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LAPLACE TRANSFORM OF DISTRIBUTION-VALUED FUNCTIONS AND ITS APPLICATIONS

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A b s t r a c t. *For the distribution-valued functions the Laplace transform is defined and some properties of it are proved. In order to illustrate the elaborated results, the conditions are proved to have the unicity and existence of solutions to a mathematical model of the vibrating rod with boundary conditions.*

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1. *Notation and notions*

We repeat some definitions and facts, we need in our exposition but for special case.

Let Q be an open set belonging to $\mathbb{R}^n$. By $\mathcal{D}(Q)$ we denote the space $\{\varphi \in \mathcal{C}^\infty(Q); \mathrm{supp}\varphi \subset K_\varphi\}$, K_φ is a compact set in Q which depends on φ. $\mathcal{D}'(Q)$ is the space of continuous linear functionals on $\mathcal{D}(Q)$ - the *space of distributions.* Every $f \in L_{loc}(Q)$ defines a distribution, called *regular distribution,* denoted by $[f]$,

$$\langle [f], \varphi \rangle = \int_Q f(t)\varphi(t)dt, \quad \varphi \in \mathcal{D}(Q).$$

Important subspaces of $\mathcal{D}'(\mathbb{R}^n)$ we use, are: $\mathcal{D}'(\overline{\mathbb{R}}^n_+) = \{f \in \mathcal{D}'(\mathbb{R}^n),\ \mathrm{supp} f \subset \overline{\mathbb{R}}^n_+\}$, the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}'(\overline{\mathbb{R}}^n_+) = \{f \in \mathcal{S}'(\mathbb{R}^n),\ \mathrm{supp} f \in \overline{\mathbb{R}}^n_+\}$. For the space $\mathcal{S}'(\mathbb{R}^n)$ and its topology cf. [8].

Definition 1. *The Laplace transform (in short LT) of* $f \in \mathcal{S}'(\overline{\mathbb{R}}^n_+)$ *is defined by*

$$\widehat{f}(z) = \mathcal{L}(f)(z) = \langle f(t), \eta(t) e^{-zt} \rangle, \quad z \in \mathbb{R}^n_+ + i\mathbb{R}^n,$$

where $\eta \in \mathcal{C}^\infty(\mathbb{R}^n),\ |\eta^{(\alpha)}(t)| \le C_\alpha,\ \alpha \in \mathbb{N}$ *and* $\eta(t) = 1,\ t \in (-\varepsilon, \infty)^n$; $t \in \mathbb{R}^n \setminus (-\infty, -2\varepsilon)^n,\ \varepsilon > 0$; *For the properties of so defined LT one can consult [8] and [9].*

Let $\mathcal{H}^{\alpha,\beta}(\mathbb{R}^n_+),\ \alpha \ge 0,\ \beta \ge 0$, denote the sets of holomorphic functions on $\mathbb{R}^n_+ + i\mathbb{R}^n$ which satisfy the following growth condition:

$$|f(z)| \le M(1 + |z|^2)^{\alpha/2}(1 + |x|^{-\beta}),\ z = x + iy \in \mathbb{R}^n_+ + i\mathbb{R}^n.$$

For the applications of the LT the following is very important:

Proposition A. *([8] ,p. 191). The algebras* $\mathcal{S}'(\overline{\mathbb{R}}^n_+)$ *and* $\mathcal{H}(\mathbb{R}^n_+) = \bigcup\limits_{\alpha \ge 0} \bigcup\limits_{\beta \ge 0} \mathcal{H}^{\alpha,\beta}(\mathbb{R}^n_+)$ *are isomorphic. This isomorphism is accomplished via the LT.*

We consider the space $e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+) \subset \mathcal{D}'(\overline{\mathbb{R}}^n_+) \subset \mathcal{D}'(\mathbb{R}^n),\ \ \omega > 0$.

Definition 2. *If* $f \in e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+)$, *then the LT of it is*

$$\widehat{f}(z) = \mathcal{L}(f)(z) = \langle f(t), \eta(t) e^{-(z-\omega)t} \rangle, \quad z \in \omega + \mathbb{R}^n_+ + i\mathbb{R}^n.$$

Let $b > 0$. We denote by $\mathcal{A}_b$ the space

$$\mathcal{A}_b = \{f \in e^{\omega t}\mathcal{D}'(\overline{\mathbb{R}}^n_+),\ \ \mathrm{supp} f \subset [b, \infty)^n\}.$$

Now we can define an equivalent relation in $e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+) : f \sim g \Longleftrightarrow f - g \in \mathcal{A}_b$. Then the space $\mathcal{B}_b$ by definition is:

$$\mathcal{B}_b = e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+)/\mathcal{A}_b.$$

Definition 3. *The space* $\mathcal{D}'_\omega([0, b)^n)$ *is by definition:*

$$\mathcal{D}'_\omega([0, b)^n) = \{T \in \mathcal{D}'([0, b)^n); \exists \overline{T} \in e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+),\ \overline{T}|_{(-\varepsilon, b)^n} = T\},$$

where $\overline{T}|_{(-\varepsilon,b)^n}$ *is the restriction of* $\overline{T}$ *on* $(-\varepsilon, b)^n,\ \varepsilon > 0$.

Proposition B. *([6], [7])* $\mathcal{D}'_\omega([0,b)^n)$ *is algebraically isomorphic to* $\mathcal{B}_b$.

2. *Distribution-valued function and its Laplace transform*

Our definition of the distribution-valued functions (in short d-v-f) (cf. [5]) is appropriate to the mathematical models in mechanics.

In this section Ω denotes an open set belonging to $\mathbb{R}^m$.

Definition 4. *Let* ω *be a positive real number. The function* $[u(x,.)],\ x \in \Omega$, *with values in* $e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+) \subset \mathcal{D}'(\overline{\mathbb{R}}^n_+)$:

$$\Omega \ni x \to [u(x,.)] \in e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+),\ \ \omega t = \omega t_1 + ... + \omega t_n$$

will be refered to as distribution-valued function (d-v-f).

The space of d-v-f defined on Ω we denote by $(e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+))^\Omega$. A d-v-f $[u(x,.)] = [u_0(x,.)]e^{\omega t}$, where $[u_0(x,.)] \in \mathcal{S}'(\overline{\mathbb{R}}^n_+)$ for every $x \in \Omega$, hence $[u(x,.)]e^{-\omega t} = [u_0(x,.)] \in \mathcal{S}'(\overline{\mathbb{R}}^n_+)$.

