Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 9 Issue 2(2017), Pages 24-29.

AN EXAMINATION ON HELIX AS INVOLUTE, BERTRAND MATE AND MANNHEIM PARTNER OF ANY CURVE α IN $\mathbf{E}^3$

SÜLEYMAN ŞENYURT, ŞEYDA KILIÇOĞLU

Abstract. In this study we consider three offset curves of a curve α such as the involute curve α^* , Bertrand mate α_1 and Mannheim partner α_2 . We examined and find the conditions of Frenet apparatus of any curve α which has the involute curve α^* , Bertrand mate α_1 and Mannheim partner α_2 are the general helix.

1. Introduction and Preliminaries

In science and nature helix is very famous and fascinating curve. A curve α with $\tau(s) \neq 0$ is called a cylindrical helix if the tangent lines of make a constant angle with a fixed direction. Also cylindrical helix or general helix is a helix which lies on the cylinder. If the curve is a general helix, the ratio of the first curvature of the curve to the torsion of the curve must be constant. Further if both τ and κ are non-zero constant, we call a curve a circular helix. In [1] general Helices in the Sol Space Sol^3 are examined.The quantities $\{T, N, B, \kappa, \tau\}$ are collectively Frenet-Serret apparatus of a curve α. The Frenet formulae are also well known as

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{1.1}$$

1.1. **Involute curve and Frenet apparatus.** The involute of a given curve is a well-known concept in Euclidean $3-space$. Let α and α^* are the arclengthed curves with the arcparametres s and s^*, respectively. The quantities $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$ are collectively Frenet-Serret apparatus of the curve α and α^*, respectively. If the curve α^* which lies on the tangent surface intersect the tangent lines orthogonally is called an involute of α. If a curve α^* is an involute of α.

$$\alpha^*(s) = \alpha(s) + (c-s)T(s) \tag{1.2}$$

is the equation of involute of the curve α. For more detail see in [2, 5].

2000 *Mathematics Subject Classification.* 53A04, 53A05.
Key words and phrases. Involute curves; Bertrand curves; Mannheim curves.

Submitted December 20, 2016. April 17, 2017.
Communicated by Krishan Lal Duggal.

Theorem 1.1. *The Frenet vectors of the involute α^*, based on the its evolute curve α [2] are*

$$\begin{cases} T^* = N, \\ N^* = \dfrac{-\kappa}{\sqrt{\kappa^2+\tau^2}}T + \dfrac{\tau}{\sqrt{\kappa^2+\tau^2}}B \\ B^* = \dfrac{\tau}{\sqrt{\kappa^2+\tau^2}}T + \dfrac{\kappa}{\sqrt{\kappa^2+\tau^2}}B. \end{cases} \tag{1.3}$$

The first and second curvature of involute α^*, respectively, are

$$\kappa^* = \frac{\sqrt{\kappa^2+\tau^2}}{(c-s)\kappa}, \quad \tau^* = \frac{-\tau^2\left(\frac{\kappa}{\tau}\right)'}{(c-s)\kappa\left(\kappa^2+\tau^2\right)}. \tag{1.4}$$

Also

$$\frac{ds}{ds^*} = \frac{1}{(c-s)\kappa}. \tag{1.5}$$

1.2. **Bertrand curve and Frenet apparatus.** The curves $\{\alpha, \alpha_1\}$ defined Bertrand pairs curve if they have common principal normal lines. If the α_1 is called Bertrand mate of α, then we have

$$\alpha_1(s) = \alpha(s) + \lambda N(s). \tag{1.6}$$

If α is a Bertrand curve if and only if there exist non-zero real numbers λ and β such that constant

$$\lambda\kappa + \beta\tau = 1, \ \beta = \frac{1-\lambda\kappa}{\tau} \tag{1.7}$$

for any $s \in I$. It follows from this fact that a circular helix is a Bertrand curve, [2, 5, 6].

