown by the new simple 3-D discrete map (1) which is the chaotic behavior in the map (1) is periodic, i.e., that the chaos repeats itself regular after cycles are called periods. Furthermore the chaotic attractors obtained for the map (1) are symmetric about the origin.

Here we consider essentially the following modified 3-D map (1):

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$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} y_t \\ \sin z_t \\ a + bx_t + cy_t - \sin z_t \end{pmatrix} \tag{1}$$

Where$(a, b, c) \in \mathbb{R}^3$ are bifurcation parameters and $(x_t, y_t, z_t) \in \mathbb{R}^3$ are the state variables. The choice of sinusoidal map has an important role, is to guarantee the boundedness of the orbits of the map (1) for all values of a, b and c. Generally, the map (1) is not symmetric and the associated map of the new 3-D map (1) is continuous and differentiable on $\mathbb{R}^3$. Furthermore the Jacobian matrix of the map (1) is not constant and equal to $b \cos z$ ($b \neq 0$ and $\cos z \neq 0$).

The map (1) can be transform into a third-order difference equation as follows:

$$z_{t+1} = a + b \sin z_{t-2} + c \sin z_{t-1} - \sin z_t \tag{2}$$

2. Analytical analysis of Parameters

In this section we will show that the all orbits of the map (1) are bounded and are lies inside in a cuboid and we investigate domains for the bifurcation parameters $(a, b, c) \in \mathbb{R}^3$ in which the fixed points of the map (1) are asymptotically stable.

Theorem 1. *The all orbits of the map (1) are bounded for every* $(a, b, c) \in \mathbb{R}^3$ *and* $t > 2$, *and for all finite initial conditions* (x_0, y_0, z_0).

Proof. We use the following standard results: The real sequence $(z_n)_n$ is bounded if there is one positive real k such that $|z_n| \leq k$ for every $n \in \mathbb{N}$. In our case the sequence $(z_t)_t$ given in (2) satisfies the following inequality: $|z_t| \leq 1 + |a| + |b| + |c|$ because $|\sin z| \leq 1$ for every $z \in \mathbb{R}$. Since the real $1 + |a| + |b| + |c|$ is positive, thus the sequence $(z_t)_t$ is bounded for every $(a, b, c) \in \mathbb{R}^3$ and $t > 2$. Thus implies the all orbits of the map (1) are bounded for every $(a, b, c) \in \mathbb{R}^3$ and $t > 2$, and for all finite initial conditions $(x_0, y_0, z_0) \in \mathbb{R}^3$. ∎

It was shown in [16] that the all bounded orbits of the Hénon map are lies inside in a square, and it was shown in [15] that the all bounded orbits of the volume preserving map are lies inside in a cube. Similarly, we will show that the all orbits of the map (1) are lies inside in a cuboid.

Theorem 2. *The all orbits of the map (1) are lies inside in the following cuboid:*

$$\left\{ (x, y, z) \in \mathbb{R}^3 : |x| \leq 1, |y| \leq 1, |z| \leq 1 + |a| + |b| + |c| \right\} \tag{3}$$

Proof. It's very easy to prove this theorem, since the map (1) is equivalent to:

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} \sin z_{t-1} \\ \sin z_t \\ a + b \sin z_{t-2} + c \sin z_{t-1} - \sin z_t \end{pmatrix} \tag{4}$$

∎

Theorem 3. *The fixed point* $A(x, y, z)$ *of the map (1) is asymptotically stable for all* $a \in \mathbb{R}$ *if and only if* $(b, c) \in \cup_{i=1}^{i=2} \Omega_i$, *where:*

$$\Omega_1 : \begin{cases} -1 < b < 1 \\ \frac{b(b+1)\cos^2 z - 1}{\cos z} < c < \frac{1 - (1-b)\cos z}{\cos z} \end{cases} \tag{5}$$

$$\Omega_2 : \begin{cases} -1 < b < 1 \\ \frac{1+(1-b)\cos z}{\cos z} < c < \frac{b(b+1)\cos^2 z - 1}{\cos z} \end{cases} \tag{6}$$