It is easily seen that $[u(x,.)] \in (e^{\omega_1 t}\mathcal{S}'(\overline{\mathbb{R}}^n_+))^\Omega$ for every $\omega_1 \geq \omega$.

If $[u_1(x,.)]$ and $[u_2(x,.)]$ belong to $(e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+))^\Omega$, then $[u_1(x,.)] = [u_2(x,.)]$ if and only if $\langle [u_1(x,t) - [u_2(x,t)],\ \varphi(t)\rangle = 0$ for every $\varphi \in \mathcal{D}(\mathbb{R}^n),\ x \in \Omega$.

Definition 5. *A d-v-f (1) is called continuous if for every* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *the function* $x \to \langle [u(x,t)]e^{-\omega t}, \varphi(t)\rangle$ *is continuous.*

Let $x_0 \in \partial\Omega$ *(*$\partial\Omega$ *is the boundary of* Ω*). By definition:*

$$\lim_{x\to x_0, x\in\Omega}[u(x,.)] = [u_0(t)]e^{\omega t} \in e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+)$$

if $[u(x,.)]e^{-\omega t}$ *converges to* $[u_0(t)]$ *in* $\mathcal{S}'(\mathbb{R}^n)$.

Remarks: a) The definition of the continuity and the limit do not depend on $\omega_1 \geq \omega$.

b) The defined continuity of a d-v-f is equivalent to the statement that the mapping $x \to [u(x,.)]$ of Ω into $e^{\omega t}\mathcal{S}'(\mathbb{R}^n)$ is continuous ($\mathcal{S}'(\mathbb{R}^n)$ provided with the usual topology).

Definition 6. *A distribution* $u \in \mathcal{D}'(\Omega \times \mathbb{R}^n)$ *is called the distribution induced by the d-v-f (1) if*

$$\langle u, \psi\rangle = \int_\Omega \langle [u(x,.)],\ \psi(x,.)\rangle dx,\ \ \psi \in \mathcal{D}(\Omega \times \mathbb{R}^n),$$

provided the integral exists for every $\psi \in \mathcal{D}(\Omega \times \mathbb{R}^n)$.

The next Proposition gives conditions that a distribution, induced by a d-v-f, has been induced by a unique d-v-f.

Proposition 1. *Let $u \in \mathcal{D}'(\Omega \times \mathbb{R}^n_+) \subset \mathcal{D}'(\Omega \times \mathbb{R}^n)$ be the distribution induced by a continuous d-v-f $[u(x,.)]$, $x \in \Omega$. Then $[u(x,.)]$, $x \in \Omega$, is uniquely determined by u in the class of continuous d-v-fs: $\Omega \to e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+)$.*

P r o o f. Assume that there are two d-v-fs $[u_1(x,.)]$ and $[u_2(x,.)]$ which induce the distribution u. Put $[w(x,.)] = [u_1(x,.)] - [u_2(x,.)]$ and let

$$\psi(x,t) \in \mathcal{D}(\Omega \times \mathbb{R}^n),\ \ \psi(x,t) = \alpha(x)\beta(t),\ \ \alpha \in \mathcal{D}(\Omega)\ \text{ and } \beta \in \mathcal{D}(\mathbb{R}^n).$$

By Definition 6 we have for every α and β :

$$\int_\Omega \langle [w(x,.)], \beta(t)\rangle \alpha(x) dx = 0.$$

It follows that

$$\langle [w(x,.)]e^{-\omega t},\ e^{\omega t}\beta(t)\rangle = 0 \quad \text{for a.a. } \ x \in \Omega. \tag{1}$$

Since $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$, (1) is also true for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$, i.e.,

$$\langle [w(x,.)]e^{-\omega t},\ \varphi(t)\rangle = 0 \quad \text{for a.a. } \ x \in \Omega.$$

From the Definition on continuity it follows that the distribution $[w(x,.)] = 0$ for $x \in \Omega$. Consequently, $[u_1(x,.)] = [u_2(x,.)]$, $x \in \Omega$.

Definition 7. *Let $\eta = (\eta_1, ..., \eta_m)$, $\eta_s = 0$ for $s \neq j$ and $\eta_j = 1$. Let $\varepsilon_0 > 0$ be such that for $x_0 \in \Omega$, $\ x_0 + \eta\varepsilon \in \Omega$, $|\varepsilon| < \varepsilon_0$. If for $[u(x,.)] \in (e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+))^\Omega$ the limit*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}([u(x_0 + \varepsilon\eta,.)] - [u(x_0,.)])e^{-\omega t} = [v(x_0,.)]$$

exists in $\mathcal{S}'(\mathbb{R}^n)$ for every $x_0 \in \Omega$, we put by definition

$$\frac{\partial}{\partial x_j}[u(x,.)] = [v(x,.)]e^{\omega t} \in (e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+))^\Omega.$$

If the function $\Omega \ni x \to \langle [u(x,t)]e^{-\omega t},\ \varphi(t)\rangle$ is of class $\mathcal{C}^k(\Omega)$ for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then the d-v-f $[u(x,.)]$ is by definition of class $\mathcal{C}^k(\Omega)$.

Remark. Let $\mathcal{U}(x,t)$ denote $\mathcal{U}(x,t) = \langle [u(x,t)]e^{-\omega t},\ \varphi(t)\rangle$. Then $\frac{\partial}{\partial x}[u(x,.)]$ exists if and only if there exists $v(x,t) \in \mathcal{S}'(\overline{\mathbb{R}}^n_+),\ x \in \Omega$, such that

$$\frac{\partial}{\partial x}\mathcal{U}(x,t) = \frac{\partial}{\partial x}\langle [u(x,t)]e^{-\omega t},\ \varphi(t)\rangle = \langle v(x,t), \varphi(t)\rangle$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \Omega$.

The next two propositions refer to the connection between some operations on a distribution induced by a d-v-f $[u(x,.)]$ and this d-v-f $[u(x,.)]$.