Theorem 1.2. *Let α_1 be the Bertrand mate of the curve α. The quantities $\{T, N, B, \kappa, \tau\}$ and $\{T_1, N_1, B_1, \kappa_1, \tau_1\}$ are collectively Frenet-Serret apparatus of the curves α and the Bertrand mate α_1 , respectively, then* [6]

$$\begin{cases} T_1 = \frac{\beta}{\sqrt{\lambda^2+\beta^2}}T + \frac{\lambda}{\sqrt{\lambda^2+\beta^2}}B \\ N_1 = N \\ B_1 = \frac{-\lambda}{\sqrt{\lambda^2+\beta^2}}T + \frac{\beta}{\sqrt{\lambda^2+\beta^2}}B, \end{cases} \tag{1.8}$$

and the first and second curvatures of the offset curve α_1 are given by

$$\kappa_1 = \frac{\beta\kappa - \lambda\tau}{\left(\lambda^2+\beta^2\right)\tau}, \quad \tau_1 = \frac{1}{\left(\lambda^2+\beta^2\right)\tau}. \tag{1.9}$$

Also

$$\frac{ds}{ds_1} = \frac{1}{\tau\sqrt{\lambda^2+\beta^2}}. \tag{1.10}$$

1.3. **Mannheim curve and Frenet apparatus.** Let $T_2(s_2), N_2(s_2), B_2(s_2)$ be the Frenet frames of the α_2, respectively. If the principal normal vector N of the curve α is linearly dependent on the binormal vector B^* of the curve α^*, then the pair $\{\alpha, \alpha_2\}$ is said to be Mannheim pair, then α is called a Mannheim curve and α^* is called Mannheim partner curve of α where $\langle T, T_2\rangle = \cos\theta$ and besides the equality $\frac{\kappa}{\kappa^2+\tau^2} = \textit{constant}$ is known the offset property, for some non-zero constant [3]. Mannheim partner curve of α can be represented

$$\alpha_2(s) = \alpha(s) - \lambda^* N(s) \tag{1.11}$$

where

$$\lambda^* = -\frac{\kappa}{\kappa^2+\tau^2}. \tag{1.12}$$

Frenet-Serret apparatus of Mannheim partner curve α^*, based in Frenet-Serret vectors of Mannheim curve α are

$$\begin{cases} T_2 = \cos\theta\ T - \sin\theta\ B \\ N_2 = \sin\theta\ T + \cos\theta\ B \\ B_2 = N. \end{cases} \tag{1.13}$$

The curvature and the torsion have the following equalyties,

$$\begin{cases} \kappa_2 = -\dfrac{d\theta}{ds^*} = \dfrac{\theta'}{\cos\theta} \\ \\ \tau_2 = \dfrac{\kappa}{\lambda^*\tau} = \dfrac{\kappa^2+\tau^2}{-\tau} \end{cases} \tag{1.14}$$

we use dot to denote the derivative with respect to the arc length parameter of the curve α. Also

$$\frac{ds}{ds_2} = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1+\lambda^*\tau}}. \tag{1.15}$$

For more detail see in [4].

2. Helices as Involute, Bertrand and Mannheim pairs of any curve

Let $\{\alpha, \alpha^*\}$ be evolute-involute curves. If involute α^* is an general helix, lets say α^* is involute helix.

Theorem 2.1. *Let $\{\alpha, \alpha^*\}$ be evolute-involute curves. Involute α^* is a general helix under the condition*

$$\tau^2\left(\kappa^2+\tau^2\right)\left(\frac{\kappa}{\tau}\right)'' + \left(2\kappa^2\tau\tau' - 3\tau^2\tau' + 2\tau^3\tau' - 3\tau^2\kappa'\right)\left(\frac{\kappa}{\tau}\right)' = 0 \tag{2.1}$$

Proof. Involute α^* is a general helix if and only if $\frac{\tau^*}{\kappa^*}$ is constant. From the equation (1.4), we can write

$$\frac{\tau^*}{\kappa^*} = \frac{\dfrac{-\tau^2\left(\frac{\kappa}{\tau}\right)'}{(c-s)\kappa\left(\kappa^2+\tau^2\right)}}{\dfrac{\sqrt{\kappa^2+\tau^2}}{(c-s)\kappa}} = \frac{-\tau^2\left(\frac{\kappa}{\tau}\right)'}{\left(\kappa^2+\tau^2\right)^{\frac{3}{2}}}.$$

