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THE MAXIMAL EXCEPTIONAL GRAPHS WITH MAXIMAL DEGREE LESS THAN 28

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A b s t r a c t. A graph is said to be exceptional *if it is connected, has least eigenvalue greater than or equal to* -2*, and is not a generalized line graph. Such graphs are known to be representable in the root system* E_8*. The 473 maximal exceptional graphs were found initially by computer, and the 467 with maximal degree* 28 *have subsequently been characterized. Here we use constructions in* E_8 *to prove directly that there are just six maximal exceptional graphs with maximal degree less than 28.*

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1. *Introduction*

A graph is said to be *exceptional* if it is connected, has least eigenvalue greater than or equal to -2, and is not a generalized line graph. Generalized line graphs have been studied in [9, 14], while exceptional graphs first appeared in the context of spectral characterizations of certain classes of line graphs by A. J. Hoffman and others in the 1960s (see, for example, [12, pp.

12-14]). The key paper [5] introduced root systems as a means of investigating graphs with least eigenvalue -2; in particular it was shown by this technique that an exceptional graph has at most 36 vertices and each vertex has degree at most 28. The regular exceptional graphs, 187 in number, were found in [2, 3], but the problem of finding a suitable description of all the exceptional graphs remained open. Much information on these topics can be found in the monographs [1, 6, 8] and in the expository paper [4]. We described in [11] the results of an exhaustive computer search for the exceptional graphs which are maximal in the sense that every exceptional graph is an induced subgraph of (at least) one such graph. These graphs, 473 in all, were found as maximal extensions of appropriate star complements (cf. [10, 13, 14, 17] and below). An independent means of constructing those with maximal degree 28 was included in [11]: the crucial property is that the neighbours of a vertex of degree 28 induce a subgraph which is switching-equivalent to the line graph $L(K_8)$ and hence determined by a 2-colouring of the edges of K_8. In [15] we used a variant of this approach to obtain various constructions for the maximal exceptional graphs with maximal degree less than 28, but the number of isomorphism classes was not verified. Here we determine these 'exceptional maximal exceptional graphs' directly from the root system E_8 and prove that there are precisely six of them. They are necessarily the graphs labelled $M001, M002, M417, M428, M437, M462$ in [11], and definitions of them appear below. The graph $M001$ was identified in [16] as a non-regular graph with just three distinct eigenvalues.

It is well known that an exceptional graph G is representable in the root system E_8 (see [6, Chapter 3] or [1, Chapter 3]). This means that if G has A as a $(0,1)$-adjacency matrix then $I + \frac{1}{2}A$ is the Gram matrix of a set of normalized vectors in E_8; explicitly, if $\{\mathbf{e}_1, \ldots, \mathbf{e}_8\}$ is an orthonormal basis for $\mathbb{R}^8$ then $8I + 4A$ is the Gram matrix of a subset of the following set of 240 vectors (cf. [2, 11]):

type a: 28 vectors of the form $\mathbf{a}_{ij} = 2\mathbf{e}_i + 2\mathbf{e}_j;\ \ i,j = 1,\ldots,8,\ \ i<j;$

type a′: 28 vectors opposite to those of type a;

type b: 28 vectors of the form $\mathbf{b}_{ij} = -2\mathbf{e}_i - 2\mathbf{e}_j + \sum_{k=1}^{8}\ \mathbf{e}_k;$

type b′: 28 vectors opposite to those of type b;

type c: 56 vectors of the form $\mathbf{c}_{ij} = 2\mathbf{e}_i - 2\mathbf{e}_j;\ \ i,j = 1,\ldots,8,\ \ i \neq j;$

type d: 70 vectors of the form $\mathbf{d}_{ijkl} = -2\mathbf{e}_i - 2\mathbf{e}_j - 2\mathbf{e}_k - 2\mathbf{e}_l + \sum_{s=1}^{8}\ \mathbf{e}_s$ with distinct $i,j,k,l \in \{1,\ldots,8\}$;

type e: 2 vectors $\mathbf{e}$ and $-\mathbf{e}$, where $\mathbf{e} = \ \sum_{i=1}^{8} \mathbf{e}_i$.

These 240 vectors determine 120 lines at $60°$ or $90°$. Let Γ denote the graph which has these lines as vertices, with two vertices adjacent if and only if the corresponding lines are orthogonal. We recall from [7, p.85] some properties of the automorphism group of Γ (communicated by P. J. Cameron). This group has as a subgroup of index 2 the orthogonal group $O^+(8,2)$, which is transitive on the vertices of Γ. Moreover the stabilizer of a vertex v acts as a rank 3 group on the subgraph $\Gamma(v)$ induced by the neighbours of v; in particular, the stabilizer of v is edge-transitive on $\Gamma(v)$.

By a *representation* of the exceptional graph G we mean a subset $\mathcal{R}(G)$ of E_8 whose Gram matrix is a scalar multiple of $8I + 4A$, where A is the adjacency matrix of G. Note that if $\mathcal{R}(G)$ is a representation of G then so is $-\mathcal{R}(G) = \{-\mathbf{u} : \mathbf{u} \in \mathcal{R}(G)\}$. In view of the transitivity of $\mathrm{Aut}(\Gamma)$, we can therefore assume that $\mathbf{e}$ represents a vertex of maximal degree, and in this case we call $\mathcal{R}(G)$ a *standard* representation. Note that then no vector of type a', b' features in $\mathcal{R}(G)$; moreover a second standard representation is given by $\phi(\mathcal{R}(G))$ where the involutory map ϕ is defined by: $\phi(e) = e, \phi(\mathbf{a}_{ij}) = \mathbf{b}_{ij}, \phi(\mathbf{b}_{ij}) = \mathbf{a}_{ij}, \phi(\mathbf{c}_{ij}) = \mathbf{c}_{ji}(-\mathbf{c}_{ij}), \phi(\mathbf{d}_{ijkl}) = \mathbf{d}_{\overline{ijkl}}(= -\mathbf{d}_{ijkl})$. We refer to $\mathcal{R}(G)$ and $\phi(\mathcal{R}(G))$ as *dual* representations. (Accordingly we may assume if necessary that the number of vectors of type b in $\mathcal{R}(G)$ does not exceed the number of vectors of type a.) We give standard representations of the graphs $M001, M002, M417, M428, M437$ and $M462$:

- $M001$ (22 vertices, with degrees $16^{14}, 7^8$; the vertices of degree 16 induce the cocktail-party graph $\overline{7K_2}$, while those of degree 7 form a coclique)
  $\mathbf{a}_{ij}(ij = 12, 13, 14, 15, 23, 24, 26, 34, 37, 48)$;
  $\mathbf{b}_{ij}(ij = 56, 57, 58, 67, 68, 78)$;
  $\mathbf{c}_{ij}(ij = 15, 26, 37, 48)$; $\mathbf{d}_{5678}$; $\mathbf{e}$.

- $M002$ (28 vertices, with degrees $22^7, 16^{14}, 10^7$; the vertices of degree 10 form a coclique)
  $\mathbf{a}_{ij}(ij = 12, 13, 14, 17, 18, 23, 25, 27, 28, 36, 37, 38, 78)$;
  $\mathbf{b}_{ij}(ij = 45, 46, 47, 48, 56, 57, 58, 67, 68)$;
  $\mathbf{c}_{ij}(ij = 14, 25, 36)$; $\mathbf{d}_{4567}$, $\mathbf{d}_{4568}$; $\mathbf{e}$.

- $M417$ (29 vertices, with degrees $26^1, 24^2, 18^{16}, 12^8, 10^2$)
  $\mathbf{a}_{ij}(ij = 12, 15, 16, 17, 18, 25, 26, 27, 28, 57, 68)$;
  $\mathbf{b}_{ij}(ij = 13, 24, 34, 35, 36, 37, 38, 45, 46, 47, 48, 56, 58, 67, 78)$;
  $\mathbf{c}_{13}, \mathbf{c}_{24}$; $\mathbf{e}$.

- $M428$ (29 vertices, with degrees $26^2, 22^1, 18^{16}, 14^6, 10^4$)
  $\mathbf{a}_{ij}(ij = 12, 15, 16, 17, 18, 25, 26, 27, 28)$;

$\mathbf{b}_{ij}(ij = 13, 24, 34, 35, 36, 37, 38, 45, 46, 47, 48, 56, 57, 58, 67, 68, 78)$;
$\mathbf{c}_{13}, \mathbf{c}_{24}$; $\mathbf{e}$.

- $M437$ (30 vertices, with degrees $26^2, 24^1, 20^8, 17^8, 16^1, 14^2, 13^4, 11^4$)
  $\mathbf{a}_{ij}(ij = 12, 15, 16, 17, 18, 25, 26, 27, 28, 56)$;
  $\mathbf{b}_{ij}(ij = 13, 24, 34, 35, 36, 37, 38, 45, 46, 47, 48, 57, 58, 67, 68, 78)$;
  $\mathbf{c}_{13}$, $\mathbf{c}_{24}$; $\mathbf{d}_{3478}$; $\mathbf{e}$.