Proposition 2. *Let $u \in \mathcal{D}'(\Omega \times \mathbb{R}^n)$ be the distribution induced by a d-v-f $[u(x,.)] \in (e^{\omega t}\mathcal{S}(\overline{\mathbb{R}}^n_+))^{\Omega}$ of class $\mathcal{C}^1$. Then the distribution derivative $D_{x_j}u,\ j \in (1, ..., m)$, is indiced by the d-v-f:*

$$\Omega \ni x \to \frac{\partial}{\partial x_j}[u(x,.)] = [q_j(x,.)]e^{\omega t} \in e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+),$$

i.e., for every $\psi \in \mathcal{D}(\Omega \times R^n)$ we have

$$\langle D_{x_j}u, \psi\rangle = \int\limits_{\Omega} \Big\langle \frac{\partial}{\partial x_j}u(x,.),\ \ \psi(x,.)\Big\rangle dx.$$

P r o o f. By Definition 6 for every $\psi \in \mathcal{D}(\Omega \times R^n)$:

$$\begin{aligned}\langle D_{x_j}u, \psi\rangle &= -\Big\langle u, \frac{\partial}{\partial x_j}\psi\Big\rangle \\ &= -\int\limits_{\Omega} \Big\langle u(x,.)],\ \frac{\partial}{\partial x_j}\psi(x,.)\Big\rangle dx.\end{aligned}$$

It is well-knowm (cf. [4, Chapter IV, Theorem III]) that the subspace of functions of the form $\sum\limits_i v_i(x)w_i(t)$ is dense in $\mathcal{D}(\Omega \times \mathbb{R}^n)$, where $v_i \in \mathcal{D}(\Omega)$ and $w_i \in \mathcal{D}(\mathbb{R}^n)$. For every $\psi \in \mathcal{D}(\Omega \times \mathbb{R}^n)$ there exists a sequence $\psi_\nu(t) = \sum\limits_i^{\nu} v_i(x)w_i(t)$ which converges in $\mathcal{D}(\Omega \times \mathbb{R}^n)$ to $\psi(x,t)$.

Hence, there exist compact sets $K_v \subset \Omega$ and $K_w \subset \mathbb{R}^n$ such that supp$\psi \subset K_v \times K_w,\ v_i \in \mathcal{C}_0^\infty(K_v)$ and $w_i \in \mathcal{C}_0^\infty(K_w)$ for $i \in \mathbb{N}$; $\{\psi_\nu(x,t)\}_{\nu\in\mathbb{N}}$ converges uniformly on $K_v \times K_w$. Thus

$$\begin{aligned}\langle D_{x_j}u, \psi\rangle &= -\lim_{\nu\to\infty}\sum_i^{\nu}\int\limits_{\Omega}\langle [u(x,.)], w_i(.)\rangle \frac{\partial}{\partial x_j}v_i(x)dx \\ &= \lim_{\nu\to\infty}\sum_i^{\nu}\int\limits_{\Omega}\frac{\partial}{\partial x_j}\langle [u_0(x,.)],\ e^{\omega t}w_i(.)\rangle v_i(x)dx\end{aligned}$$

$$= \lim_{\nu\to\infty} \sum_i^{\nu} \int_\Omega \langle [q(x,.)]e^{\omega t}, w_i(.)\rangle v_i(x)dx$$

$$= \int_\Omega \langle [q(x,.)]e^{\omega t}, \psi(x,.)\rangle dx$$

$$= \int_\Omega \Big\langle \frac{\partial}{\partial x_j}[u(x,.)], \psi(x,.)\Big\rangle dx.$$

Proposition 3. *Let Ω be an open interval $(a,b) \subset \mathbb{R}$. Suppose that $u \in e^{\omega t}\mathcal{S}'((a,b)\times\mathbb{R}^n)$ satisfies the equation $D_x u = 0$. The distribution u is induced by the constant d-v-f defined on (a,b). This is the unique continuous d-v-f on Ω which induces u.*

P r o o f. If $D_x u = 0$, then u does not depend on $x \in (a,b)$. Let ψ be any function belonging to $\mathcal{D}((a,b)\times\mathbb{R}^n)$. As in the previous proof we use the sequence $\psi_\nu(x,t) = \sum_i^{\nu} v_i(x)w_i(t)$ and we have

$$\begin{aligned}\langle u,\psi\rangle &= \lim_{\nu\to\infty}\sum_i^{\nu}\langle\langle u(.), w_i(.)\rangle, v_i(x)\rangle\\ &= \lim_{\nu\to\infty}\sum_i^{\nu}\int_\Omega \langle u(.), w_i(.)\rangle v_i(x)dx\\ &= \int_\Omega \langle u(.), \psi(x,.)\rangle dx.\end{aligned}$$

By Definition 6 u is defined by the d-v-f: $(a,b)\ni x \to [u(.)]$. By Proposition 1 this is the unique continuous d-v-f which induces u.

We are going now to prove some properties of the regular d-v-f.

Let $\mathcal{C}_{\Omega,\overline{\mathbb{R}}^n_+}$ denote the class of functions $f(x,t),\ x\in\Omega,\ t\in\mathbb{R}^n$ such that:

1. $\mathrm{supp} f(x,t) \subset \overline{\mathbb{R}}^n_+,\ \ x\in\Omega$;
2. $|f(x,t)e^{-\omega t}/(1+|t|^2)^{m/2}| \le g(t) \in L^1(\overline{\mathbb{R}}^n_+)$ for an $m\in\mathbb{N}_0$ and for $x\in\Omega$.