Then

$$\left(\frac{\tau^*}{\kappa^*}\right)'_{s^*} = 0.$$

Hence

$$\frac{\left(\kappa^2+\tau^2\right)^{\frac{3}{2}}\left(-\tau^2\left(\frac{\kappa}{\tau}\right)'\right)' - \left(-\tau^2\left(\frac{\kappa}{\tau}\right)'\right)\left(\left(\kappa^2+\tau^2\right)^{\frac{3}{2}}\right)'}{\left(\kappa^2+\tau^2\right)^3(c-s)\kappa} = 0$$

$$\Rightarrow \quad \left(\kappa^2+\tau^2\right)^{\frac{3}{2}}\left(-\tau^2\left(\frac{\kappa}{\tau}\right)'\right)' - \left(-\tau^2\left(\frac{\kappa}{\tau}\right)'\right)\left(\left(\kappa^2+\tau^2\right)^{\frac{3}{2}}\right)' = 0$$

$$\Rightarrow \quad \tau^2\left(\kappa^2+\tau^2\right)\left(\frac{\kappa}{\tau}\right)'' + \left(2\kappa^2\tau\tau' - 3\tau^2\tau' + 2\tau^3\tau' - 3\tau^2\kappa'\right)\left(\frac{\kappa}{\tau}\right)' = 0.$$

□

Corollary 2.2. *If the curve α is a general helix, then the involute α^* of the curve α is a planar curve. Hence involute α^* cant be a general helix.*

Proof. It has been known that the curve $\alpha(s)$ is a general helix if and only if $\frac{\kappa}{\tau} = d$ is constant, then $\left(\frac{\kappa}{\tau}\right)' = 0$. It is trivial since

$$\frac{\tau^*}{\kappa^*} = \frac{-\tau^2\left(\frac{\kappa}{\tau}\right)'}{\left(\kappa^2+\tau^2\right)^{\frac{3}{2}}}.$$

□

Let $\{\alpha, \alpha_1\}$ be Bertrand curve and Bertrand mate If Bertrand mate α_1 is a general helix, lets say α_1 is Bertrand mate helix.

Theorem 2.3. *Let $\{\alpha, \alpha_1\}$ be Bertrand curve and Bertrand mate. Bertrand mate α_1 is a general helix under the condition*

$$\lambda = \frac{\left(\frac{\tau}{\kappa}\right)'}{\left(\frac{\kappa^2+\tau^2}{\tau}\right)'}, \quad \beta = \frac{\left(\frac{\kappa^2+\tau^2}{\tau}\right)' - \left(\frac{\kappa}{\tau}\right)'\kappa}{\left(\frac{\kappa^2+\tau^2}{\tau}\right)'\tau}$$

Proof. Bertrand mate α_1 is a general helix if and only if $\dfrac{\tau_1}{\kappa_1}$ is constant. From the equation (1.9), we can write

$$\frac{\tau_1}{\kappa_1} = \frac{\dfrac{1}{\left(\lambda^2+\beta^2\right)\tau}}{\dfrac{\beta\kappa-\lambda\tau}{\left(\lambda^2+\beta^2\right)\tau}} = \frac{1}{\beta\kappa-\lambda\tau}$$