Proof. The characteristic polynomial of the Jacobian matrix of the map (1) calculated at the fixed point $A(x, y, z)$, which takes the form: $P_A(\lambda) = \lambda^3 + \cos z\lambda^2 - c\cos z\lambda - b\cos z$, according to the result available in [17], we conclude that the fixed point A of the map (1) is asymptotically stable if and only if the following conditions holds: (1) $|b\cos z| < 1$, (2) $1 + \cos z - c\cos z - b\cos z > 0$, (3) $1 - \cos z - c\cos z + b\cos z > 0$ and (4) $1 - b^2\cos^2 z > b\cos^2 z - c\cos z$. From (1) we have (5) $|b| < 1$, and from (2), (3) and (5) we have (6) $c\cos z < 1 - (1-b)|\cos z|$, and from (4) we have (7) $c\cos z > b(b+1)\cos^2 z - 1$, and from (6) and (7) we get (8) $b(b+1)\cos^2 z - 1 < c\cos z < 1 - (1-b)|\cos z|$. Finally, by the conditions (5) and (8) we can obtain the conditions of asymptotic stability for the fixed point A. ∎

For example, the fixed points of the map (1) are the real solutions of the system:

$$x = y,\ y = \sin z,\ z = a + b\sin z + c\sin z - \sin z$$

Hence, one may easily obtain the equation:

$$z - (b + c - 1)\sin z - a = 0$$

Can't be compute the fixed points of the map (1) analytically, we remark if $a = 0$, the point $(0, 0, 0)$ it is fixed point of the map (1) for all values of the bifurcation parameters $(b, c) \in \mathbb{R}^2$. Thus one has the following theorem:

Theorem 4. *If $a = 0$, the fixed point $(0, 0, 0)$ of the map (1) is asymptotically stable if and only if the following condition is satisfied:*

$$\begin{cases} -1 < b < 1 \\ b(b+1) - 1 < c < b \end{cases} \tag{7}$$

If we choose $a = 0, b = 0.8$ and $c = 0.1$. Then with this values the map (1) has only one fixed point $(0, 0, 0)$, the fixed point is asymptotically stable, since we have the following three eigenvalues: $\lambda_1 = -0.8566 - 0.6229i$, $\lambda_2 = -0.8566 + 0.6229i$ and $\lambda_3 = -0.3380$, thus $|\lambda_{1,2,3}| < 1$.

3. Numerical Analysis of Parameters

In this section, we will illustrate some observed chaotic attractors, the dynamical behaviors of the map (1) are investigated numerically. Figures 1 shows the bifurcation diagram and the diagram of the variation of the largest Lyapunov exponent of the map (1) that are obtained at different values of parameter a, $a \in [-4, 4]$. However, we deduce from the bifurcation diagram Fig.1(a), that the proposed map (1) exhibit a quasi-periodic bifurcation scenario route to chaos for the selected values of the bifurcation parameter a.

First, we fix the initial condition $x_0 = y_0 = z_0 = 0.01$ and $b = 0.8$, $c = 0.9$ and let the parameter a vary in the interval $[-4, 4]$, the map (1) exhibits the following dynamical behaviors as shown in Fig.1(a) and Fig.1(b): For the range $-1.68 < a \leq 0$ the dynamical behavior of the map (1) is periodic, which is verified by the corresponding largest Lyapunov exponent is negative as shown in Fig.1(b), for the range $-2.32 < a \leq -1.68$ the dynamical behavior the map (1) is quasi-periodic with periodic windows, at the point $a = -2.24$ the dynamical behavior the map (1) is in the quasi-periodic-19 attractor as shown in Fig.4(e), Fig.4(f) shows the quasi-periodic attractor of the map (1) when $a = 1.84$, for the range $-2.72 < a \leq -2.32$ the dynamical behavior of the map (1) is chaotic with periodic windows in the chaotic band which is verified by the corresponding largest Lyapunov exponent is positive. For the range $-3.6 < a \leq -2.72$ the dynamical behavior of the map (1) is quasi-periodic with periodic windows, at the point $a = -2.72$ the dynamical behavior the map (1) is in the quasi-periodic-40 as shown in Fig.4(d), and at the point $a = -3.52$ the behavior is quasi-periodic, as shown in Fig.4(c), for the range $-4 \leq a \leq -3.6$ the behavior of the map (1) is chaotic with periodic windows in the chaotic band which is verified by the corresponding largest Lyapunov exponent is positive, Fig.4(a) and Fig.4(b) shows respectively the chaotic attractors of the map (1) when $a = -3.76$ and $a = -3.6$.