- $M462$ (31 vertices, with degrees $26^3, 22^4, 19^8, 16^4, 15^6, 12^6$)
  $\mathbf{a}_{ij}(ij = 12, 15, 16, 17, 18, 25, 26, 27, 28, 56, 67)$;
  $\mathbf{b}_{ij}(ij = 13, 24, 34, 35, 36, 37, 38, 45, 46, 47, 48, 57, 58, 68, 78)$;
  $\mathbf{c}_{13}$, $\mathbf{c}_{24}$; $\mathbf{d}_{3458}$, $\mathbf{d}_{3478}$; $\mathbf{e}$.

These standard representations are not unique; indeed others arise in the course of our constructions, and in such cases we specify an isomorphism with one of the above graphs. The isomorphisms are found by means of star complements, as we now explain. We write $V(G)$ for the set of vertices of the graph G, and $\Delta(v)$ for the set of neighbours of the vertex v. Further, if H is a subgraph of G then $\Delta_H(v) = \Delta(v) \cap V(H)$.

Recall that if μ is an eigenvalue of G with multiplicity k, then a star complement for μ is an induced subgraph $H = G - X(X \subseteq V(G))$ such that $|X| = k$ and μ is not an eigenvalue of $G - X$. If $\mu \notin \{-1, 0\}$ then the H-neigbourhoods $\Delta_H(v)(v \notin V(H))$ are distinct [12, Corollary 7.3.6]. Now let G, G' be graphs with H, H' respectively as star complements for μ, where $\mu \neq -1, 0$. If ψ is an isomorphism $H \to H'$ such that ψ maps the neigbourhoods $\Delta_H(v)(v \notin V(H))$ onto the neighbourhoods $\Delta_{H'}(v')(v' \notin V(H'))$ then by the Reconstruction Theorem [12, Theorem 7.4.1], ψ extends to an isomorphism $G \to G'$, defined outside H by $\psi(\Delta_H(v)) = \Delta_{H'}(\psi(v))$. For the six graphs above, the isomorphisms required in Section 4 are constructed using star complements for -2 isomorphic to $\overline{K_{1,2} \cup 5K_1}$, the graph labelled $E443$ in [11].

In a standard representation $\mathcal{R}(G)$ of an exceptional graph G, the following are the pairs of vectors which are incompatible because they have inner product -4.

(CC) $\mathbf{c}_{ij}$ and $\mathbf{c}_{jk}$,

(DD) $\mathbf{d}_{ijkl}$ and $\mathbf{d}_{i'j'k'l'}$ whenever $|\{i, j, k, l\} \cap \{i', j', k', l'\}| \leq 1$,

(AB) $\mathbf{a}_{ij}$ and $\mathbf{b}_{ij}$,

(AC) $\mathbf{a}_{ij}$ and $\mathbf{c}_{hj}(h \neq i, j)$, $\mathbf{a}_{ij}$ and $\mathbf{c}_{hi}(h \neq i, j)$, (AD) $\mathbf{a}_{uv}$ and $\mathbf{d}_{ijkl}$ whenever $\{u,v\} \subseteq \{i,j,k,l\}$,

(BC) $\mathbf{b}_{ij}$ and $\mathbf{c}_{ik}(k \neq i,j)$, $\mathbf{b}_{ij}$ and $\mathbf{c}_{jk}(k \neq i,j)$,

(BD) $\mathbf{b}_{uv}$ and $\mathbf{d}_{ijkl}$ whenever $\{u,v\} \cap \{i,j,k,l\} = \emptyset$,

(CD) $\mathbf{c}_{uv}$ and $\mathbf{d}_{ijkl}$ whenever $\{u,v\} \cap \{i,j,k,l\} = \{u\}$.

The following consequence is a reformulation of [11, Theorem 3.6].

Lemma 1.1. *If for some pair i,j the vectors $\mathbf{a}_{ij}$ and $\mathbf{b}_{ij}$ are absent from a standard representation $\mathcal{R}(G)$ of a maximal exceptional graph G then $\mathcal{R}(G)$ includes vectors $\mathbf{v}$ and $\mathbf{w}$ such that $\mathbf{e},\mathbf{v},\mathbf{w}$ are pairwise orthogonal.*

P r o o f. By the maximality of G, $\mathbf{a}_{ij}$ and $\mathbf{b}_{ij}$ are excluded by the presence of certain vectors, which in view of the complete list of incompatibilities above, are of type c or d. Now the vectors of type c or d which exclude $\mathbf{a}_{ij}$ are those in the set A_{ij} comprising $\mathbf{c}_{hi}(h \neq i,j)$, $\mathbf{c}_{hj}(h \neq i,j)$ and the vectors $\mathbf{d}_{pqrs}$ for which $\{i,j\} \subseteq \{p,q,r,s\}$. Those which exclude $\mathbf{b}_{ij}$ are those in the set B_{ij} comprising $\mathbf{c}_{ik}(k \neq i,j)$, $\mathbf{c}_{jk}(k \neq i,j)$ and the vectors $\mathbf{d}_{pqrs}$ for which $\{i,j\} \cap \{p,q,r,s\} = \emptyset$. Note that the inner product of any vector in A_{ij} with any vector in B_{ij} is non-positive; in particular, two orthogonal vectors $\mathbf{v}$ and $\mathbf{w}$, each of type c or d, must be present. Since these vectors are orthogonal to $\mathbf{e}$ the Lemma is proved. □

Henceforth we consider a standard representation $\mathcal{R}(G)$ of a maximal exceptional graph G with maximal degree less than 28.

In $\mathrm{Aut}(\Gamma)$, the stabilizer of the line $\langle \mathbf{e} \rangle$ is edge-transitive on the subgraph induced by the neighbours of $\langle \mathbf{e} \rangle$ and so in view of Lemma 1.1 we may assume that $\mathcal{R}(G)$ contains two orthogonal vectors $\mathbf{v},\mathbf{w}$ of type c. (Alternative representations, in which at least one of the vectors $\mathbf{v},\mathbf{w}$ is of type d, will be described elsewhere.) Let θ be the maximum number of pairwise orthogonal vectors of type c in $\mathcal{R}(G)$, and note that $2 \leq \theta \leq 4$. We analyze the cases $\theta = 4,3,2$ in Sections 2,3,4, respectively. When $\theta = 4$ we find that G is $M001$; when $\theta = 3$ we find that G is $M002$; and when $\theta = 2$ we find that G is one of $M001, M002, M417, M428, M437, M462$. We may summarize the results as follows.

Main Theorem. *If G is a maximal exceptional graph in which every vertex has degree less than 28 then G is isomorphic to one of $M001, M002$, $M417, M428, M437$ and $M462$.*

In the sequel we identify vertices of G with corresponding vectors in

$\mathcal{R}(G)$.

2. *The case* $\theta = 4$

Without loss of generality, $\mathcal{R}(G)$ contains the vectors $\mathbf{c}_{15}, \mathbf{c}_{26}, \mathbf{c}_{37}, \mathbf{c}_{48}$.

In view of the incompatibilities (AC), (BC), (CC) the further possible vectors of types a, b, c in $\mathcal{R}(G)$ are

$$\mathbf{a}_{ij}(ij = 12, 13, 14, 23, 24, 34, 15, 26, 37, 48);$$

$$\mathbf{b}_{ij}(ij = 15, 26, 37, 48, 56, 57, 58, 67, 68, 78);$$

and

$$\mathbf{c}_{ij}(ij = 16, 17, 18, 25, 27, 28, 35, 36, 38, 45, 46, 47).$$

Moreover, if $\mathbf{d}_{ijkl}$ is present then $|\{i, j, k, l\} \cap \{5, 6, 7, 8\}| \geq 2$. (For example, neither $\mathbf{d}_{1234}$ nor $\mathbf{d}_{2345}$ is compatible with $\mathbf{c}_{26}$.) It follows that $\mathbf{d}_{5678}$ is compatible with all possible vectors, hence is present by maximality. Now $\mathbf{d}_{5678}$ is adjacent to each of $\mathbf{c}_{15}, \mathbf{c}_{26}, \mathbf{c}_{37}, \mathbf{c}_{48}$, and is adjacent to all possible neighbours of $\mathbf{e}$ except $\mathbf{a}_{ij}$ and $\mathbf{b}_{ij}(ij = 15, 26, 37, 48)$.

Recall now that $\deg(e) \geq \deg(\mathbf{d}_{5678})$, while for given ij at most one of $\mathbf{a}_{ij}, \mathbf{b}_{ij}$ is present. It follows that (i) $\deg(\mathbf{e}) = \deg(\mathbf{d}_{5678})$; (ii) one of $\mathbf{a}_{ij}, \mathbf{b}_{ij}$ is present for each $ij = 15, 26, 37, 48$; (iii) no further vectors of type c are present (for any such vector would be adjacent to $\mathbf{d}_{5678}$); (iv) similarly, if another vector $\mathbf{d}_{ijkl}$ is present then $|\{i, j, k, l\} \cap \{5, 6, 7, 8\}| = 2$.

It follows from (iv) by (CD) that the only possible vectors of type d are $\mathbf{d}_{ijkl}$ for $ijkl = 1256, 1357, 1458, 2367, 2468, 3478$.