Then differentiating, we find

$$\begin{aligned}
\left(\frac{\tau_1}{\kappa_1}\right)'_{s_1} &= 0 \\
&\Rightarrow \left(\frac{\tau_1}{\kappa_1}\right)'_{s}\frac{ds}{ds_1} = 0 \\
&\Rightarrow \left(\frac{1}{\beta\kappa-\lambda\tau}\right)'_{s}\frac{1}{\tau\sqrt{\lambda^2+\beta^2}} = 0, \quad \frac{1}{\tau\sqrt{\lambda^2+\beta^2}} \neq 0 \\
&\Rightarrow \left(\frac{1}{\beta\kappa-\lambda\tau}\right)'_{s} = 0 \\
&\Rightarrow \frac{-\left(\beta\kappa-\lambda\tau\right)'}{\left(\beta\kappa-\lambda\tau\right)^2} = 0 \\
&\Rightarrow \left(\beta\kappa-\lambda\tau\right)' = 0 \\
&\Rightarrow \left(\frac{1-\lambda\kappa}{\tau}\kappa-\lambda\tau\right)' = 0
\end{aligned}$$

$$\Rightarrow \quad \frac{\left(\kappa-\lambda\left(\kappa^2+\tau^2\right)\right)'\tau-\tau'\left(\kappa-\lambda\left(\kappa^2+\tau^2\right)\right)}{\tau^2}=0$$

$$\Rightarrow \quad \lambda=\frac{\tau\kappa'-\kappa\tau'}{\left(\tau\left(\kappa^2+\tau^2\right)'-\left(\kappa^2+\tau^2\right)\tau'\right)}=\frac{\left(\frac{\kappa}{\tau}\right)'}{\left(\frac{\kappa^2+\tau^2}{\tau}\right)'}$$

and

$$\beta=\frac{\left(\frac{\kappa^2+\tau^2}{\tau}\right)'-\left(\frac{\kappa}{\tau}\right)'\kappa}{\left(\frac{\kappa^2+\tau^2}{\tau}\right)'\tau}.$$

□

Let $\{\alpha,\alpha_2\}$ be Mannheim curve and Mannheim partner. Mannheim partner α_2 is a general helix, lets say α_2 is Mannheim partner helix.

Theorem 2.4. *Let $\{\alpha,\alpha_2\}$ be Mannheim curve and Mannheim partner. Mannheim partner α_2 is a general helix under the condition*

$$\tan\theta=\frac{-\left(\tau'\theta'+\tau\theta''\right)\left(\kappa^2+\tau^2\right)+\tau\theta'\left(\kappa^2+\tau^2\right)'}{2\tau\theta'\left(\kappa^2+\tau^2\right)}$$

or

$$2\theta'\tan\theta-\theta''=\left(\frac{\tau}{\kappa^2+\tau^2}\right)'\frac{\left(\kappa^2+\tau^2\right)}{\tau}$$

Proof. Mannheim partner α_2 is a general helix if and only if

$$\frac{\tau_2}{\kappa_2}=\frac{-\tau\theta'}{\left(\kappa^2+\tau^2\right)\cos\theta}=constant$$