Proposition 4. *Let Ω be an open set in $\mathbb{R}^m$.*

a) If $f\in\mathcal{C}_{\Omega,\overline{\mathbb{R}}^n_+}$, then f defines a d-v-f $[f(x,.)]$ which is called regular d-v-f.

b) If $f\in\mathcal{C}_{\Omega,\overline{\mathbb{R}}^n_+}$ and f is continuous in $x\in\Omega$ for almost all $t\in\overline{\mathbb{R}}^n_+$, then $[f(x,.)]$ is a continuous d-v-f on Ω. It defines a regular distribution on $\Omega\times\mathbb{R}^n$, as well.

c) If f is a function defined on $\Omega \times \mathbb{R}^n$ such that $\dfrac{\partial^i}{\partial x_j} f(x,t),\ i = 0, ..., k,$ are continuous in $x \in \Omega$ for a.a. $t \in \overline{\mathbb{R}}^n_+$. and belong to $\mathcal{C}_{\Omega,\overline{\mathbb{R}}^n_+}$, then

$\dfrac{\partial^i}{\partial x^i_j} f(x,t)$ defines a regular continuous d-v-f on Ω :

$\Big[\dfrac{\partial^i}{\partial x^i_j} f(x,t)\Big],\ \ i = 0, ..., k,\ \ j \in (1, ..., m)$ and

$$\frac{\partial^i}{\partial x^i_j}\Big[f(x,.)\Big] = \Big[\frac{\partial^i}{\partial x^i_j} f(x,.)\Big], \quad x \in \Omega.$$

P r o o f. a) By Proposition 3, p. 158 in [5], $[f(x,.)] \in (e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+))^{\Omega}$.

b) By definition of a regular d-v-f defined by $f(x,t)$ depending on the parameter x, we have for $x \in \Omega$ and $\varphi \in S(\mathbb{R}^n)$:

$$\langle [f(x,.)]e^{-\omega t}, \varphi(\cdot)\rangle = \int\limits_{\overline{\mathbb{R}}^n_+} f(x,t)e^{-\omega t}\varphi(t)dt.$$

The proof of b) follows now by the properties of the integral (cf. [4], Proposition 45).

c) We prove only if $i = 1$. For $i > 1$ we have only to repeat the procedure.

By a) f defines a regular d-v-f

$$[f(x,.)] = e^{\omega t}[e^{-\omega t} f(x,t)].$$

Following Definition 7 of the derivative in x_j of $[f(x,.)]$ and the Remark after it we analyse the limit:

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \tfrac{1}{\varepsilon}\langle [e^{-\omega t} f(x+\varepsilon\eta, t)] - [e^{-\omega t} f(x,t)], \varphi(t)\rangle \\
&= \lim_{\varepsilon\to 0} \tfrac{1}{\varepsilon} \int\limits_{\overline{\mathbb{R}}^n_+} e^{-\omega t}(f(x+\varepsilon\eta, t) - f(x,t))\varphi(t)dt \\
&= \int\limits_{\overline{\mathbb{R}}^n_+} e^{-\omega t}\frac{\partial}{\partial x_j} f(x,t)\varphi(t)dt \\
&= \langle e^{-\omega t}\frac{\partial}{\partial x_j} f(x,t), \varphi(t)\rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^n)
\end{aligned}$$

(cf. Proposition 46, p. 62 in [4]).

We apply a) once more to $\dfrac{\partial}{\partial x_j} f(x,t)$. Then this partial derivative defines a d-v-f, as well, and

$$e^{-\omega t}\Big[\frac{\partial}{\partial x_j} f(x,.)] \in \mathcal{S}'(\overline{\mathbb{R}}^n_+).$$

Hence

$$\frac{\partial}{\partial x_j}[f(x,.)] = \Big[\frac{\partial}{\partial x_j} f(x,.)\Big].$$

Proposition 5. *Let $\Omega \subset \mathbb{R}^m$ be an open set and Ω_1 any open subset of Ω, $\overline{\Omega}_1 \subset \Omega$.. Let $f \in \mathcal{C}_{\Omega,\overline{\mathbb{R}}^n_+}$. If for any $x_0 \in \partial\Omega_1$ we have:*

a) There exist

$$\lim_{x\to x_0, x\in\Omega_1} f(x,t) = v_1(t) \quad \text{and} \quad \lim_{x\to x_0, x\in\Omega\setminus\overline{\Omega}_1} f(x,t) = v_2(t)$$

for a.a. $t \in \overline{\mathbb{R}}^n_+$;

b) there exist $p_i \in \mathbb{N}_0$, such that $e^{-\omega t} v_i(t)/(|t|^{p_i}+1) \in L^1(\overline{\mathbb{R}}^n_+)$, $i = 1,2$;

c) $[f(x,.)]$ is a continuous d-v-f on Ω.

Then $f(x,t)$ is continuous on Ω for a.a. $t \in \overline{\mathbb{R}}^n_+$.

P r o o f. By Definition 5

$$\begin{aligned}\lim_{x\to x_0, x\in\Omega_1}[f(x,.)] &= \lim_{x\to x_0, x\in\Omega_1} \int_{\overline{\mathbb{R}}^n_+} f(x,t)e^{-\omega t}\varphi(t)dt \\ &= \int_{\overline{\mathbb{R}}^n_+} v_1(t)e^{-\omega t}\varphi(t)dt\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to x_0, x\in\Omega\setminus\overline{\Omega}_1}[f(x,.)] &= \lim_{x\to x_0, x\in\Omega\setminus\overline{\Omega}_1} \int_{\overline{\mathbb{R}}^n_+} f(x,t)e^{-\omega t}\varphi(t)dt \\ &= \int_{\overline{\mathbb{R}}^n_+} v_2(t)e^{-\omega t}\varphi(t)dt\end{aligned}$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Since $[f(x,.)]$ is continuous, we have:

$$\int_{\overline{\mathbb{R}}^n_+} e^{-\omega t}(v_1(t) - v_2(t))\varphi(t)dt$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Consequently, $v_1(t) = v_2(t)$ for a.a. $t \in \overline{\mathbb{R}}^n_+$.

Proposition 6. *Let $\partial\Omega$ denote the boundary of Ω. If $f \in \mathcal{C}_{\Omega,\overline{\mathbb{R}}^n_+}$ and for an $x_0 \in \partial\Omega$ there exists*

$$\lim_{x \to x_0, x \in \Omega} f(x,t) = f_0(t) \quad \textit{for a.a.} \quad t \in \overline{\mathbb{R}}^n_+,$$

then we have

$$\lim_{x \to x_0, x \in \Omega} [f(x,.)] = [f_0(t)e^{-\omega t}]e^{\omega t} = [f(\cdot)].$$

P r o o f. By Definition 5 we have first to find

$$\begin{aligned} \lim_{\substack{x \to x_0 \\ x \in \Omega}} \langle [f(x,t)]e^{-\omega t}, \varphi(t) \rangle &= \lim_{\substack{x \to x_0 \\ x \in \Omega}} \int_{\overline{\mathbb{R}}^n_+} f(x,t)e^{-\omega t}\varphi(t)dt \\ &= \int_{\overline{\mathbb{R}}^n_+} f_0(t)e^{-\omega t}\varphi(t)dt \end{aligned}$$

because $f \in \mathcal{C}_{\Omega, \overline{\mathbb{R}}^n_+}$. This proves the assertion of Proposition 6.

