

THE CIRCLE AND DIVISOR PROBLEM

A. IVIĆ

(Presented at the 2nd Meeting, held on March 26, 2004)

A b s t r a c t. New proofs for the classical bounds

$$P(x) \ll x^{1/3}, \quad \Delta(x) \ll x^{1/3} \log x$$

are given. Here $P(x)$ denotes the error term in the classical circle, and $\Delta(x)$ in the classical divisor problem.

AMS Mathematics Subject Classification (2000): 11N37

Key Words: circle problem, divisor problem, Voronoi summation formula, Selberg class

Recently S.D. Miller and W. Schmid [7] gave a proof of the bound

$$P(x) \ll_{\varepsilon} x^{1/3+\varepsilon} \tag{1}$$

in the classical circle problem. Here as usual $P(x) = \sum_{n \leq x} r(n) - \pi x$, $r(n)$ is the number of representations of the natural number n as a sum of two integer squares, and $\varepsilon > 0$ denotes arbitrarily small numbers. The bound (1) was obtained by the use of the Voronoi summation formula

$$\sum'_{a \leq n \leq b} r(n) f(n) = \pi \int_a^b f(x) dx + \sum_{n=1}^{\infty} r(n) \int_a^b f(x) J_0(2\pi\sqrt{xn}) dx, \tag{2}$$

where $f(x)$ is a suitable smooth function, J is the Bessel function, and \sum' denotes that at $n = a$ and $n = b$ the summand is to be halved if a or b is an integer. In [7], (2) was proved by a two-dimensional Poisson summation formula, but it may be proved analogously like the classical Voronoi formula (see e.g., [2, Chapter 3]). Then, on p. 20, the authors say: “With more effort, one can remove ε from these bounds (i.e. (1), and the analogous bound in the divisor problem), and get Voronoi’s result

$$P(x) \ll x^{1/3}.” \quad (3)$$

The aim of this note is to give a new, simple proof of (3) by using (2). We start by noting that, in view of the non-negativity of $r(n)$, we have

$$\sum_{n=1}^{\infty} f_-(n)r(n) \leq \sum_{X < n \leq 2X} r(n) \leq \sum_{n=1}^{\infty} f_+(n)r(n), \quad (4)$$

where $f_-(x)$ is a smooth, non-negative function supported in $[X, 2X]$ such that $f_-(x) = 1$ for $x \in [X + G, 2X - G]$ ($X^\varepsilon \leq G \leq \sqrt{X}$), while similarly $f_+(x)$ is supported in $[X - G, 2X + G]$ and satisfies $f_+(x) = 1$ for $x \in [X, 2X]$. If henceforth we denote by $f(x)$ either $f_-(x)$ or $f_+(x)$, then $f^{(r)}(x) \ll_r G^{-r}$ ($r = 0, 1, 2, \dots$), and by (2) we have

$$\sum_{n=1}^{\infty} f(n)r(n) = \pi X + O(G) + \sum_{n=1}^{\infty} r(n) \int_{X-G}^{2X+G} f(x) J_0(2\pi\sqrt{xn}) dx. \quad (5)$$

From the theory of Bessel functions we need only the relation (see e.g., N.N. Lebedev [6])

$$\frac{d}{dz} [z^\nu J_\nu(z)] = z^\nu J_{\nu-1}(z) \quad (6)$$

and the asymptotic expansion ($k \in \mathbb{N}$ is arbitrary, but fixed, and $|\arg z| \leq \pi - \varepsilon$)

$$J_\nu(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right) \left(\sum_{j=0}^k c_j(\nu) z^{-2j} + O(|z|^{-2k-2})\right) \\ + \left(\frac{2}{\pi z}\right)^{1/2} \sin\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right) \left(\sum_{j=1}^k d_j(\nu) z^{1-2j} + O(|z|^{-2k-1})\right), \quad (7)$$

with suitable constants $c_j(\nu)$, $d_j(\nu)$. By using (7) and the first derivative test (see e.g., [2, Lemma 2.1]) it is seen that the integral on the right-hand

side of (5) is $\ll X^{1/4}n^{-3/4}$, hence for $Y \geq 2$ we have

$$\sum_{n \leq Y} r(n) \int_{X-G}^{2X+G} f(x) J_0(2\pi\sqrt{xn}) \, dx \ll \sum_{n \leq Y} r(n) X^{1/4} n^{-3/4} \ll (XY)^{1/4}$$

on using partial summation and $\sum_{n \leq x} r(n) \ll x$. Furthermore, by using (6) (with $\nu = 1, 2$), performing two integrations by parts and noting that the support of f'' has measure $\ll G$, we obtain that

$$\begin{aligned} & \sum_{n > Y} r(n) \int_{X-G}^{2X+G} f(x) J_0(2\pi\sqrt{xn}) \, dx \\ &= \sum_{n > Y} \frac{r(n)}{\pi^2 n} \int_{X-G}^{2X+G} f''(x) x J_2(2\pi\sqrt{xn}) \, dx \\ &\ll \sum_{n > Y} r(n) n^{-5/4} G^{-1} X^{3/4} \ll X^{3/4} G^{-1} Y^{-1/4}. \end{aligned}$$

