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NEW RESULTS ON SIMULTANEOUS APPROXIMATION PRESERVING MAPS

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Abstract. We are going to study best simultaneous approximation preserving maps in a conditionally complete lattice Banach space X with a strong unit. Also we develop a theory of best simultaneous approximations for closed upward sets.

1. Introduction

The linear preserver problem on X is to characterize certain properties which maybe functions, relations, subsets, etc should be preseved. The study of this problem has attracted the attention of many mathematicians in recent years. Many results which have been obtained on this topic in recent decades reveal both algebraic and geometric structures of the operator algebras from some new aspects.

Best approximation is an important problem on X which were considered recently in a series papers (for instance [3-8]). In this paper we would like consider best simultaneous approximation preserve property on Babach lattice. Best simultaneous approximation is a generalization of the best approximation and has a variety of applications for instance control theory and perturbation theory. We will consider some property in best simultaneous approximation in Banach lattice.

Let X be a normed linear space, W a non-empty subset of X and S a non-empty bounded subset of X. Define

$$d(S, W) = \inf_{w \in W} \sup_{s \in S} \|s - w\|$$

We recall a point $w_0 \in W$ is called a best simultaneous approximation to S from W if

$$\sup_{s \in S} \|s - w_0\| = d(S, W)$$

and also

$$S_W(S) := \{w \in W : \sup_{s \in S} \|s - w\| = d(S, W)\}.$$

It is clear that $S_W(S)$ is a closed and bounded subset of X.

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If there exists at least one best simultaneous approximation to S from W, then W is called a simultaneous proximinal subset of X. If there exists a unique best simultaneous approximation to S from W, then W is called a simultaneous Chebyshev of X.

Recall that the set X endowed with partially ordered relation $\leq$ is said to lattice if for every $x, y \in X$, $\sup\{x, y\}$ and $\inf\{x, y\}$ exist in X that denoted by $(X, \leq)$. Also vector lattice $(X, \leq, +, .)$ is a lattice $(X, \leq)$, with a binary operation $+$ and scalar product . such that $(X, +, .)$ is a vector space.

Definition 1.1. *([9]) A lattice $(X, \leq)$ is said to be conditionally complete if it satisfies one of the following equivalent conditions*
(1) Every non-empty lower bounded set admits an infimum.
(2) Every non-empty upper bounded set admits a supremum.
(3) There exists a complete lattice $\overline{X} := X \cup \{\perp, \top\}$, which we shall call the minimal completion of X, with bottom element $\perp$ and top element $\top$, such that X is a sublattice of $\overline{X}$, $inf X = \perp$ and $sup X = \top$.

Definition 1.2. *([9]) A vector lattice $(X, \leq, +, .)$ such that $(X, \leq)$ is a conditionally complete lattice is called a conditionally complete vector lattice.*

Let X be a vector lattice. Recall that an element $\mathbf{1} \in X$ is called a strong unit if for every $x \in X$ there exists a $0 < \lambda \in \mathbf{R}$ such that $x \leq \lambda \mathbf{1}$. Using a strong unit $\mathbf{1}$, we can prove that

$$\|x\| = \inf\{\lambda > 0 : \ |x| \leq \lambda \mathbf{1}\} \ \forall x \in X.$$

is a norm on X. Also

$$|x| \leq \|x\| \mathbf{1} \ \forall x \in X.$$

Well-know examples of vector lattices with the strong units are the lattice of all bounded functions defined on a set X and the lattice $L^\infty(S, \Sigma, \mu)$ of all essentially bounded functions on a space S with a σ-algebra of measurable sets Σ and a measure μ.

Definition 1.3. *([9]) A conditionally complete lattice Banach space X is a real Banach space which is also a conditionally complete vector lattice and such that*

$$|x| \leq |y| \Rightarrow \|x\| \leq \|y\| \ \forall x, y \in X.$$

Also every bounded set S define

$$B(S, r) = \{y \in X : \sup_{s \in S} \|s - y\| \leq r\}$$

$$= \{y \in X : \sup S - r\mathbf{1} \leq y \leq inf S + r\mathbf{1}\},$$

where $0 < r \in \mathbb{R}$.

Example 1.4. *Suppose $X = \ell^2_\infty$ is defined by*

$$\ell^2_\infty = \{x = (t_1, t_2) : t_1, t_2 \in \mathbb{R}\}$$

with the natural order and the norm defined by

$$\|x\| = \max\{|t_1|, |t_2|\} \ for \ every \ x = (t_1, t_2) \in \ell^2_\infty.$$

Then X is a conditional complete lattice Banach space with the strong unit $\mathbf{1} = (1,1)$. Suppose

$$W = \{(t, t-1) : t \in [0,1]\},\ \ S = \{(t, -t) : t \in [-1, 0]\}.$$

Hence W is a closed convex subset of X and S is a bounded subset of X. It is clear that $w_0 = (\frac{1}{2}, -\frac{1}{2}) \in S_W(S)$.