The product of a numerical function $c(x),\ x \in \Omega$, and a d-v-f $[u(x,.)] \in (e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+))^\Omega,\ c(x)[u(x,.)]$, we define in every point $x_0 \in \Omega:$

$$x_0 \to c(x_0)[u(x_0,.)]. \tag{2}$$

It is easily seen that if $c(x)$ is continuous on Ω, then $c(x)\ [u(x,.)]$ is a continuous d-v-f on Ω.

We cite the next Proposition which gives the relation between the distributional and classical derivatives of a function.

Proposition C. *(cf. [3]) Let $f \in \mathcal{C}^{(p)}((-\infty,b)),\ p \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$, and H_a be a function such that $H_a(x) = 0,\ -\infty < x < a < b \le \infty;\ H_a(x) = 1,\ 0 \le a \le x < b$. Denote by $[\mathbb{H}_a f]$ the regular distribution defined by $H_a f$. Hence, $[H_a f] \in \mathcal{D}'((-\infty,b)),\ \mathrm{supp}[H_a f] \subset [a,b)$ or $[H_a f] \in \mathcal{D}'([a,b))$, as well. By $[f_a^{(p)}],\ p \in \mathbb{N}$, we denote the distribution defined by the function $f_a^{(p)}$ equals $f^{(p)}(x),\ x \in (a,b)$ and equals zero for $x \in (-\infty,a)$ and is not defined for $x = a$.*

Since the function $(H_a f)^{(k)}$ has in general a discontinuity of the first kind in $x = a,\ k = 0,1,...,p$, by the well known formula (cf. [51])

$$\begin{aligned} D^p[H_a f] &= [f_a^{(p)}] + f^{(p-1)}(a)\delta(x-a) + ... + f(a)\delta^{(p-1)}(x-a) \\ &= [f_a^{(p)}] + R_{p,a}(f) = [H_a f^{(p)}] + R_{p,a}(f), \end{aligned}$$

where $D^p[H_a f]$ is the derivative of order p in the sense of distributions, and

$$R_{a,p}(f) = f^{(p-1)}(a)\delta(x-a) + ... + f(a)\delta^{(p-1)}(x-a).$$

We define now the *Laplace transform* (*in short LT*) of a d-v-f.

Definition 8. *Let* $[u(x,.)] \in (e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+))^{\Omega}$. *The LT of* $[u(x,.)]$ *is defined by*

$$\mathcal{L}[u(x,.)](z) \equiv \widehat{u}(x,z) =$$
$$= \langle [u(x,t)]e^{-\omega t},\ \eta(t)e^{-(z-\omega)t}\rangle,\ \ z \in \omega + \mathbb{R}^n_+ + i\mathbb{R}^n.$$

For the function $\eta(t)$ cf. Definition 1. In this way the LT of a d-v-f has been deduced to the LT of tempered distributions.

Definition 9. *The LT of elements belonging to* $\mathcal{D}'_\omega$ *is defined by*

$$\mathcal{L}(\mathcal{D}'_\omega([0,b)^n) = \mathcal{L}(e^{\omega t}\mathcal{S}(\overline{\mathbb{R}}^n_+))/\mathcal{L}(A_b).$$

Proposition 7. *If* $f(x,t)$ *belongs to* $\mathcal{C}_{\Omega,\overline{\mathbb{R}}^n_+}$ *and for every* $x \in \Omega$ *has the classical LT in* t *denoted by* $F(x,z)$, $Re\, z > \omega$, *then the regular d-v-f* $[f(x,.)]$ *has the LT* $\widehat{f}(x,z)$, $Re\, z > \omega$, *and* $\widehat{f}(x,z) = F(x,z)$, $Re\, z > \omega$.

P r o o f. By Proposition 1. a) $f(x,t)$ defines the regular d-v-f $[f(x,.)]$. Then

$$\begin{aligned}\widehat{f}(x,z) &= \langle [f(x,t)]e^{-\omega t},\ \eta(t)e^{-(z-\omega)t}\rangle \\ &= \int_{\overline{\mathbb{R}}^n_+} f(x,t)e^{-zt}dt = F(x,z),\ \ x \in \Omega,\ \ Re\, z > \omega.\end{aligned}$$

Proposition 8. *Let* $x_0 \subset \partial\Omega$. *If* $f(x,t)$ *belongs to* $\mathcal{C}_{\Omega,\overline{\mathbb{R}}^n_+}$ *and converges for almost all* $t \in \overline{\mathbb{R}}^n_+$ *to* $f_0(t)$ *when* $x \to x_0$, $x \in \Omega$, *then*

$$\lim_{x\to x_0, x\in\Omega} \mathcal{L}[f(x,t)](z) = \mathcal{L}([f_0])(z), \quad z \in \omega + \mathbb{R}^n_+ + i\mathbb{R}^n.$$

Since $\eta(t)e^{-(z-\omega)t} \in \mathcal{S}(\mathbb{R}^n)$, $Re\, z > \omega$, the proof of Proposition 8 follows from Proposition 6 and Proposition 7.

3. *Application to mathematical models appeared in mechanics*

In this Section 3 our aim is only to illustrate the method elaborated in Section 2 analysing the generalized solutions to a partial differential equation.

The mathematical model of the vibrating rod is

$$\frac{\partial^4}{\partial x^4}u(x,t)+\frac{\partial^2}{\partial t^2}u(x,t)=0, \quad 0<x<1, \;\; t\geq 0. \tag{3}$$

We shall examine first the uniqueness of generalized solutions to (3) with boundary conditions (4) of clamped ends:

$$\begin{aligned}\lim_{x\to 0^+} u(x,t) &= \lim_{x\to 1^-} u(x,t) = \lim_{x\to 0^+} \frac{\partial}{\partial x}u(x,t)\\ &= \lim_{x\to 1^-} \frac{\partial}{\partial x}u(x,t) = 0, \;\; t\in\overline{\mathbb{R}}_+.\end{aligned} \tag{4}$$

First we have to find the equation in $(e^{-\omega t}\mathcal{S}'(\overline{\mathbb{R}}_+)^{(0,1)}$ which corresponds to equation (3) in such a way that if there exists a solution $[u(x,.)] \in (e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}_+))^{(0,1)}$ of this equation and is given by a function $u(x,t)$ which has the continuous classical partial derivatives $\dfrac{\partial^4}{\partial x^4}u(x,t)$ and $\dfrac{\partial^2}{\partial t^2}u(x,t)$, then $u(x,t)$ has to be a solution to (3). Let us do it.