Secondly, for the range $0 \leq a < 1.68$ the dynamical behavior of the map (1) is periodic, which is verified by the corresponding largest Lyapunov exponent is negative as shown in Fig.1(b), for the range $1.68 \leq a < 2.32$ the dynamical behavior of the map (1) is quasi-periodic orbits with periodic windows, at the point $a = 2.24$ the dynamical behavior of the map (1) is in the quasi-periodic-19 as shown in Fig.5(e), Fig.5(f) shows the quasi-periodic attractor of the map (1) when $a = 1.84$, for the range $2.32 \leq a < 2.72$ the dynamical behavior of the map (1) is chaotic with periodic windows in the chaotic band which is verified by the corresponding largest Lyapunov exponent is positive. For the range $2.72 \leq a < 3.6$ the dynamical behavior of the map (1) is quasi-periodic with periodic windows, at the point $a = 2.72$ the dynamical behavior of the map (1) is in the quasi-periodic-40 as shown in Fig.5(d), and at the point $a = 3.52$ the dynamical behavior is quasi-periodic as shown in Fig.5(c), for the range $3.6 \leq a \leq 4$ the dynamical behavior of the map (1) is chaotic with periodic windows in the chaotic band which is verified by the largest Lyapunov exponent is positive, Fig.5(a) and Fig.5(b) respectively show the chaotic attractors of the map (1) when $a = 3.76$ and $a = 3.6$.

From the bifurcation diagrams in Fig.1(a), Fig.2(a), Fig.3(a) and Fig.4(a), we deduce that the chaotic behavior in the map (1) is regular periodic and has a horizontal stretch, i.e., that the chaos repeats itself as it moves along the $a - axis$ and the cycles of this regular repeating are called periods**,** and the amplitude of this chaotic bands will be 1 (it ranges from -1 to $+1$) as shown in Fig.1(a), Fig.2(a), Fig.3(a) and Fig.4(a). Furthermore, the attractors given in Figures 5 and Figures 6 are respectively symmetric about the origin and inside the two finite intervals $[-4, 0]$ and $[0, 4]$. This phenomenon called symmetry with regular periodic chaos.

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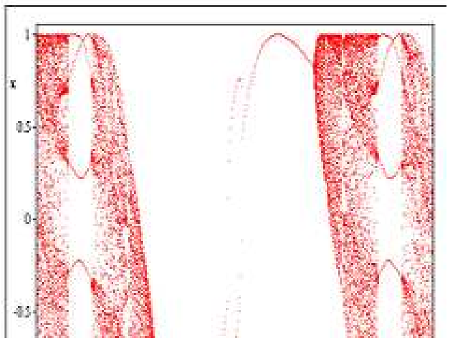

FIGURE 1. Fig.1(a): Bifurcation diagram, the symmetry quasi-periodic route to chaos for the map (1) obtained for $b = 0 : 8$, $c = 0 : 9$ and $-4 \leq a \leq 4$

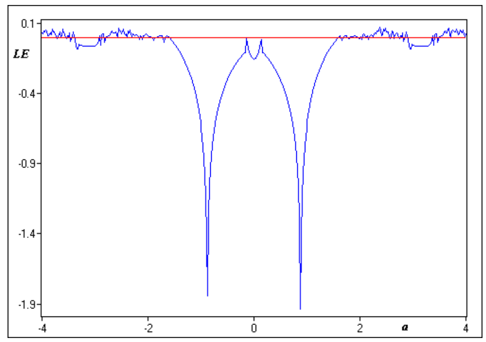

FIGURE 2. Fig.1(b): Variation symmetry of the largest Lyapunov exponent of the map (1) for $b = 0 : 8$, $c = 0 : 9$ and $-4 \leq a \leq 4$.

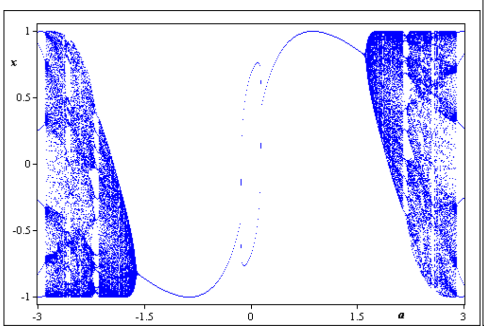

FIGURE 3. Fig.2(a): Bifurcation diagram, the symmetry quasi-periodic route to chaos for the map (1) obtained for $b = 0 : 8$, $c = 0 : 9$ and $-3 \leq a \leq 3$

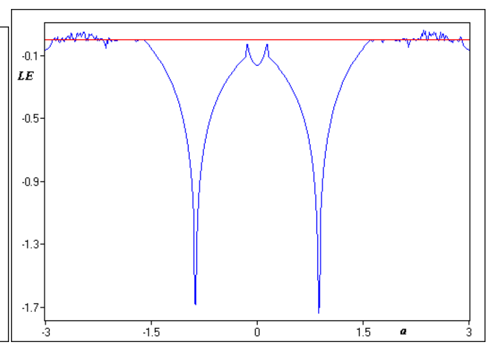

FIGURE 4. Fig.2(b): Variation symmetry of the largest Lyapunov exponent of the map (1) for $b = 0 : 8$, $c = 0 : 9$ and $-3 \leq a \leq 3$.