In this paper we shall introduce and discuss the concept of approximation simultaneous preserving operators on Banach lattices with strong unit $\mathbf{1}$; Also different conditions for maps which preserve best approximation by subsets of Banach lattices is obtained.

2. Main Results

A linear mapping from a Banach lattice to a Banach lattice is positive if it carries positive vectors to positive vectors. This is equivalent to saying that the mapping is order preserving. It is easy to see a positive linear mapping is continuous, so we call them positive operators.

Definition 2.1. *Suppose X, Y are two Banach lattices. A map $T : X \to Y$ is called simultaneous approximation preserving map if for subspace W of X, $T(S_W(S)) \subseteq S_{T(W)}(T(S))$.*

The following theorem characterizes simultaneous approximation preserving maps.

Theorem 2.2. *Let X, Y are two Banach lattices with strong units $\mathbf{1}$. Let $T : X \to Y$ be an injective positive operator such that $T(\mathbf{1}) = \mathbf{1}$. Then T is a simultaneous approximation preserving operator.*

Proof. Suppose d(S,W)=r and $w_0 \in W$ is a best simultaneous approximation to S from W, then

$$\begin{aligned} & \sup_{s\in S} \|s - w_0\| \leq r \\ \Rightarrow\ & \sup S - r\mathbf{1} \leq w_0 \leq \inf S + r\mathbf{1} \\ \Rightarrow\ & T(\sup S) - T(r\mathbf{1}) \leq T(w_0) \leq T(\inf S) + T(r\mathbf{1}) \\ \Rightarrow\ & \sup T(S) - r\mathbf{1} \leq T(w_0) \leq \inf T(S) + r\mathbf{1} \\ \Rightarrow\ & \sup_{s\in T(S)} \|T(s) - T(w_0)\| \leq r. \end{aligned}$$

On the other hand

$$d(T(S), T(W)) = \inf_{w\in W} \sup_{s\in S} \|Ts - Tw\| = \inf_{w\in W} \sup_{s\in S} \|s - w\| = d(S, W) = r.$$

Therefore $T(w_0)$ is a best simultaneous approximation for $T(S)$ and so T is simultaneous approximation preserving. ☐

Corollary 2.3. *Let X, Y are two Banach lattices with strong units $\mathbf{1}$. Let $T : X \to Y$ be an isometry positive operator such that $T(\mathbf{1}) = \mathbf{1}$. Then*

$$T(S_W(S)) = S_{T(W)}(T(S)).$$

Example 2.4. *Let X be a Banach lattice with strong unit $\mathbf{1}$ and $a \in X$. Define $T_a : X \to X$, $T_a(x) = a + x$. It is clear that T_a is an simultaneous approximation preserving operator on X, for every $a \in X$.*

Let X be a conditionally complete lattice Banach space with a strong unit $\mathbf{1}$. In the rest of the paper we shall assume that S is a non-empty bounded set in X. Note that $U \subseteq X$ is an upward set if $u \in U$ and $u \leq x$, then $x \in U$. For instance suppose

$x \in X$ and $U = \{y \in X : \ x \leq y\}$. Then U is an upward set of X.

Theorem 2.5. *Let W be a closed upward subset of X. Then W is a simultaneous proximinal subset of X.*

Proof. Suppose S is an arbitrary bounded set in X and $r = d(S,W) > 0$. Let $w_0 = \inf S + r\mathbf{1} \in X$. Then $w_0 \in B(S,r)$,

$$\sup_{s\in S} \|s - w_0\| \leq r = d(S,W).$$

Since $r = d(S,W) = \inf_{w\in W} \sup_{s\in S} \|s - w\|$, it follows that for every $\epsilon > 0$, there exists $w_\epsilon \in W$ such that $\sup_{s\in S} \|s - w_\epsilon\| \leq r + \epsilon$. Then

$$w_\epsilon - s \leq (r + \epsilon)\mathbf{1} \ \forall s \in S.$$

This implies that $w_\epsilon \leq \inf S + r\mathbf{1} + \epsilon\mathbf{1}$, and so $w_\epsilon \leq w_0 + \epsilon\mathbf{1}$ for every $\epsilon > 0$. Since W is upward and $w_\epsilon \in W$, it follows that $w_0 + \epsilon\mathbf{1} \in W$ for every $\epsilon > 0$. Hence, since W is closed, $w_0 \in W$. Therefore

$$d(S,W) \leq \sup_{s\in S} \|s - w_0\|.$$

We get $w_0 \in S_W(S)$. Hence, W is a simultaneous proximinal in X. □

Corollary 2.6. *Let W be a closed upward subset of X. Then there exists the least element $w_0 := minS_W(S)$ of the set $S_W(S)$, namely, $w_0 = \inf S + r\mathbf{1}$, where $r :=$ d(S,W)*