The function $u(x,t)$, a solution to (3), has its support in $(0,1)\times[0,\infty)$. To use the Laplace transform we have to extend it on $(0,1)\times(-\infty,\infty)$ in such a way that $u(x,t)=0$ on $(0,1)\times(-\infty,0)$. Then by Proposition C

$$\Big[\frac{\partial^2}{\partial t^2}u(x,t)\Big] = D_t^2[u(x,t)] - u(x,0)\delta^{(1)}(t) - \frac{\partial}{\partial t}u(x,t)\Big|_{t=0}\delta(t).$$

Now to equation (3) it corresponds in the space $(e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+))^{(0,1)}$ the equation (cf. Proposition 4 c) and Proposition C):

$$\frac{\partial^4}{\partial x^4}[u(x,.)] + D_t^2[u(x,.)] = B_1(x)\delta(t) + B_0(x)\delta^{(1)}(t), \tag{5}$$

where

$$B_0(x) = u(x,0) \;\text{ and }\; B_1(x) = \frac{\partial}{\partial t}u(x,t)\Big|_{t=0}. \tag{6}$$

The functions B_0 and B_1 are continuous in $x\in(0,1)$ which can be continuously extended to $[0,1]$. D_t^2 is the second partial derivative in the sense of distributions.

Because of the equality in the space $(e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}_+))^{(0,1)}$ equation (5) can be written in the form:

$$\Big\langle \frac{\partial^4}{\partial x^4}[u(x,t)] + D_t^2[u(x,t)] - B_1(x)\delta(t) - B_0(x)\delta^{(1)}(t), \;\; \varphi(t)\Big\rangle = 0$$

for every $\varphi \in \mathcal{S}(\mathbb{R})$. Since $\eta(t)e^{-zt} \in \mathcal{S}(\mathbb{R})$, $z \in \omega + \mathbb{R}_+ + i\mathbb{R}$, by Definition 8 and the Remark after Definition 7 the last equation gives

$$\frac{\partial^4}{\partial x^4}\langle u(x,t)e^{-\omega t}, e^{-(z-\omega)t}\rangle + \langle D_t^2 u(x,t), e^{-zt}\rangle = B_0(x)z + B_1(x),$$

$z \in \omega + \mathbb{R}_+ + i\mathbb{R}$, and finally

$$\frac{\partial^4}{\partial x^4}\widehat{u}(x,z) + z^2\widehat{u}(x,z) = B_1(x) + B_0(x)z. \tag{7}$$

Let us remark that (7) is a classical differential equation in x with a parameter z. Hence we can apply the well known methods to solve it.

3.1 *Uniqueness of a solution to (3)*

Suppose that we have two solutions $u_1(x,t)$ and $u_2(x,t)$ to (3), (4) with the same initial condition (6). Then $W(x,t) = u_1(x,t) - u_2(x,t)$ satisfies the initial condition with $B_0(x) = 0$ and $B_1(x) = 0$, the boundary condition (4) and the homogenuous part of equation (3). Then the corresponding equation (5) is also homogeneous and the LT of it is

$$\frac{\partial^4}{\partial x^4}\widehat{W}(x,z) + z^2\widehat{W}(x,z) = 0, \quad 0 < x < 1, \quad Re\, z > \omega. \tag{8}$$

The general solution to (8) is

$$\widehat{W}(x,z) = \sum_{j=1}^{4} C_j(z)e^{r_j x}, \quad 0 < x < 1, \quad Re\, z > \omega, \tag{9}$$

where $C_j(z),\ j = 1,...,4$, are functions in z, $Re\, z > \omega$ and r_j, $j = 1,...,4$ are solutions to equation $r^4 + z^2 = 0$:

$$r_1 = \sqrt{\frac{z}{2}}(1+i), \quad r_2 = \sqrt{\frac{z}{2}}(-1+i), \quad r_3 = \sqrt{\frac{z}{2}}(-1-i), \quad r_4 = \sqrt{\frac{z}{2}}(1-i). \tag{10}$$

In order to apply Proposition 8 we give an other form to $\widehat{W}(x,z)$ given in (9):

$$\begin{aligned}\widehat{W}(x,z) \quad &= D_1(z)e^{-r_1(1+a-x)} + D_2(z)e^{r_2(x+a)} \\ &+D_3(z)e^{r_3(x+a)} + D_4(z)e^{-r_4(1+a-x)},\end{aligned}$$

where $a > 0$ and $D_i(z),\ i = 1,...,4$ are holomorphic functions for $Re\, z > \omega > 0$. Now we can apply Proposition 8 when we use boundary conditions (4).

In this way the functions $D_j,\ j = 1, ..., 4$ have to satisfy the homogeneous system:

$$\begin{aligned} D_1(z)e^{-r_1(1+a)} + D_2(z)e^{r_2 a} + D_3(z)e^{r_3 a} + D_4(z)e^{-r_4(1+a)} &= 0 \\ D_1(z)e^{-r_1 a} + D_2(z)e^{r_2(1+a)} + D_3(z)e^{r_3(1+a)} + D_4(z)e^{-r_4 a} &= 0 \\ D_1(z)r_1 e^{-r_1(1+a)} + D_2(z)r_2 e^{r_2 a} + D_3(z)r_3 e^{r_3 a} + D_4(z)r_4 e^{-r_4(1+a)} &= 0 \\ D_1(z)r_1 e^{-r_1 a} + D_2(z)r_2 e^{r_2(1+a)} + D_3(z)r_3 e^{r_3(1+a)} + D_4(z)r_4 e^{-r_4 a} &= 0. \end{aligned} \tag{11}$$

We will prove that there is no number $\omega > 0$ such that $D_i(z),\ i = 1, ..., 4$, are solutions to the system (11) and are not identically zero for $Re\, z > \omega$.