Next we show that either all $\mathbf{a}_{ij}(ij = 15, 26, 37, 48)$ are present or all $\mathbf{b}_{ij}(ij = 15, 26, 37, 48)$ are present. Without loss of generality, suppose by way of contradiction that $\mathbf{a}_{15}$ and $\mathbf{b}_{26}$ are present. Then the vectors $\mathbf{d}_{ijkl}(ijkl = 1256, 1357, 1458, 3478)$ are excluded and $\mathbf{d}_{2678}$ is compatible with all of the possible vectors remaining; but then by maximality $\mathbf{d}_{2678}$ is present, a contradiction.

The presence of $\mathbf{a}_{ij}(ij = 15, 26, 37, 48)$ or $\mathbf{b}_{ij}(ij = 15, 26, 37, 48)$ now excludes all possible vectors of type d other than $\mathbf{d}_{5678}$, and so there remain just two possible maximal sets of 22 pairwise compatible vectors. By duality we may assume that the number of vectors of type b does not exceed the number of vectors of type a. Accordingly just one graph arises, namely the graph $M001$ defined in Section 1.

3. *The case* $\theta = 3$

Without loss of generality suppose that $\{\mathbf{c}_{14}, \mathbf{c}_{25}, \mathbf{c}_{36}\}$ is a largest set of pairwise orthogonal vectors of type c. In view of the incompatibilities (AC), (BC), (CC) the further possible vectors of types a, b, c in $\mathcal{R}(G)$ are

$$\mathbf{a}_{ij}(ij = 12, 13, 17, 18, 23, 27, 28, 37, 38, 78, 14, 25, 36);$$

$$\mathbf{b}_{ij}(ij = 45, 46, 47, 48, 56, 57, 58, 67, 68, 78, 14, 25, 36);$$

and

$$\mathbf{c}_{ij}(ij = 15, 16, 17, 18, 24, 26, 27, 28, 34, 35, 37, 38, 74, 75, 76, 84, 85, 86).$$

Moreover, if $\mathbf{d}_{ijkl}$ is present then by (CD) either $|\{i, j, k, l\} \cap \{4, 5, 6\}| \geq 2$ or $ijkl \in \{1478, 2578, 3678\}$.

Now the compatible vectors $\mathbf{d}_{4567}, \mathbf{d}_{4568}$ are compatible with all possible vectors, and are therefore present by maximality.

We show next that the vectors $\mathbf{a}_{12}, \mathbf{a}_{13}, \mathbf{a}_{23}$ and $\mathbf{b}_{45}, \mathbf{b}_{46}, \mathbf{b}_{56}$ are all present. If $\mathbf{a}_{12}$ is absent it must be excluded by $\mathbf{d}_{1245}$, and if $\mathbf{b}_{45}$ is absent it must be excluded by $\mathbf{d}_{3678}$. (The reasons are that $\mathbf{a}_{12}, \mathbf{b}_{45}$ are compatible with all possible vectors of type c, while any vector of type d which is present must be compatible with $\mathbf{c}_{14}$ and $\mathbf{c}_{25}$.) If $\mathbf{d}_{1245}$ is present then $\mathbf{d}_{3678}$ is absent and so $\mathbf{a}_{45}$ is present; but then $\deg(\mathbf{b}_{45}) > \deg(\mathbf{e})$ since $\mathbf{a}_{14}, \mathbf{a}_{25}, \mathbf{b}_{67}, \mathbf{b}_{68}, \mathbf{b}_{78}, \mathbf{b}_{36}$ are excluded. Similarly, if $\mathbf{d}_{3678}$ is present then $\mathbf{b}_{12}$ is present and $\deg(\mathbf{a}_{12}) > \deg(\mathbf{e})$. In either case we have a contradiction and so $\mathbf{a}_{12}, \mathbf{b}_{45}$ are present. Similarly, $\mathbf{a}_{13}, \mathbf{b}_{46}$ are present and $\mathbf{a}_{23}, \mathbf{b}_{56}$ are present.

Let

$$S = \{\mathbf{a}_{ij} : ij = 37, 38, 78, 14, 25, 36\} \cup \{\mathbf{b}_{ij} : ij = 67, 68, 78, 14, 25, 36\},$$

$$T = \{\mathbf{a}_{ij} : ij = 13, 17, 18, 23, 27, 28\} \cup \{\mathbf{b}_{ij} : ij = 46, 47, 48, 56, 57, 58\},$$

and let α be the number of adjacencies between $\{\mathbf{a}_{12}, \mathbf{b}_{45}\}$ and vectors of type c or d.

Note that the elements of $T \cap \Delta(\mathbf{e})$, together with $\mathbf{e}, \mathbf{c}_{14}, \mathbf{c}_{25}, \mathbf{d}_{4567}, \mathbf{d}_{4568}$, are adjacent to both $\mathbf{a}_{12}$ and $\mathbf{b}_{45}$; while those of $S \cap \Delta(\mathbf{e})$, together with $\{\mathbf{a}_{12}, \mathbf{b}_{45}\}$, are adjacent to just one of $\mathbf{a}_{12}$, $\mathbf{b}_{45}$. It follows that

$$\deg(\mathbf{a}_{12}) + \deg(\mathbf{b}_{45}) = |S \cap \Delta(\mathbf{e})| + 2|T \cap \Delta(\mathbf{e})| + 4 + \alpha.$$

Now

$$2\deg(\mathbf{e}) = 2|S \cap \Delta(\mathbf{e})| + 2|T \cap \Delta(\mathbf{e})| + 4 \geq \deg(\mathbf{a}_{12}) + \deg(\mathbf{b}_{45})$$

and so $|S \cap \Delta(\mathbf{e})| \geq \alpha$. On the other hand, $\alpha \geq 8$ and $|S \cap \Delta(\mathbf{e})| \leq 8$, whence $|S \cap \Delta(\mathbf{e})| = \alpha = 8$.

It follows that (i) $\deg(\mathbf{e}) = \deg(\mathbf{a}_{12}) = \deg(\mathbf{b}_{45})$; (ii) $\mathbf{a}_{37}, \mathbf{a}_{38}, \mathbf{b}_{67}, \mathbf{b}_{68}$ are present, and either $\mathbf{a}_{ij}$ or $\mathbf{b}_{ij}$ is present for each $ij = 14, 25, 36, 78$.

If both $\mathbf{a}_{14}$ and $\mathbf{a}_{25}$ are present then so are $\mathbf{a}_{36}$ and $\mathbf{a}_{78}$, because $\deg(\mathbf{e}) = \deg(\mathbf{a}_{12}) = \deg(\mathbf{b}_{45})$. Similarly, if both $\mathbf{b}_{14}$ and $\mathbf{b}_{25}$ are present then so are $\mathbf{b}_{36}$ and $\mathbf{b}_{78}$. Identical arguments hold when we apply the permutation (123)(456) to subscripts, and we conclude that either $\mathbf{a}_{ij}(ij = 14, 25, 36, 78)$ are present or $\mathbf{b}_{ij}(ij = 14, 25, 36, 78)$ are present. Moreover all of $\mathbf{a}_{12}, \mathbf{a}_{13}, \mathbf{a}_{23}, \mathbf{b}_{45}, \mathbf{b}_{46}, \mathbf{b}_{56}$ have the same degree as $\mathbf{e}$. It follows that there are no further vectors of type c, and no vectors of type d other than $\mathbf{d}_{4567}, \mathbf{d}_{4568}$. The 28 vectors which remain are $\mathbf{a}_{12}, \mathbf{b}_{45}, \mathbf{c}_{14}, \mathbf{c}_{25}, \mathbf{c}_{36}, \mathbf{d}_{4567}, \mathbf{d}_{4568}, \mathbf{e}$, the 8 vectors in $S \cap \Delta(\mathbf{e})$ and the 12 vectors in T. By duality we may assume that the number of vectors of type b does not exceed the number of vectors of type a. Accordingly just one graph arises, namely the graph $M002$ defined in Section 1.

4. *The case* $\theta = 2$

Without loss of generality, suppose that $\{\mathbf{c}_{13}, \mathbf{c}_{24}\}$ is a largest set of pairwise orthogonal vectors of type c. In view of the incompatibilities (AC), (BC), (CC) the further possible vectors of types a, b, c in $\mathcal{R}(G)$ are

$$\mathbf{a}_{ij}(ij = 13, 24; 12; 15, 16, 17, 18, 25, 26, 27, 28; 56, 57, 58, 67, 68, 78);$$

$$\mathbf{b}_{ij}(ij = 13, 24; 34; 35, 36, 37, 38, 45, 46, 47, 48; 56, 57, 58, 67, 68, 78);$$

and

$$\mathbf{c}_{ij}(ij = 14, 15, 16, 17, 18, 23, 25, 26, 27, 28, 53, 54, 63, 64, 73, 74, 83, 84).$$