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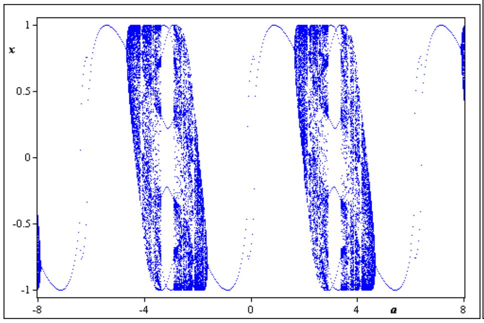

FIGURE 5. Fig.3(a): Bifurcation diagram, the symmetry quasi-periodic route to chaos for the map (1) obtained for $b = 0:8$, $c = 0:9$ and $-8 \leq a \leq 8$

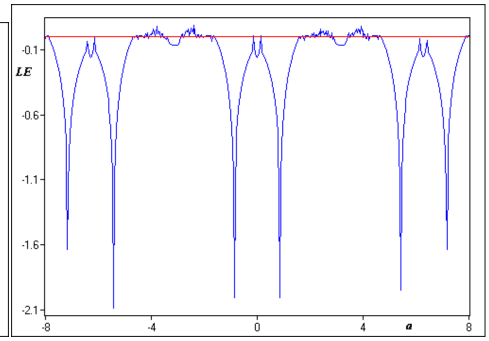

FIGURE 6. Fig.3(b): Variation symmetry of the largest Lyapunov exponent of the map (1) for $b = 0:8$, $c = 0:9$ and $-8 \leq a \leq 8$.

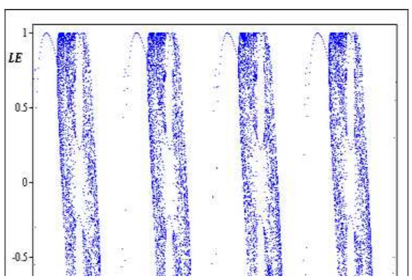

FIGURE 7. Fig.4(a): Bifurcation diagram, the symmetry quasi-periodic route to chaos for the map (1) obtained for $b = 0:8$, $c = 0:9$ and $-12 \leq a \leq 12$

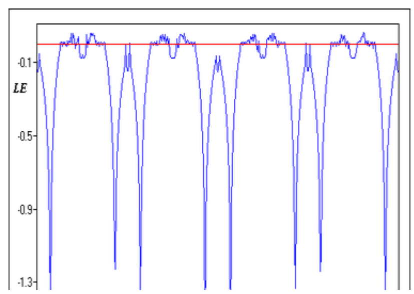

FIGURE 8. Fig.4(b): Variation symmetry of the largest Lyapunov exponent of the map (1) for $b = 0:8$, $c = 0:9$ and $-12 \leq a \leq 12$.

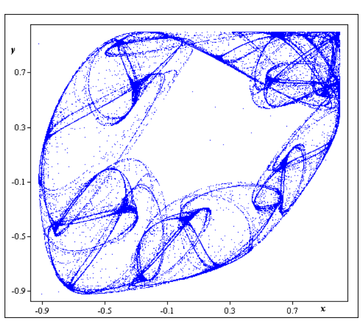

FIGURE 9. Fig.5(a): $a = -3.76$

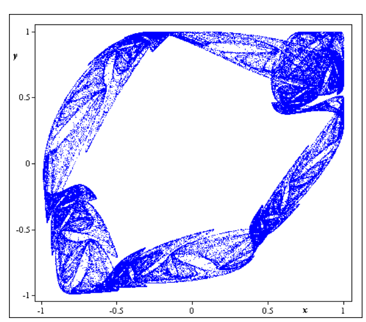

FIGURE 10. Fig.5(b): $a = -3.6$.