The determinant of the system (11) has the form

$$\Delta(z) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ e^{r_1} & e^{r_2} & e^{-r_1} & e^{-r_2} \\ r_1 & r_2 & -r_1 & -r_2 \\ r_1 e^{r_1} & r_2 e^{r_2} & -r_1 e^{-r_1} & -r_2 e^{-r_2} \end{vmatrix} e^{-r_1(1+2a)+r_2(1+2a)} \tag{12}$$

$$= \Delta'(z)e^{-r_1(1+2a)+r_2(1+2a)}.$$

In this calculations we used the relations between r_j : $r_3 = -r_1$ and $r_4 = -r_2$.

Let $k_0 \in \mathbb{N}$ be such that $8k_0^2\pi^2 > \omega$ for a given $\omega > 0$. Then if $z = 8k^2\pi^2$, then $\sqrt{\frac{z}{2}} = 2k\pi$. It is easily seen that $\Delta'(8k^2\pi^2) \neq 0,\ \ k > k_0,\ \ k \in \mathbb{N}$. Since $\Delta'(\xi)$ is continuous function for $\xi > 0$, then $\Delta'(\xi) \neq 0$ in a sequence of open neighbourhoods Ω_k of the points $\xi = 8k^2\pi^2$. Consequently, every function $D_i(z),\ i = 1, ..., 4$, equals zero in each open set in which it is holomorphic, if this open set contains an Ω_k. Hence, there is no $\omega > 0$ such that $\Delta(z) \neq 0$ for $Re\, z > \omega$. This prove that the system (11) has the unique solution $D_j(z) \equiv 0,\ j = 1, ..., 4,\ \ Re\, z > \omega$.

Theorem 1. *A generalized solution $[u(x, .)]$ to (3), (4), (6), such that the d-v-f $\dfrac{\partial^i}{\partial x^i}[u(x, .)] \in (e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}_+))^{(0,1)}$, $\ i = 0, 1, ..., 4$, and $u_x^{(i)}(x, t) \in C_{(0,1)\times\overline{\mathbb{R}}_+}$, $i = 0, 1$, if it exists, it is unique.*

Remark. It is well known that (3), (4) has a family of solutions of the form $u_k(x,t) = V_k(x)T_k(t),\ k \in \mathbb{N}$, (cf. [1]). Then $B_0(x) = V(x)T(0)$ and $B_1(x) = V(x)T^{(1)}(0)$ for a fixed $k \in \mathbb{N}$. With such initial condition the corresponding solution is unique. Hence the boundary conditions of the form (4) is not sufficient for unicity.

3.2. *Existence of the solution to (3), (13), (6)*

We have seen that the existence of a solution to (3) depends not only on the boundary conditions but it also depends on the initial condition. We are interested in finding the form of the initial condition (6) such that equation (3) has a solution in case of supported ends of the vibrating rod. Then the boundary condition is

$$\begin{aligned} \lim_{x\to 0^+} u(x,t) &= \lim_{x\to 1^-} u(x,t) = \lim_{x\to 0^+} \frac{\partial^2}{\partial x^2} u(x,t) = \\ &= \lim_{x\to 1^-} \frac{\partial^2}{\partial x^2} u(x,t) = 0, \quad t \in \mathbb{R}_+. \end{aligned} \tag{13}$$

For this purpose we have first to find a particular solution $\widehat{u}_p(x,z)$ to differential equation (7) which satisfies condition (13) (cf. Proposition 8). This is very easy if $B_0(x)$ and $B_1(x)$ are of some special form. Then by Proposition A we have to show that $\widehat{u}_p(x,z)$ is the LT of a solution to (3).

To reduce the routine work, let us suppose that $B_0 = 0$ in (6). The solution of the homogeneous part of equation (7) is

$$\widehat{u}_0(x,z) = \sum_{j=1}^{4} C_j(z) e^{r_j x}, \quad 0 < x <, \quad Re\, z > \omega, \tag{14}$$

where $C_j(z),\ j = 1,...,4$, have the same properties as in (9) and $r_j,\ j = 1,...,4$ have been given in (10).

In order to find a particular solution $\widehat{u}_1(x,z)$ to (7) with $B_0 = 0$, we use the method named "Variation of Constants".

Suppose that $C_j(x,z),\ j = 1,...,4$ are functions with the continuous first partial derivative in x for every $z,\ Re\, z > \omega$. Then we have to solve the system

$$\begin{aligned} \sum_{j=1}^{4} \frac{\partial}{\partial x} C_j(x,z) e^{r_j x} &= 0 \\ \sum_{j=1}^{4} \frac{\partial}{\partial x} C_j(x,z) r_j e^{r_j x} &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ \sum_{j=1}^{4} \frac{\partial}{\partial x} C_j(x,z) r_j^3 e^{r_j x} &= B_1(x). \end{aligned} \tag{15}$$

The determinant of the system (15) is

$$\Delta(z) = \begin{vmatrix} e^{r_1 x} & \dots & e^{r_4 x} \\ r_1 e^{r_1 x} & \dots & r_4 e^{r_4 x} \\ \dots & \dots & \dots \\ r_1^3 e^{r_1 x} & \dots & r_4^3 e^{r_4 x} \end{vmatrix} = \begin{array}{l} -2(\sqrt{2z})^6 \neq 0, \\ Re\, z > \omega > 0. \end{array} \tag{16}$$

Hence we can find

$$\frac{\partial}{\partial x} C_j(x,z) = \frac{1}{2(\sqrt{2z})^3} A_j B_1(x) e^{-r_j x}, \quad j = 1, ..., 4,$$

where $A_1 = (-1)(1+i)$, $A_2 = -1+i$, $A_3 = 1+i$ and $A_4 = 1-i$. The looked - for functions $C_j(x,z),\ j = 1, ..., 4$ are

$$C_j(x,z) = \frac{1}{2(\sqrt{2z})^3} A_j \Big(\int_0^x B_1(\xi) e^{-r_j \xi} d\xi + E_j(z) \Big), \tag{17}$$

where $E_j(z)$ are undefined functions.