Lemma 4.1 *The vectors* $\mathbf{a}_{12}$ *and* $\mathbf{b}_{34}$ *are present.*

P r o o f. If $\mathbf{a}_{12}$ is absent then it is excluded by a vector of type d compatible with $\mathbf{c}_{13}$ and $\mathbf{c}_{24}$, and this is necessarily $\mathbf{d}_{1234}$. Similarly, if $\mathbf{b}_{34}$ is absent then it is excluded by $\mathbf{d}_{5678}$. Since $\mathbf{d}_{1234}$ and $\mathbf{d}_{5678}$ are incompatible at least one of $\mathbf{a}_{12}, \mathbf{b}_{34}$ is present. If only $\mathbf{a}_{12}$ is present then $\mathbf{d}_{5678}$ is present and so the further possible vectors of type a or b are:

$$\mathbf{a}_{ij}(ij = 13, 24, 12, 15, 16, 17, 18, 26, 26, 27, 28)$$

and

$$\mathbf{b}_{ij}(ij = 35, 36, 37, 38, 45, 46, 47, 48; 56, 57, 58, 67, 68, 78).$$

Thus $\mathbf{a}_{12}$ is adjacent to all other neighbours of $\mathbf{e}$, as well as to $\mathbf{d}_{5678}$ and $\mathbf{c}_{13}$. Then $\deg(\mathbf{a}_{12}) > \deg(\mathbf{e})$, a contradiction. If only $\mathbf{b}_{34}$ is present then we obtain similarly the contradiction $\deg(\mathbf{b}_{34}) > \deg(\mathbf{e})$. Consequently both $\mathbf{a}_{12}$ and $\mathbf{b}_{34}$ are present. □

Let us now introduce some more notation: γ_1 (resp. γ_2) is the number of vectors of type c adjacent to one (resp. both) of $\mathbf{a}_{12}, \mathbf{b}_{34}$, while δ_1 (resp. δ_2) is the number of vectors of type d adjacent to one (resp. both) of $\mathbf{a}_{12}, \mathbf{b}_{34}$. Note that $\gamma_2 \geq 2$. Let

$$S=\{\mathbf{a}_{ij}\colon ij = 13, 24, 56, 57, 58, 67, 68, 78\} \cup \{\mathbf{b}_{ij}\colon ij = 13, 24, 56, 57, 58, 67, 68, 78\},$$

$$T=\{\mathbf{a}_{ij}\colon ij = 15, 16, 17, 18, 25, 26, 27, 28\} \cup \{\mathbf{b}_{ij} : ij = 35, 36, 37, 38, 45, 46, 47, 48\}.$$

Lemma 4.2 *With the above notation, the following holds:*

$$\gamma_1 + 2\gamma_2 + \delta_1 + 2\delta_2 \leq |S \cap \Delta(\mathbf{e})|. \tag{1}$$

Proof. We have

$$\deg(\mathbf{a}_{12}) + \deg(\mathbf{b}_{45}) = 4 + 2|S \cap \Delta(\mathbf{e})| + |T \cap \Delta(\mathbf{e})| + \gamma_1 + 2\gamma_2 + \delta_1 + 2\delta_2$$

and

$$\deg(\mathbf{e}) = 2 + |S \cap \Delta(\mathbf{e})| + |T \cap \Delta(\mathbf{e})|.$$

The lemma follows because $\deg(\mathbf{a}_{12}) + \deg(\mathbf{b}_{45}) \leq 2\deg(\mathbf{e})$. □

Lemma 4.3 *At most one of the vectors* $\mathbf{c}_{14}$ *and* $\mathbf{c}_{23}$ *is present.*

P r o o f. If both $\mathbf{c}_{14}$ and $\mathbf{c}_{23}$ are present then $\delta_2 \geq 4$ and so $|S \cap \Delta(\mathbf{e})| \geq 8$ by Lemma 4.2. On the other hand, $\mathbf{a}_{24}$ and $\mathbf{b}_{13}$ are excluded by $\mathbf{c}_{14}$, while $\mathbf{b}_{24}$ and $\mathbf{a}_{13}$ are excluded by $\mathbf{c}_{23}$. Thus $|S \cap \Delta(\mathbf{e})| \leq 6$, a contradiction. □

The next three lemmas are symmetric in 5,6,7,8.

Lemma 4.4 *If* $\mathbf{a}_{78}$ *and* $\mathbf{b}_{56}$ *are absent then either (a)* $\mathbf{d}_{3478}$ *is present* *or (b)* $\mathbf{b}_{78}$ *and* $\mathbf{a}_{56}$ *are absent. In particular, if* $\mathbf{b}_{78}$ *and* $\mathbf{a}_{56}$ *are present then* $\mathbf{d}_{3478}$ *is present.*

P r o o f. The vector $\mathbf{d}_{3478}$ is compatible with all remaining possible vectors of type a, b or c. Accordingly if (a) does not hold then $\mathbf{d}_{3478}$ is excluded by some vector $\mathbf{d}_{ijkl}$. Since $\mathbf{b}_{34}$ is present (by Lemma 4.1), it follows from (DD) and (BD) that $\{i,j,k,l\} \cap \{3,4,7,8\}$ is $\{3\}$ or $\{4\}$. In the former case, $ijkl = 1356$ since $\mathbf{c}_{24}$ excludes $\mathbf{d}_{2356}$: and in the latter case, $ijkl = 2456$ since $\mathbf{c}_{13}$ excludes $\mathbf{d}_{1456}$. In both cases, $\mathbf{b}_{78}$ and $\mathbf{a}_{56}$ are excluded. $\square$

Note that the assertions of Lemmas 4.1 to 4.4 remain true of $\phi(\mathcal{R}(G))$ when we apply the permutation (13)(24) to subscripts, and this justifies the duality arguments used in the sequel.

Lemma 4.5 *If* $\mathbf{a}_{56}$ *and* $\mathbf{b}_{56}$ *are absent then there exist vectors* $\mathbf{d}_{ijkl}$ *and* $\mathbf{d}_{i'j'k'l'}$ *with* $\{5,6\} \subseteq \{i,j,k,l\}$ *and* $\{5,6\} \cap \{i',j',k',l'\} = \emptyset$.

P r o o f. The vector $\mathbf{a}_{56}$ can be excluded by $\mathbf{c}_{15}, \mathbf{c}_{16}, \mathbf{c}_{25}, \mathbf{c}_{26}$ or $\mathbf{d}_{ijkl}$ where $\{5,6\} \subseteq \{i,j,k,l\}$; and $\mathbf{b}_{56}$ can be excluded by $\mathbf{c}_{53}, \mathbf{c}_{54}, \mathbf{c}_{63}, \mathbf{c}_{64}$ or $\mathbf{d}_{i'j'k'l'}$ where $\{5,6\} \cap \{i',j',k',l'\} = \emptyset$. By duality it suffices to exclude two possibilities: (i) each of $\mathbf{a}_{56}, \mathbf{b}_{56}$ is excluded by a vector of type c, (ii) $\mathbf{a}_{56}$ is excluded by a vector of type c and $\mathbf{b}_{56}$ is excluded by a vector of type d.

In case (i) we may assume without loss of generality first that $\mathbf{a}_{56}$ is excluded by $\mathbf{c}_{15}$, and then that $\mathbf{b}_{56}$ is excluded by $\mathbf{c}_{63}$ (since $\mathbf{c}_{15}$ excludes $\mathbf{c}_{53}$ and $\mathbf{c}_{54}$). In view of (AC) and (BC) the possible vectors in $S \cap \Delta(\mathbf{e})$ are $\mathbf{a}_{ij} (ij = 67, 68, 78; 13, 24)$ and $\mathbf{b}_{ij} (ij = 57, 58, 78; 13, 24)$. Note that by (AB), $|S \cap \Delta(\mathbf{e})| \leq 7$. Since $\gamma_1 \geq 2$ and $\gamma_2 \geq 2$ we have $|S \cap \Delta(\mathbf{e})| \geq 6$ by Lemma 4.2. Hence at most one of the vectors $\mathbf{b}_{57}, \mathbf{b}_{58}, \mathbf{a}_{67}, \mathbf{a}_{68}$ is absent. Wihtout loss of generality, $\mathbf{a}_{67}$ and $\mathbf{b}_{58}$ are present. By Lemma 4.4, $\mathbf{d}_{3467}$ is present, and so $\delta_2 \geq 1$. Now Lemma 4.2 yields the contradiction $|S \cap \Delta(\mathbf{e})| \geq 8$.

In case (ii) we may suppose without loss of generality that $\mathbf{a}_{56}$ is excluded by $\mathbf{c}_{15}$ and $\mathbf{b}_{56}$ is excluded by $\mathbf{d}_{i'j'k'l'}$. This last vector must be compatible with $\mathbf{c}_{13}, \mathbf{c}_{24}, \mathbf{c}_{15}$, and hence is one of $\mathbf{d}_{3478}, \mathbf{d}_{2478}, \mathbf{d}_{2347}, \mathbf{d}_{2348}$.