Proof. Assume that $d(S,W) > 0$ and $w_0 = \inf S + r\mathbf{1}$. Then, by Theorem 2.5., $w_0 \in S_W(S)$. Since $w_0 = infS + r1 \in B(S,r)$, that $y \geq w_0$ for every $y \in B(S,r)$. Therefore, w_0 is the least element of B(S, r).

Now, let $w \in S_W(S)$ be arbitrary. Then

$$\sup_{s\in S} \|s - w\| = d(S,W) = r$$

and so $w \in B(S,r)$. Therefore, $w \geq w_0$. Hence, w_0 is the least element of the set $S_W(S)$, which completes the proof. □

3. Application

The main intention of this section is devoted to solution of the nonlinear optimal feedback control problem under worst perturbation. as follows:

$$\inf_{u\in U} \sup_{s\in S} f(t, x(t), u(t, x(t)), s(t))dt$$

where $x : J \to A \subseteq R^n$ is an absolutely continuous function as trajectory and the set A is a compact set, $u : J \times A \to U \subseteq R^m$ is a measurable function as a control function, where the set U is a connected and compact set in R^m, and

$f : J \times A \times U \times S \to R^n$ is a bounded differentiable function.

Theorem 3.1. *Let $f_0 \in C(\Omega)$, Q be a w^*-compact subset of $M^*(\Omega)$ and S is bounded set. If for every $s \in S$ there exist $\mu_s^* \in Q$ such that*

$$\inf_{\mu \in Q} \sup_{s \in S} \mu(f_0) = \sup_{s \in S} \mu_w^*(f_0).$$

Moreover if S is compact there exist $\mu_{s_0}^*$ such that

$$\inf_{\mu \in Q} \sup_{s \in S} \mu(f_0) = \mu_{s_0}^*(f_0).$$

Proof. Since $f_0 \in C(\Omega)$ and Q is w^*-compact for every $w \in W$ there exist $\mu_s^* \in Q$ such that

$$\mu_s^*(f_0) = \inf_{\mu \in Q} \mu(f_0) \ (*)$$

Hence, in view of (*), we have

$$\begin{aligned} \inf_{\mu \in Q} \sup_{s \in S} \mu(f_0) &\leq \sup_{s \in S} \mu_s^*(f_0) \\ &= \sup_{s \in S} \inf_{\mu \in Q} \mu(f_0) \\ &\leq \inf_{\mu \in Q} \sup_{s \in S} \mu(f_0). \end{aligned}$$

Moreover if W is compact there exist $\mu_{s_0}^*$ such that $\sup_{s \in S} \mu_s^*(f_0) = \mu_{s_0}^*(f_0)$ and so

$$\inf_{\mu \in Q} \sup_{s \in S} \mu(f_0) = \mu_{s_0}^*(f_0).$$

□

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References

[1] H. Alizadeh, Sh. Rezapour and S. M. Vaezpour, Simultanous Approximations of Downward Sets, Bulletin of the Iranian Mathematical Society V. 39, No. 1(2013), pp 193-203

[2] H. Mazaheri and M. Hossein Zadeh, The maps preserving approximation, International Mathematical Forum, 2 (2007) 905-909.

[3] P. D. Milman, On the best simultaneous approximation in normed linear spaces, J. Approx. Theory, 20 (1977), 223-238.

[4] H. Mohebi, Downward sets and their best simultaneous approximation properties with applications, Numer. Funct. Anal. and Optim., 25 (2004), 685-705.

[5] H. Mohebi and A. M. Rubinov, Best approximation by downward sets with applications, Anal. in Theory and Appl., 22 (2006), 20-40.

[6] H. Mohebi and E. Naraghirad , Closed convex sets and their best simultaneous approxiamtion properties with applications, Optimi Letters, 1 (2006), 1-16.

[7] B. N. Sahney and S. P. Singh, On the best simultaneous approximation in Banach spaces, J. Approx. Theory, 35 (1982), 222-224.

[8] I. Singer, Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces, Springer-Verlag, New York-Berlin, 1970.

[9] B. Z. Vulikh, Introduction to the theory of partially ordered vector spaces, Wolters-Noordhoff, Groningen, 1967.

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