A general solution to (7) with $B_0(x) = 0$ is:

$$\widehat{u}(x,z) = \frac{1}{2(\sqrt{2z})^3} \sum_{j=1}^{4} A_j \Big(\int_0^x B_1(\xi) e^{-r_j \xi} d\xi + E_j(z) \Big) e^{r_j x}. \tag{18}$$

We have two limitations on $\widehat{u}(x,z)$. One of them comes from the boundary condition (13) and the other requires that there exists $[u(x,.)] \in (e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}_+))^{(0,1)}$ such that $\mathcal{L}([u(x,.)]) = \widehat{u}(x,z)$. This limitations give the conditions for $B_1(x)$ to have a solution to (5), (13). For practical use we shall prove

Theorem 2. *A sufficient condition that the differential problem (3), (13) has a solution $[u(x,.)]$ such that the d-v-f*

$$\frac{\partial^i}{\partial x^i}[u(x,.)] \in (e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}_+))^{(0,1)}, \ i = 0, 1, ..., 4,$$

and $u^{(i)}(x,t) \in C_{(0,1)\times\overline{\mathbb{R}}_+}$, $i = 1, 2$, is that we can find functions $E_i(z)$ such that:

1. $\int\limits_0^x B_1(\xi)e^{-r_i\xi}d\xi + E_i(z) = \widehat{F}_i(x,z)e^{-r_i x},\ i = 1,...,4,$ *for* $0 \le x \le 1,$ *where*

$$E_1(z) = E_3(z),\ \ E_2(z) = E_4(z),\ Re\, z > \omega.$$

2. $\widehat{F}_1(x,z) - \widehat{F}_3(x,z) = \widehat{F}_2(x,z) - \widehat{F}_4(x,z) = 0$ *for* $x = 0,$ *and* $x = 1.$

3. $\widehat{F}_i(x,z)/(\sqrt{z})^3$ *is holomorphic for* $Re\, z > \omega$, *for an* $\omega > 0$ *and tends to zero when* z *tends to infinity belonging to the half plane* $\{Re\, z \ge \omega + \delta,\ \delta > 0\}$ *uniformly in* $x \in [0,1]$. *Also*

$$\int\limits_{-\infty}^{\infty} |\widehat{F}(x,z)/(\sqrt{z})^3| d(Imz) < \infty,\ \ 0 < x < 1,\ Re\, z > \omega.$$

P r o o f. Let us suppose that conditions 1., 2. and 3. are satisfied. Then $\widehat{u}(x,z)$, given by (8) can have the form

$$\widehat{u}(x,z) = \frac{1}{2(\sqrt{2z})^3}\sum_{j=1}^{4} A_j \widehat{F}(x,z). \tag{19}$$

By Theorem 3 in [2, I, p. 263] there exist functions $F_i(x,t)$ such that $\mathcal{L}^{-1}(\widehat{F}_i(x,z)\ /(2\sqrt{2z})^2)(t) = F_i(x,t),\ i = 1,...,4,$ and the function $u(x,t) = \mathcal{L}^{-1}\widehat{u}(x,z)(t)$,

$$u(x,t) = \sum_{j=1}^{4} A_j F_j(x,t),\ \ 0 \le x \le 1,\ \ t > 0. \tag{20}$$

Now we can use Proposition 8 to satisfy boundary condition (13). We start with (19) and

$$\frac{\partial^2}{\partial x^2}\widehat{u}(x,z) = \frac{1}{2(\sqrt{2z})^3}\sum_{j=1}^{4} A_j(r_j^{-2}\widehat{F}_j(x,z) + r_1^{-1}B_1(x) + B_1^{(1)}(x)).$$

In order to satisfy boundary condition (13) we obtain the following system:

$$\begin{aligned} \sum_{j=1}^{4} A_j \widehat{F}_j(0,z) \ &= 0 \\ \sum_{j=1}^{4} A_j \widehat{F}_j(1,z) \ &= 0 \\ \sum_{j=1}^{4} A_j(r_j^{-2}\widehat{F}_j(0,z) + B_1^{(1)}(0)) \ &= 0 \\ \sum_{j=1}^{4} A_j(r_j^{-2}\widehat{F}_j(1,z) + B_1^{(1)}(1)) \ &= 0. \end{aligned} \tag{21}$$

Since

$$\sum_{j=1}^{4} A_j = 0 \quad \text{and} \quad \sum_{j=1}^{4} A_j r_j^{-2} = (-1)^{j+1} A_j,$$

system (21) becomes:

$$\begin{aligned} \sum_{j=1}^{4} A_j \widehat{F}_j(0,z) &= 0 \\ \sum_{j=1}^{4} A_j \widehat{F}_j(1,z) &= 0 \\ \sum_{j=1}^{4} (-1)^{j+1} A_j \widehat{F}_j(0,z) &= 0 \\ \sum_{j=1}^{4} (-1)^{j+1} A_j \widehat{F}_j(1,z) &= 0. \end{aligned} \tag{22}$$

From (22) it follows that $E_1(z) = E_3(z), E_2(z) = E_4(z)$ and

$$\widehat{F}_1(x,z) - \widehat{F}_3(x,z) = \widehat{F}_2(x,z) - \widehat{F}_4(x,z) = 0$$

for $x = 0$ and $x = 1$ which is our supposition 2. This completes the proof of Theorem 2.

As an example we consider the case $B_1(x) = \sin k\pi x$, for a $k \in \mathbb{N}$. Then equation (7) becomes

$$\frac{\partial^4}{\partial x^4} \widehat{u}(x,z) + z^2 \widehat{u}(x,z) = \sin k\pi x,\ 0 < x < 1,\ Re\, z > \omega \tag{23}$$

In order to find a solution to (23), we suppose that $\widehat{u}(x,z) = A(z) \sin k\pi x$. In that case (23) gives

$$(k\pi)^4 A(z) \sin k\pi x + z^2 A(z) \sin k\pi x = \sin k\pi x.$$

Hence,

$$A(z) = 1/((k\pi)^4 + z^2),\ Re\, z > 0, \ \text{ and } \ \widehat{u}(x,z) = \sin k\pi x/((k\pi)^4 + z^2).$$

It is easily seen that $\widehat{u}(x,z)$ satisfies the boundary condition (13). A solution to (3), (13) with initial condition (6), expressed by $B_0(x) = 0,\ B_1(x) = \sin k\pi x$ is $u(x,t) = (k\pi)^{-2} \sin k\pi x\ \sin(k\pi)^2 t$. That is a well known result. It is also easy to verify that $B_1(x) = \sin k\pi x$ satisfies the cited conditions in Theorem 2.

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