Since $\mathbf{a}_{12}$ is adjacent to $\mathbf{e}, \mathbf{b}_{34}, \mathbf{c}_{13}, \mathbf{c}_{24}$ and $\mathbf{c}_{15}$, we know that

$$\deg(\mathbf{a}_{12}) \geq 5 + |S \cap \Delta(\mathbf{a}_{12})| + |T \cap \Delta(\mathbf{e})|,$$

while

$$\deg(\mathbf{e}) = 2 + |S \cap \Delta(\mathbf{a}_{12})| + |S \cap \Delta(\mathbf{b}_{34})| + |T \cap \Delta(\mathbf{e})|.$$

Since $\deg(\mathbf{e}) \geq \deg(\mathbf{a}_{12})$, it follows that $|S \cap \Delta(\mathbf{b}_{34})| \geq 3$. Since $S \cap \Delta(\mathbf{b}_{34}) \subseteq \{\mathbf{b}_{24}, \mathbf{a}_{67}, \mathbf{a}_{68}, \mathbf{a}_{78}\}$ we conclude that not both $\mathbf{b}_{67}$ and $\mathbf{b}_{68}$ are present. If say $\mathbf{b}_{67}$ is absent then we may apply Lemma 4.4 to $\mathbf{a}_{58}, \mathbf{b}_{67}$ to deduce that either (a) $\mathbf{d}_{3458}$ is present or (b) $\mathbf{b}_{58}, \mathbf{a}_{67}$ are absent.

In subcase (a), $|S \cap \Delta(\mathbf{e})| = 7$ and either $\mathbf{a}_{58}, \mathbf{b}_{57}$ are present or $\mathbf{a}_{68}, \mathbf{b}_{57}$ are present. By Lemma 4.4 either $\mathbf{d}_{3467}$ or $\mathbf{d}_{3468}$ is present, and so $\delta_2 \geq 2$. By Lemma 4.2, $|S \cap \Delta(\mathbf{e})| \geq 8$, a contradiction. In subcase (b), $|S \cap \Delta(\mathbf{e})| = 5$, $S \cap \Delta(\mathbf{b}_{34}) = \{\mathbf{b}_{24}, \mathbf{a}_{68}, \mathbf{a}_{78}\}$, $\delta_1 = 0$ and $\delta_2 = 0$. Then $\mathbf{d}_{i'j'k'l'} = \mathbf{d}_{2478}$, a contradiction because this vector is not compatible with $\mathbf{a}_{78}$. $\square$

Lemma 4.6 *If* $\mathbf{a}_{56}$ *and* $\mathbf{b}_{56}$ *are absent then so are* $\mathbf{a}_{78}$ *and* $\mathbf{b}_{78}$*.*

P r o o f. We suppose that the conclusion does not hold, and obtain a contradiction. By Lemma 4.4, either $\mathbf{a}_{78}$ and $\mathbf{d}_{3456}$ are present or $\mathbf{b}_{78}$ and $\mathbf{d}_{3478}$ are present. By duality, we may assume that the former is the case. By Lemma 4.5, a vector $\mathbf{d}_{ijkl}$ is present, with $\{5,6\} \cap \{i,j,k,l\} = \emptyset$. This vector must be compatible with $\mathbf{a}_{12}$ and $\mathbf{a}_{78}$, and so $ijkl$ is one of 1347,1348,2347,2348. Without loss of generality, suppose that $\mathbf{d}_{1347}$ is present. Now

$$S \cap \Delta(\mathbf{e}) \subseteq \{\mathbf{b}_{13}, \mathbf{a}_{24}, \mathbf{b}_{24}, \mathbf{a}_{57}, \mathbf{b}_{57}, \mathbf{a}_{58}, \mathbf{a}_{67}, \mathbf{b}_{67}, \mathbf{a}_{68}, \mathbf{a}_{78}\}$$

and so $|S \cap \Delta(\mathbf{a}_{12})| \leq 3$. Also, in view of (AB), we have $|S \cap \Delta(\mathbf{e})| \leq 7$. On the other hand, $\gamma_2 \geq 2$, $\delta_1 \geq 1$ and $\delta_2 \geq 1$, whence $|S \cap \Delta(\mathbf{e})| \geq 7$ by Lemma 4.2.

Next, $\mathbf{b}_{34}$ is adjacent to $\mathbf{a}_{12}, \mathbf{c}_{13}, \mathbf{c}_{24}, \mathbf{d}_{3456}, \mathbf{d}_{1347}$ and $\mathbf{e}$ and so

$$\deg(\mathbf{b}_{34}) \geq 6 + |S \cap \Delta(\mathbf{b}_{34})| + |T \cap \Delta(\mathbf{e})|.$$

Now, arguing as in Lemma 4.5, we obtain the contradiction $|S \cap \Delta(\mathbf{a}_{12})| \geq 4$. $\square$

We are now in a position to determine the graphs which can arise. It is convenient to discuss the various possibilites in terms of the graph Q on $\{5,6,7,8\}$ in which i and j are joined by a red edge if $\mathbf{a}_{ij}$ is present, and by a blue edge if $\mathbf{b}_{ij}$ is present. By duality we may assume that $n_b(Q)$, the number of blue edges of Q, is not less than $n_r(Q)$, the number of red edges. We distinguish five cases: (1) Q is incomplete, (2) $(n_b(Q), n_r(Q)) = (6,0)$, (3) $(n_b(Q), n_r(Q)) = (5,1)$, (3) $(n_b(Q), n_r(Q)) = (4,2)$, (5) $(n_b(Q), n_r(Q)) = (3,3)$.

Case 1*: H is incomplete.*

Without loss of generality, suppose that $\mathbf{a}_{56}$ and $\mathbf{b}_{56}$ are absent. By Lemma 4.5, a vector $\mathbf{d}_{ijkl}$ is present, with $\{5,6\} \cap \{i,j,k,l\} = \emptyset$. If also

$\{1,2\} \cap \{i,j,k,l\} = \emptyset$ then $\mathbf{d}_{ijkl} = \mathbf{d}_{3478}$ and $\delta_2 \geq 2$. Since also $\gamma_2 \geq 2$, Lemma 4.2 yields $|S \cap \Delta(\mathbf{e})| \geq 8$, a contradiction. Since $\mathbf{d}_{ijkl}$ must be compatible with $\mathbf{c}_{13}$ and $\mathbf{c}_{24}$ the possibilities for $ijkl$ are 1378, 2478.

By Lemma 4.6, $\mathbf{a}_{78}$ and $\mathbf{b}_{78}$ are absent, and so similarly either $\mathbf{d}_{1356}$ or $\mathbf{d}_{2456}$ is present. Since $\mathbf{d}_{1356}$ and $\mathbf{d}_{2478}$ are incompatible, and $\mathbf{d}_{2456}$ and $\mathbf{d}_{1378}$ are incompatible, we may assume without loss of generality that $\mathbf{d}_{2456}$ and $\mathbf{d}_{2478}$ are present. Then $\mathbf{a}_{24}$ and $\mathbf{b}_{13}$ are excluded and we note that $\mathbf{c}_{14}$ is compatibIe with all possible vectors of type a, b or c. It follows that $\mathbf{c}_{14}$ is present, for otherwise it is excluded by a vector of type d compatible with $\mathbf{a}_{12}$ and $\mathbf{b}_{34}$: such a vector has the form $\mathbf{d}_{13uv}$ where $\{u,v\} \subseteq \{5,6,7,8\}$, and is therefore not compatible with both $\mathbf{d}_{2456}$ and $\mathbf{d}_{2478}$.

Now $\mathbf{a}_{24}, \mathbf{b}_{13}$ are excluded, and we have $\gamma_2 \geq 3$. Since $|S \cap \Delta(\mathbf{e})| \leq 6$ it follows from Lemma 4.2 that $|S \cap \Delta(\mathbf{e})| = 6$, $\gamma_2 = 3$, $\gamma_1 = \delta_1 = \delta_2 = 0$ and $\deg(\mathbf{a}_{12}) = \deg(\mathbf{b}_{34}) = \deg(\mathbf{e})$. In view of Lemma 4.4 (applied to non-adjacent edges of H), there are just two possibilities for $S \cap \Delta(\mathbf{e})$, namely $\{\mathbf{a}_{13}, \mathbf{a}_{57}, \mathbf{a}_{68}, \mathbf{b}_{24}, \mathbf{b}_{67}, \mathbf{b}_{58}\}$ and $\{\mathbf{a}_{13}, \mathbf{b}_{57}, \mathbf{b}_{68}, \mathbf{b}_{24}, \mathbf{a}_{67}, \mathbf{a}_{58}\}$.

Since $\gamma_2 = 3$ and $\gamma_1 = 0$ there can be no vectors of type c other than $\mathbf{c}_{13}, \mathbf{c}_{24}, \mathbf{c}_{14}$. Moreover there are no vectors of type d other than $\mathbf{d}_{2456}, \mathbf{d}_{2478}$: the only possible vectors of type d compatible with $\mathbf{a}_{12}, \mathbf{b}_{34}, \mathbf{c}_{24}$, $\mathbf{d}_{2456}, \mathbf{d}_{2478}$ are $\mathbf{d}_{i457}, \mathbf{d}_{i458}, \mathbf{d}_{i467}, \mathbf{d}_{i468} (i = 1,2)$, but each of these is incompatible with both candidates for $S \cap \Delta(\mathbf{e})$. (For example, $\mathbf{d}_{i467}$ is incompatible with both $\mathbf{b}_{58}$ and $\mathbf{a}_{67}$.)