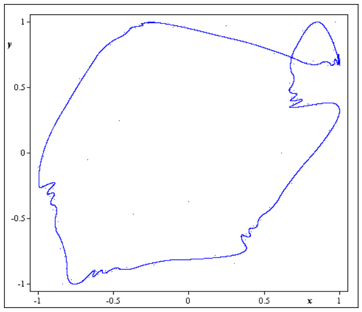

FIGURE 11. Fig.5(c): $a = -3.52$

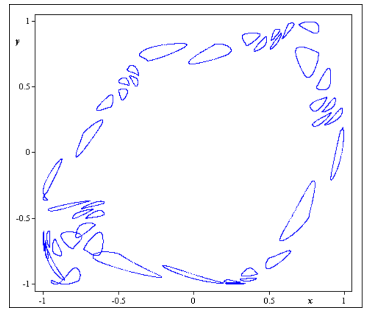

FIGURE 12. Fig.5(d): $a = -2.72$.

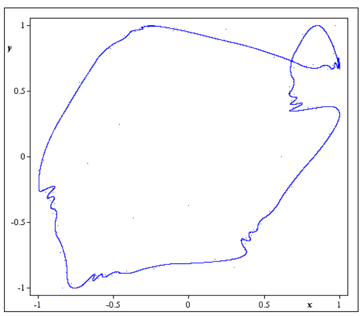

FIGURE 13. Fig.5(e): $a = -2.24$

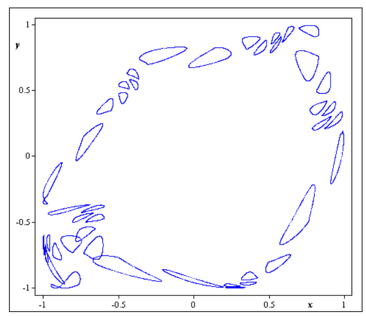

FIGURE 14. Fig.5(f): $a = -1.84$.

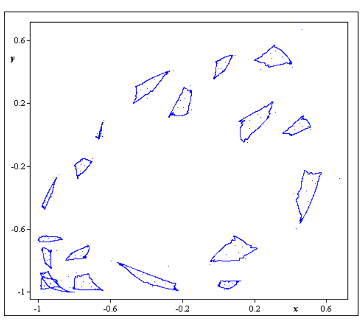

FIGURE 15. Fig.6(a): $a = 3.76$

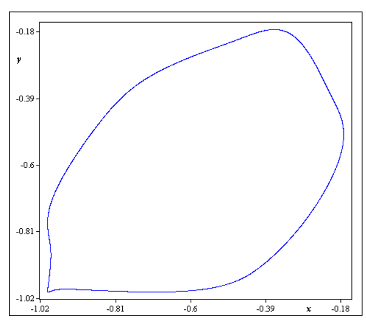

FIGURE 16. Fig.6(b): $a = 3.6$.

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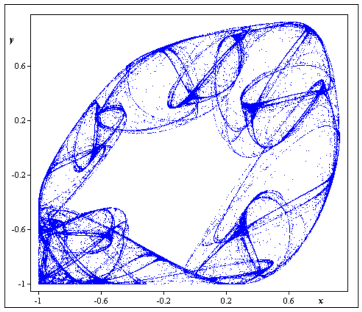

FIGURE 17. Fig.6(c): $a = 3.52$

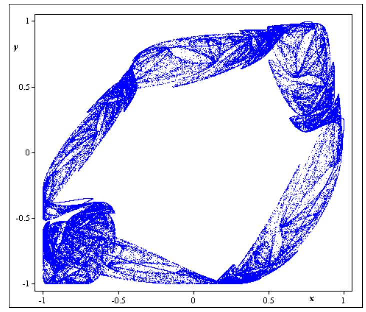

FIGURE 18. Fig.6(d): $a = 2.72$.

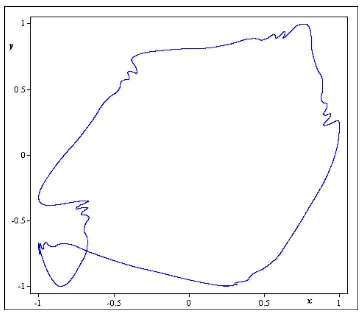

FIGURE 19. Fig.6(e): $a = 2.24$

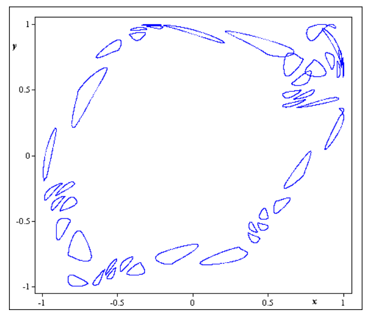

FIGURE 20. Fig.6(f): $a = 1.84$.

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