Therefore (5) yields

$$\begin{aligned} \sum_{n=1}^{\infty} f(n) r(n) &= \pi X + O(G) + O((XY)^{1/4}) + O(X^{3/4} G^{-1} Y^{-1/4}) \\ &= \pi X + O(X^{1/3}) \end{aligned}$$

on choosing $G = Y = X^{1/3}$. From (4) and the above estimates we have then

$$\sum_{X < n \leq 2X} r(n) = \pi X + O(X^{1/3}), \quad (8)$$

and (3) follows from (8) on replacing X by $2^{-j}X$ and summing over $j = 1, 2, \dots$

An analogous reasoning gives also the classical bound

$$\Delta(X) \ll X^{1/3} \log X, \quad \Delta(X) = \sum_{n \leq X} d(n) - X(\log X + 2\gamma - 1),$$

in the Dirichlet divisor problem, where $d(n)$ is the number of divisors of n and $\gamma = -\Gamma'(1)$ is Euler's constant. This follows with the use of the corresponding Voronoi formula (see [2, Chapter 3]) for the divisor function, instead of (2). Namely we have

$$\sum'_{a \leq n \leq b} d(n)f(n) = \int_a^b (\log x + 2\gamma)f(x) dx + \sum_{n=1}^{\infty} d(n) \int_a^b f(x)\alpha(xn) dx, \quad (9)$$

where $0 < a < b < \infty$, $f(x) \in C^2[a, b]$, and in standard notation of Bessel functions

$$\begin{aligned} \alpha(x) &= 4K_0(4\pi\sqrt{x}) - 2\pi Y_0(4\pi\sqrt{x}) \\ &= -\sqrt{2}x^{-1/4} \left(\sin(4\pi\sqrt{x} - \pi/4) - (32\pi)^{-1} \cos(4\pi\sqrt{x} - \pi/4) \right) + O(x^{-5/4}). \end{aligned} \quad (10)$$

In case we are dealing with a Voronoi formula for a multiplicative function which is not necessarily non-negative, then often one can use the theorem of P. Shiu [8] on sums of multiplicative functions in short intervals. For example, such is the function $a(n)$, the n -th Fourier coefficient of a holomorphic cusp form with respect to the full modular group, which is a normalized eigenfunction for the Hecke operators (see e.g., M. Jutila [4] for Voronoi-type formulas for these functions). More generally, Voronoi-type formulas for smooth functions f can be obtained for a wide class of arithmetic functions (e.g., the so-called \mathcal{S} -class of A. Selberg; see Kaczorowski-Perelli [5] for an comprehensive account). The author [3] obtained such a formula for the divisor function $d_r(n)$, generated by $\zeta^r(s)$, $r \in \mathbb{N}$. The key to obtaining Voronoi-type formulas in such situations is the relation

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F(s)G(s) ds = \int_0^{\infty} f(x)g(x) dx$$

which holds (see e.g., E.C. Titchmarsh [9]) under suitable conditions if $F(s)$ and $G(s)$ are Mellin transforms of $f(x)$ and $g(x)$, respectively. The ideas used in this note in proving (1) work well also in the general case, enabling one to get bounds for corresponding error terms without the “ ε ”-factor. The simple (explicit) form of the right-hand side of (2) (and (9)-(10)) is due to the fact that

$$\frac{2^{s-p-1}\Gamma(\frac{1}{2}s)}{\Gamma(p - \frac{1}{2}s + 1)} \quad (0 < \Re s = \sigma < p + \frac{3}{2})$$

is the Mellin transform of $x^{-p}J_p(x)$, and a corresponding Mellin pair exists also in the case of the Voronoi formula (9). In the general case the function appearing in the summation formula is a ‘generalized’ Bessel function (see e.g., J.L. Hafner [1]), and its asymptotics may be found by the method developed in Hafner’s paper or by the author in [3].

REFERENCES

- [1] J. L. H a f n e r, *On the representation of summatory function of a class of arithmetic functions*, in “Analytic Number Theory” (ed. M.I. Knopp), LNM 899, Springer Verlag, Berlin etc., 1981, 145-165.
- [2] A. I v i ć, *The Riemann zeta-function*, Dover, Mineola (New York), 2003.
- [3] A. I v i ć, *On the ternary additive divisor problem and the sixth moment of the zeta-function*, in “Sieve Methods, Exponential Sums, and their Applications in Number Theory” (eds. G.R.H. Greaves et al.), Cambridge University Press, Cambridge, 1996, 205-243.
- [4] M. J u t i l a, *A method in the theory of exponential sums*, TATA Institute of Fundamental Research Lecture Notes 80, Springer Verlag, Berlin etc., 1987.
- [5] J. K a c z o r o w s k i and A. P e r e l l i, *The Selberg class; a survey*, in “Number Theory in Progress” (eds. K. Györy et al.), Walter de Gruyter, Berlin–New York, 1999, 953-992.
- [6] N. N. L e b e d e v, *Special functions & their applications*, Dover, New York, 1972.
- [7] S. D. M i l l e r and W. S c h m i d, *Summation formulas, from Poisson and Voronoi to the present*, [arXiv:math.NT/0304187](https://arxiv.org/abs/math/0304187), 2003.
- [8] P. S h i u, *A Brun–Titchmarsh theorem for multiplicative functions*, J. reine angew. Math. 31(1980), 161-170.
- [9] E. C. T i t c h m a r s h, *Introduction to the Theory of Fourier Integrals*, Clarendon Press, Oxford, 1948.

Aleksandar Ivić
Katedra Matematike RGF-a
Universiteta u Beogradu
Džušina 7, 11000 Beograd
Serbia (Yugoslavia)
e-mail: aivic@mat.bg.ac.yu, ivic@rgf.bg.ac.yu