Since $\mathbf{d}_{2456}$ excludes $\mathbf{a}_{25}, \mathbf{a}_{26}, \mathbf{b}_{37}, \mathbf{b}_{38}$ and $\mathbf{d}_{2478}$ excludes $\mathbf{b}_{35}, \mathbf{b}_{36}, \mathbf{a}_{27}$, $\mathbf{a}_{28}$, we have

$$T \cap \Delta(\mathbf{e}) = \{\mathbf{a}_{15}, \mathbf{a}_{16}, \mathbf{a}_{17}, \mathbf{a}_{18}, \mathbf{b}_{45}, \mathbf{b}_{46}, \mathbf{b}_{47}, \mathbf{b}_{48}\}.$$

By applying the permutation (56) if necessary we may assume that

$$S \cap \Delta(\mathbf{e}) = \{\mathbf{a}_{13}, \mathbf{a}_{57}, \mathbf{a}_{68}, \mathbf{b}_{24}, \mathbf{b}_{58}, \mathbf{b}_{67}\}.$$

The vectors which remain are $\mathbf{a}_{12}, \mathbf{b}_{34}, \mathbf{c}_{13}, \mathbf{c}_{24}, \mathbf{c}_{14}, \mathbf{d}_{2456}, \mathbf{d}_{2478}, \mathbf{e}$ together with those in $S \cap \Delta(\mathbf{e})$ and $T \cap \Delta(\mathbf{e})$; they are pairwise compatible and determine a maximal graph with 22 vertices. An isomorphism ψ from the resulting graph to $M001$ is given by:

$$\begin{array}{r|cccccccc|} \mathbf{u} & \mathbf{a}_{13} & \mathbf{c}_{24} & \mathbf{b}_{34} & \mathbf{a}_{12} & \mathbf{b}_{46} & \mathbf{b}_{47} & \mathbf{b}_{45} & \mathbf{b}_{48} \\ \psi(\mathbf{u}) & \mathbf{d}_{5678} & \mathbf{a}_{15} & \mathbf{e} & \mathbf{a}_{12} & \mathbf{a}_{13} & \mathbf{a}_{14} & \mathbf{b}_{67} & \mathbf{b}_{68} \end{array}$$

$$\begin{array}{r|cccccccccccccc|} \mathbf{u} & \mathbf{a}_{17} & \mathbf{a}_{16} & \mathbf{b}_{24} & \mathbf{c}_{13} & \mathbf{a}_{57} & \mathbf{a}_{68} & \mathbf{e} & \mathbf{a}_{18} & \mathbf{a}_{15} & \mathbf{c}_{14} & \mathbf{b}_{67} & \mathbf{b}_{58} & \mathbf{d}_{2456} & \mathbf{d}_{2478} \\ \psi(\mathbf{u}) & \mathbf{a}_{23} & \mathbf{a}_{24} & \mathbf{a}_{34} & \mathbf{a}_{26} & \mathbf{a}_{37} & \mathbf{a}_{48} & \mathbf{b}_{56} & \mathbf{b}_{57} & \mathbf{b}_{58} & \mathbf{b}_{78} & \mathbf{c}_{15} & \mathbf{c}_{26} & \mathbf{c}_{37} & \mathbf{c}_{48} \end{array}.$$

Here, and in isomorphisms exhibited subsequently, the first eight vectors induce a subgraph H isomorphic to $E443$, with degrees in H equal to 5,6,6,7,7,7,7,7.

Case 2*: $n_b(Q) = 6$, $n_r(Q) = 0$.*

Arguing as in Lemma 4.5, we find that $|S \cap \Delta(\mathbf{b}_{34})| \geq 2$. Since $S \cap \Delta(\mathbf{e})$ contains the six vectors $\mathbf{b}_{ij}(\{i,j\} \subseteq \{5,6,7,8\})$ it follows that

$$S \cap \Delta(\mathbf{e}) = \{\mathbf{b}_{ij} : ij = 13, 24, 56, 57, 58, 67, 68, 78\}.$$

In view of (BC) the only vectors of type c which are present are $\mathbf{c}_{13}\mathbf{c}_{24}$; and in view of (BD) there are no vectors of type d. The 29 vectors which remain are $\mathbf{a}_{12}, \mathbf{b}_{34}, \mathbf{c}_{13}, \mathbf{c}_{24}, \mathbf{e}$ together with those in $S \cap \Delta(\mathbf{e})$ and T. They are pairwise compatible and determine a maximal graph which is the graph $M428$ defined in Section 1.

Case 3*: $n_b(Q) = 5$, $n_r(Q) = 1$.*

We may suppose that the red edge of Q is 56. Thus $\mathbf{a}_{56}$ and $\mathbf{b}_{78}$ are present, while $\mathbf{b}_{56}$ and $\mathbf{a}_{78}$ are absent. By Lemma 4.4, $\mathbf{d}_{3478}$ is present, and so $\deg(\mathbf{a}_{12}) \geq 5 + |S \cap \Delta(\mathbf{a}_{12})| + |T \cap \Delta(\mathbf{e})|$. Since $\deg(\mathbf{e}) \geq \deg(\mathbf{a}_{12})$, it follows that $|S \cap \Delta(\mathbf{b}_{34})| \geq 3$, and hence that $\mathbf{b}_{13}$ and $\mathbf{b}_{24}$ are present. Since also the vectors $\mathbf{b}_{ij}(ij = 57, 58, 67, 68, 78)$ are present, we conclude from (BC) that $\mathbf{c}_{13}, \mathbf{c}_{24}$ are the only vectors of type c present. If another vector of type d is present then it must have the form $\mathbf{d}_{ij78}(ij \neq 34)$ for compatiblity with $\mathbf{a}_{56}$ and $\mathbf{b}_{ij}(ij = 56, 57, 67, 68, 78)$. However, no such vector is compatible with $\mathbf{a}_{12}, \mathbf{c}_{13}, \mathbf{c}_{24}, \mathbf{b}_{13}, \mathbf{b}_{24}$. The 30 vectors which remain are $\mathbf{a}_{12}, \mathbf{b}_{34}, \mathbf{c}_{13}, \mathbf{c}_{24}, \mathbf{d}_{3478}, \mathbf{e}$ together with those in $S \cap \Delta(\mathbf{e})$ and T. They are pairwise compatible and determine a maximal graph which is the graph $M437$ defined in Section 1.

Case 4*: $n_b(Q) = 4$, $n_r(Q) = 2$.*

We distinguish two subcases depending on the factorization of Q induced by the edge-colouring.

Subcase 4*a: The two red edges are non-adjacent.*

We assume, without loss of generality, that edges 57 and 68 are red, so that $S \cap \Delta(\mathbf{e})$ contains $\mathbf{a}_{57}, \mathbf{a}_{68}, \mathbf{b}_{56}, \mathbf{b}_{58}, \mathbf{b}_{67}, \mathbf{b}_{78}$. These vectors exclude all vectors of type d, and all further vectors of type c other than $\mathbf{c}_{14}, \mathbf{c}_{23}$.

By Lemma 4.3, at most one of $\mathbf{c}_{14}, \mathbf{c}_{23}$ is present. If say $\mathbf{c}_{14}$ is present then $\mathbf{b}_{13}, \mathbf{a}_{24}$ are excluded, and by maximality $\mathbf{a}_{13}, \mathbf{b}_{24}$ are present. In this case the vectors which remain are $\mathbf{a}_{12}, \mathbf{b}_{34}, \mathbf{c}_{13}, \mathbf{c}_{24}, \mathbf{c}_{14}, \mathbf{e}$ together with those in $S \cap \Delta(\mathbf{e})$ and T. They are pairwise compatible and determine a maximal graph with 30 vertices. An isomorphism ψ from this graph to $M437$ is given by:

$$\begin{array}{r|cccccccc|ccccccc|} \mathbf{u} & \mathbf{b}_{35} & \mathbf{b}_{46} & \mathbf{a}_{15} & \mathbf{a}_{12} & \mathbf{a}_{18} & \mathbf{b}_{36} & \mathbf{a}_{17} & \mathbf{e} & \mathbf{a}_{16} & \mathbf{b}_{37} & \mathbf{b}_{48} & \mathbf{a}_{25} & \mathbf{b}_{38} & \mathbf{b}_{24} & \mathbf{a}_{57} \\ \psi(\mathbf{u}) & \mathbf{a}_{18} & \mathbf{a}_{25} & \mathbf{b}_{38} & \mathbf{a}_{12} & \mathbf{a}_{15} & \mathbf{b}_{36} & \mathbf{b}_{37} & \mathbf{e} & \mathbf{a}_{16} & \mathbf{a}_{17} & \mathbf{a}_{26} & \mathbf{a}_{27} & \mathbf{a}_{28} & \mathbf{a}_{56} & \mathbf{b}_{13} \end{array}$$

$$\begin{array}{r|ccccccccccccccc|} \mathbf{u} & \mathbf{a}_{68} & \mathbf{b}_{34} & \mathbf{a}_{27} & \mathbf{a}_{26} & \mathbf{a}_{28} & \mathbf{b}_{45} & \mathbf{b}_{47} & \mathbf{b}_{58} & \mathbf{b}_{78} & \mathbf{b}_{56} & \mathbf{b}_{67} & \mathbf{a}_{13} & \mathbf{c}_{13} & \mathbf{c}_{24} & \mathbf{c}_{14} \\ \psi(\mathbf{u}) & \mathbf{b}_{24} & \mathbf{b}_{34} & \mathbf{b}_{35} & \mathbf{b}_{45} & \mathbf{b}_{46} & \mathbf{b}_{47} & \mathbf{b}_{48} & \mathbf{b}_{57} & \mathbf{b}_{58} & \mathbf{b}_{67} & \mathbf{b}_{68} & \mathbf{b}_{78} & \mathbf{c}_{13} & \mathbf{c}_{24} & \mathbf{d}_{3478} \end{array}.$$