If the derivative is taken, we can say

$$\left(\frac{\tau_2}{\kappa_2}\right)'_{s_2} = 0$$

Hence,

$$\begin{aligned}
\left(\frac{\tau_2}{\kappa_2}\right)'_{s}\frac{ds}{ds_2}=0 \quad &\Rightarrow \quad \left(\frac{-\tau\theta'}{\left(\kappa^2+\tau^2\right)\cos\theta}\right)'_{s}\frac{1}{\cos\theta}=0\\
&\Rightarrow \quad \left(\frac{-\tau\theta'}{\left(\kappa^2+\tau^2\right)\cos^2\theta}\right)'_{s}=0\\
&\Rightarrow \quad \frac{\left(-\tau\theta'\right)'\left(\kappa^2+\tau^2\right)\cos^2\theta+\tau\theta'\left(\left(\kappa^2+\tau^2\right)\cos^2\theta\right)'}{\left(\left(\kappa^2+\tau^2\right)\cos^2\theta\right)^2}=0\\
&\Rightarrow \quad -\left(\tau'\theta'+\tau\theta''\right)\left(\kappa^2+\tau^2\right)\cos^2\theta+\tau\theta'\big(\left(\kappa^2+\tau^2\right)'\cos^2\theta\\
&\qquad -2\left(\kappa^2+\tau^2\right)\cos\theta\sin\theta\big)=0
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \quad -\left(\tau'\theta'+\tau\theta''\right)\left(\kappa^2+\tau^2\right)\cos^2\theta+\tau\theta'\left(\kappa^2+\tau^2\right)'\cos^2\theta \\
&\qquad -2\tau\theta'\left(\kappa^2+\tau^2\right)\cos\theta\sin\theta=0 \\
&\Rightarrow \quad \left[-\left(\tau'\theta'+\tau\theta''\right)\left(\kappa^2+\tau^2\right)+\tau\theta'\left(\kappa^2+\tau^2\right)'\right]\cos^2\theta \\
&\qquad -2\tau\theta'\left(\kappa^2+\tau^2\right)\cos\theta\sin\theta=0 \\
&\Rightarrow \quad 2\tau\theta'\left(\kappa^2+\tau^2\right)\frac{\theta'\cos\theta\sin\theta}{\cos^2\theta}=-\left(\tau'\theta'+\tau\theta''\right)\left(\kappa^2+\tau^2\right) \\
&\qquad\qquad +\tau\theta'\left(\kappa^2+\tau^2\right)' \\
&\Rightarrow \quad 2\tau\theta'^2\left(\kappa^2+\tau^2\right)\frac{\sin\theta}{\cos\theta}=-\left(\tau'\theta'+\tau\theta''\right)\left(\kappa^2+\tau^2\right)+\tau\theta'\left(\kappa^2+\tau^2\right)' \\
&\Rightarrow \quad \tan\theta=\frac{-\left(\tau'\theta'+\tau\theta''\right)\left(\kappa^2+\tau^2\right)+\tau\theta'\left(\kappa^2+\tau^2\right)'}{2\tau\theta'^2\left(\kappa^2+\tau^2\right)} \\
&\Rightarrow \quad 2\tan\theta=\frac{-\tau\left(\kappa^2+\tau^2\right)\theta''+\left[\tau\left(\kappa^2+\tau^2\right)'-\tau'\left(\kappa^2+\tau^2\right)\right]\theta'}{\tau\left(\kappa^2+\tau^2\right)\theta'^2} \\
&\Rightarrow \quad \frac{\theta''}{\theta'}-2\tan\theta=\frac{\tau'\left(\kappa^2+\tau^2\right)-\tau\left(\kappa^2+\tau^2\right)'}{\left(\kappa^2+\tau^2\right)^2}\frac{\left(\kappa^2+\tau^2\right)}{\theta'\tau} \\
&\Rightarrow \quad 2\theta'\tan\theta-\theta''=\left(\frac{\tau}{\kappa^2+\tau^2}\right)'\frac{\left(\kappa^2+\tau^2\right)}{\tau}.
\end{aligned}$$

□

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

References

[1] Ergüt M., Körpınar T. and Turhan E., *On Normal Ruled Surfaces of General Helices In The Sol Space* Sol^3, TWMS J. Pure Appl. Math., **4 (2)** (2013), 125–130.
[2] Hacisalihoğlu H.H., *Diferensiyel Geometri*, **1** , Inönü Üniversity Publications (1994).
[3] Liu H. and Wang F., *Mannheim partner curves in 3-space*, Journal of Geometry, **88 (1-2)** (2008), 120–126(7).
[4] Orbay K. and Kasap E., *On Mannheim partner curves*, International Journal of Physical Sciences, **4 (5)**(2009), 261–264.
[5] Lipschutz M.M., *Differential Geometry*, Schaum's Outlines, 1969.
[6] Schief W.K., *On the integrability of Bertrand curves and Razzaboni surfaces* , Journal of Geometry and Physics, **45 (1-2)** (2003), 130–150.

Süleyman Şenyurt
Faculty of Arts and Sciences, Department of Mathematics, Ordu University, Ordu, Turkey
E-mail address: senyurtsuleyman@hotmail.com

Şeyda Kiliçoğlu
Faculty of Education, Department of Mathematics, Başkent University, Ankara, Turkey
E-mail address: seyda@baskent.edu.tr