If $\mathbf{c}_{14}$ and $\mathbf{c}_{23}$ are absent then $\mathbf{c}_{14}$ is excluded by $\mathbf{b}_{13}$ or $\mathbf{a}_{24}$, while $\mathbf{c}_{23}$ is exluded by $\mathbf{a}_{13}$ or $\mathbf{b}_{24}$. Thus either $\mathbf{b}_{13}, \mathbf{b}_{24}$ are present and we obtain the graph $M417$ defined in Section 1; or $\mathbf{a}_{13}, \mathbf{a}_{24}$ are present and an isomorphism ψ from the resulting graph to $M428$ is given by:

$$\begin{array}{r|cccccccc|cccccc|} \mathbf{u} & \mathbf{a}_{15} & \mathbf{b}_{35} & \mathbf{b}_{45} & \mathbf{a}_{12} & \mathbf{a}_{16} & \mathbf{a}_{17} & \mathbf{a}_{18} & \mathbf{e} & \mathbf{b}_{36} & \mathbf{b}_{37} & \mathbf{b}_{38} & \mathbf{b}_{46} & \mathbf{b}_{47} & \mathbf{b}_{48} \\ \psi(\mathbf{u}) & \mathbf{b}_{35} & \mathbf{a}_{15} & \mathbf{a}_{25} & \mathbf{a}_{12} & \mathbf{b}_{36} & \mathbf{b}_{37} & \mathbf{b}_{38} & \mathbf{e} & \mathbf{a}_{16} & \mathbf{a}_{17} & \mathbf{a}_{18} & \mathbf{a}_{26} & \mathbf{a}_{27} & \mathbf{a}_{28} \end{array}$$

$$\begin{array}{r|ccccccccccccccc|} \mathbf{u} & \mathbf{b}_{34} & \mathbf{a}_{25} & \mathbf{a}_{26} & \mathbf{a}_{27} & \mathbf{a}_{28} & \mathbf{b}_{24} & \mathbf{b}_{13} & \mathbf{b}_{78} & \mathbf{b}_{68} & \mathbf{b}_{67} & \mathbf{b}_{58} & \mathbf{b}_{57} & \mathbf{b}_{56} & \mathbf{c}_{13} & \mathbf{c}_{24} \\ \psi(\mathbf{u}) & \mathbf{b}_{34} & \mathbf{b}_{45} & \mathbf{b}_{46} & \mathbf{b}_{47} & \mathbf{b}_{48} & \mathbf{b}_{13} & \mathbf{b}_{24} & \mathbf{b}_{56} & \mathbf{b}_{57} & \mathbf{b}_{58} & \mathbf{b}_{67} & \mathbf{b}_{68} & \mathbf{b}_{78} & \mathbf{c}_{13} & \mathbf{c}_{24} \end{array}.$$

Subcase 4b: Two red edges are adjacent.

We assume, without loss of generality, that edges 56 and 67 are red, so that $S \cap \Delta(\mathbf{e})$ contains $\mathbf{a}_{56}, \mathbf{a}_{67}, \mathbf{b}_{57}, \mathbf{b}_{58}, \mathbf{b}_{68}, \mathbf{b}_{78}$. By Lemma 4.4 (applied to $\mathbf{a}_{78}, \mathbf{b}_{56}$ and to $\mathbf{a}_{58}, \mathbf{b}_{67}$), we know that $\mathbf{d}_{3478}, \mathbf{d}_{3458}$ are present. It follows that $\delta_2 \geq 2$. Since also $\gamma_2 \geq 2$, it follows from Lemma 4.2 that $\gamma_1 = \delta_1 = 0$, $\gamma_2 = \delta_2 = 2$, $|S \cap \Delta(\mathbf{e})| = 8$ and $\deg(\mathbf{a}_{12}) = \deg(\mathbf{b}_{34}) = \deg(\mathbf{e})$. Since $\deg(\mathbf{a}_{12}) = 6 + |S \cap \Delta(\mathbf{a}_{12})| + |T \cap \Delta(\mathbf{e})|$, it follows that $|S \cap \Delta(\mathbf{b}_{34})| \geq 4$, and hence that $\mathbf{b}_{13}$ and $\mathbf{b}_{24}$ is present. In view of (BD) there are no further vectors of type d. The 31 vectors which remain are $\mathbf{a}_{12}, \mathbf{b}_{34}, \mathbf{c}_{13}, \mathbf{c}_{24}, \mathbf{d}_{3458}, \mathbf{d}_{3478}, \mathbf{e}$ together with those in $S \cap \Delta(\mathbf{e})$ and T. They are pairwise compatible and determine a maximal graph which is the graph $M462$ defined in Section 1.

Case 5: $n_b(Q) = 3$, $n_r(Q) = 3$.

We show first that the three red edges form a path. Otherwise thay form a star, say with edges 56, 57 and 58. By Lemma 4.4, the vectors $\mathbf{d}_{3478}, \mathbf{d}_{3468}, \mathbf{d}_{3467}$ are present. Now we have $\gamma_2 \geq 2$ and $\delta_2 \geq 3$, contradicting Lemma 4.2.

Accordingly we assume, without loss of generality, that edges 56, 58 and 67 are red, so that $S \cap \Delta(\mathbf{e})$ contains $\mathbf{a}_{56}, \mathbf{a}_{58}, \mathbf{a}_{67}, \mathbf{b}_{57}, \mathbf{b}_{68}, \mathbf{b}_{78}$. By Lemma 4.4 (applied to $\mathbf{a}_{78}, \mathbf{b}_{56}$), we know that $\mathbf{d}_{3478}$ is present, and so $\delta_2 \geq 1$. The vectors in $S \cap \Delta(\mathbf{e})$ exclude all further possible vectors of type c other than $\mathbf{c}_{14}$ and $\mathbf{c}_{23}$.

By Lemma 4.3, at most one of $\mathbf{c}_{14}, \mathbf{c}_{23}$ is present. If say $\mathbf{c}_{14}$ is present then $\mathbf{b}_{13}, \mathbf{a}_{24}$ are excluded, and $\gamma_2 \geq 3$. By Lemma 4.2, we have $\gamma_1 = \delta_1 = 0$, $\gamma_2 = 3$, $\delta_2 = 1$, $|S \cap \Delta(\mathbf{e})| = 8$ and $\deg(\mathbf{a}_{12}) = \deg(\mathbf{b}_{34}) = \deg(\mathbf{e})$. In particular, $\mathbf{a}_{13}$ and $\mathbf{b}_{24}$ are present, but no further vectors of type c are present.

Now the only further vector of type d compatible with $S \cap \Delta(\mathbf{e}), \mathbf{a}_{12}, \mathbf{c}_{13}, \mathbf{c}_{24}$ and $\mathbf{c}_{14}$ is $\mathbf{d}_{2478}$. If this vector is present then $\mathbf{a}_{27}, \mathbf{a}_{28}, \mathbf{b}_{35}, \mathbf{b}_{36}$ are excluded. The 28 vectors which remain are $\mathbf{a}_{12}, \mathbf{b}_{34}, \mathbf{c}_{13}, \mathbf{c}_{24}, \mathbf{c}_{14}, \mathbf{d}_{2478}, \mathbf{d}_{3478}, \mathbf{e}$ together with the 8 vectors in $S \cap \Delta(\mathbf{e})$ and the 12 vectors in $T \cap \Delta(\mathbf{e})$. They are pairwise compatible and determine a maximal graph with 28 vertices. An isomorphism ψ from this graph to $M002$ is given by

$$\begin{array}{r|cccccccc|cccccc|}
\mathbf{u} & \mathbf{d}_{2478} & \mathbf{a}_{12} & \mathbf{e} & \mathbf{b}_{48} & \mathbf{a}_{15} & \mathbf{b}_{78} & \mathbf{a}_{13} & \mathbf{a}_{16} & \mathbf{b}_{34} & \mathbf{a}_{67} & \mathbf{a}_{26} & \mathbf{b}_{45} & \mathbf{c}_{13} & \mathbf{b}_{37} \\
\psi(\mathbf{u}) & \mathbf{a}_{14} & \mathbf{b}_{45} & \mathbf{b}_{46} & \mathbf{a}_{12} & \mathbf{a}_{13} & \mathbf{a}_{17} & \mathbf{a}_{18} & \mathbf{e} & \mathbf{a}_{23} & \mathbf{a}_{25} & \mathbf{a}_{27} & \mathbf{a}_{28} & \mathbf{a}_{36} & \mathbf{a}_{37}
\end{array}$$

$$\begin{array}{r|cccccccccccccc|}
\mathbf{u} & \mathbf{a}_{18} & \mathbf{b}_{57} & \mathbf{a}_{17} & \mathbf{b}_{38} & \mathbf{b}_{47} & \mathbf{c}_{14} & \mathbf{d}_{3478} & \mathbf{b}_{24} & \mathbf{a}_{56} & \mathbf{b}_{68} & \mathbf{c}_{24} & \mathbf{a}_{58} & \mathbf{b}_{46} & \mathbf{a}_{25} \\
\psi(\mathbf{u}) & \mathbf{a}_{38} & \mathbf{a}_{78} & \mathbf{b}_{47} & \mathbf{b}_{48} & \mathbf{b}_{56} & \mathbf{b}_{57} & \mathbf{b}_{58} & \mathbf{b}_{67} & \mathbf{b}_{68} & \mathbf{c}_{14} & \mathbf{c}_{25} & \mathbf{c}_{36} & \mathbf{d}_{4567} & \mathbf{d}_{4568}
\end{array}.$$

If $\mathbf{d}_{2478}$ is absent then we obtain a maximal graph with 31 vertices, and an isomorphism ψ from this graph to $M462$ is given by

$$\begin{array}{r|cccccccc|ccccccc|}
\mathbf{u} & \mathbf{b}_{35} & \mathbf{a}_{15} & \mathbf{b}_{47} & \mathbf{b}_{34} & \mathbf{a}_{18} & \mathbf{a}_{16} & \mathbf{b}_{37} & \mathbf{e} & \mathbf{b}_{46} & \mathbf{b}_{48} & \mathbf{a}_{25} & \mathbf{a}_{27} & \mathbf{a}_{13} & \mathbf{b}_{78} & \mathbf{b}_{68} \\
\psi(\mathbf{u}) & \mathbf{a}_{18} & \mathbf{b}_{38} & \mathbf{b}_{48} & \mathbf{a}_{12} & \mathbf{a}_{15} & \mathbf{a}_{16} & \mathbf{a}_{17} & \mathbf{e} & \mathbf{a}_{25} & \mathbf{a}_{26} & \mathbf{a}_{27} & \mathbf{a}_{28} & \mathbf{a}_{56} & \mathbf{a}_{67} & \mathbf{b}_{13}
\end{array}$$

$$\begin{array}{r|cccccccccccccccc|}
\mathbf{u} & \mathbf{b}_{57} & \mathbf{a}_{12} & \mathbf{b}_{38} & \mathbf{b}_{36} & \mathbf{a}_{17} & \mathbf{a}_{26} & \mathbf{a}_{28} & \mathbf{b}_{45} & \mathbf{a}_{67} & \mathbf{a}_{56} & \mathbf{a}_{58} & \mathbf{b}_{24} & \mathbf{c}_{13} & \mathbf{c}_{24} & \mathbf{c}_{14} & \mathbf{d}_{3478} \\
\psi(\mathbf{u}) & \mathbf{b}_{24} & \mathbf{b}_{34} & \mathbf{b}_{35} & \mathbf{b}_{36} & \mathbf{b}_{37} & \mathbf{b}_{45} & \mathbf{b}_{46} & \mathbf{b}_{47} & \mathbf{b}_{57} & \mathbf{b}_{58} & \mathbf{b}_{68} & \mathbf{b}_{78} & \mathbf{c}_{13} & \mathbf{c}_{24} & \mathbf{d}_{3478} & \mathbf{d}_{3458}
\end{array}.$$

If $\mathbf{c}_{14}$ and $\mathbf{c}_{23}$ are absent then (arguing as above) we find that the only possible vectors of type d in addition to $\mathbf{d}_{3478}$ are $\mathbf{d}_{1378}, \mathbf{d}_{2478}$. If both are present then the vectors $\mathbf{a}_{ij} (ij = 27, 28, 17, 18, 24, 13)$ and $\mathbf{b}_{ij} (ij = 35, 36, 45, 46, 24, 13)$ are excluded. The vectors which remain are $\mathbf{a}_{12}, \mathbf{b}_{34}$, $\mathbf{c}_{13}, \mathbf{c}_{24}$, $\mathbf{d}_{1378} \mathbf{d}_{2478}$, $\mathbf{d}_{3478}, \mathbf{e}$ together with the 6 vectors in $S \cap \Delta(\mathbf{e})$ and the 8 vectors in $T \cap \Delta(\mathbf{e})$. They are pairwise compatible and determine a maximal graph with 22 vertices. An isomorphism ψ from this graph to $M001$ is given by

$$\begin{array}{r|cccccccc|}
\mathbf{u} & \mathbf{d}_{3478} & \mathbf{e} & \mathbf{b}_{57} & \mathbf{a}_{16} & \mathbf{b}_{78} & \mathbf{b}_{47} & \mathbf{b}_{37} & \mathbf{a}_{12} \\
\psi(\mathbf{u}) & \mathbf{d}_{5678} & \mathbf{a}_{15} & \mathbf{e} & \mathbf{a}_{12} & \mathbf{a}_{13} & \mathbf{a}_{14} & \mathbf{b}_{67} & \mathbf{b}_{68}
\end{array}$$

$$\begin{array}{r|cccccccccccccc|}
\mathbf{u} & \mathbf{b}_{38} & \mathbf{b}_{34} & \mathbf{a}_{67} & \mathbf{a}_{25} & \mathbf{b}_{68} & \mathbf{a}_{58} & \mathbf{a}_{15} & \mathbf{a}_{56} & \mathbf{b}_{48} & \mathbf{a}_{26} & \mathbf{d}_{2478} & \mathbf{c}_{13} & \mathbf{d}_{1378} & \mathbf{c}_{24} \\
\psi(\mathbf{u}) & \mathbf{a}_{23} & \mathbf{a}_{24} & \mathbf{a}_{26} & \mathbf{a}_{34} & \mathbf{a}_{37} & \mathbf{a}_{48} & \mathbf{b}_{56} & \mathbf{b}_{57} & \mathbf{b}_{58} & \mathbf{b}_{78} & \mathbf{c}_{15} & \mathbf{c}_{26} & \mathbf{c}_{37} & \mathbf{c}_{48}
\end{array}.$$

If just one of $\mathbf{d}_{1378}, \mathbf{d}_{2478}$ is present we obtain a contradiction because either $\mathbf{c}_{23}$ or $\mathbf{c}_{24}$ is then not excluded. If neither $\mathbf{d}_{1378}$ nor $\mathbf{d}_{2478}$ is present then either $\mathbf{a}_{13}, \mathbf{a}_{24}$ or $\mathbf{b}_{13}, \mathbf{b}_{24}$ must be present to exclude $\mathbf{c}_{14}$ and $\mathbf{c}_{23}$. By duality we may assume that $\mathbf{b}_{13}, \mathbf{b}_{24}$ are present. The vectors which remain are $\mathbf{a}_{12}, \mathbf{b}_{34}, \mathbf{c}_{13}, \mathbf{c}_{24}, \mathbf{d}_{3478}, \mathbf{e}$ together with the 8 vectors in $S \cap \Delta(\mathbf{e})$ and the 16 vectors in T. They are pairwise compatible and determine a maximal graph with 30 vertices. An isomorphism ψ from this graph to $M437$ is given by

$$\begin{array}{r|cccccccc|ccccccc|}
\mathbf{u} & \mathbf{b}_{35} & \mathbf{a}_{25} & \mathbf{a}_{15} & \mathbf{b}_{34} & \mathbf{b}_{37} & \mathbf{b}_{36} & \mathbf{b}_{38} & \mathbf{e} & \mathbf{a}_{16} & \mathbf{a}_{18} & \mathbf{b}_{48} & \mathbf{b}_{46} & \mathbf{a}_{27} & \mathbf{b}_{78} & \mathbf{b}_{68} \\
\psi(\mathbf{u}) & \mathbf{a}_{18} & \mathbf{a}_{25} & \mathbf{b}_{38} & \mathbf{a}_{12} & \mathbf{a}_{15} & \mathbf{b}_{36} & \mathbf{b}_{37} & \mathbf{e} & \mathbf{a}_{16} & \mathbf{a}_{17} & \mathbf{a}_{26} & \mathbf{a}_{27} & \mathbf{a}_{28} & \mathbf{a}_{56} & \mathbf{b}_{13}
\end{array}$$

$$\begin{array}{r|ccccccccccccccc|}
\mathbf{u} & \mathbf{b}_{57} & \mathbf{a}_{12} & \mathbf{a}_{17} & \mathbf{b}_{45} & \mathbf{a}_{28} & \mathbf{a}_{26} & \mathbf{b}_{47} & \mathbf{a}_{67} & \mathbf{b}_{24} & \mathbf{b}_{13} & \mathbf{a}_{58} & \mathbf{a}_{56} & \mathbf{c}_{13} & \mathbf{c}_{24} & \mathbf{d}_{3478} \\
\psi(\mathbf{u}) & \mathbf{b}_{24} & \mathbf{b}_{34} & \mathbf{b}_{35} & \mathbf{b}_{45} & \mathbf{b}_{46} & \mathbf{b}_{47} & \mathbf{b}_{48} & \mathbf{b}_{57} & \mathbf{b}_{58} & \mathbf{b}_{67} & \mathbf{b}_{68} & \mathbf{b}_{78} & \mathbf{c}_{13} & \mathbf{c}_{24} & \mathbf{d}_{3478}
\end{array}.$$

This completes the proof of the Main Theorem formulated in Section 1.

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